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A PARTITION OF q-COMMUTING MATRIX

Eunmi Choi*

Abstract. We study divisibilities of elements in the q-commuting matrix $C^{(q)}$. We first make a coefficient matrix $\hat{C}$ of $C^{(q)}$ which is independent of q, study divisibilities over $\hat{C}$ and then retrieve our findings to $C^{(q)}$. Finally we partition the $C^{(q)}$ into 2×2 block matrices.

1. Introduction

A q-commuting matrix $C^{(q)} = [e^{(q)}_{i,j}]$ is an arithmetic table of a polynomial $(x+y)^i = \sum_{j=0}^{i} e^{(q)}_{i,j} x^{i-j} y^j$ having q-commuting variables $yx = qxy$ ($q \in \mathbb{Z}$) [2]. For instance, $e^{(q)}_{4,2} = 1 + q + 2q^2 + q^3 + q^4 = (1,1,2,1,1) \circ (1,\dots,q^4) = \hat{e}_{4,2} \circ (1,\dots,q^4)$ with q-coefficient $\hat{e}_{4,2} = (1,1,2,1,1)$ (often written by 11211). Let $e^{(q)}_{i,j} = \hat{e}_{i,j} \circ (1, q, q^2, \dots)$and $\hat{C} = [\hat{e}_{i,j}]$ be the q-coefficient matrix of $C^{(q)}$.

	q-commuting matrix $C^{(q)} = [c^{(q)}_{i,j}]$				coefficient matrix $\hat{C} = [\hat{c}_{i,j}]$				
$i\backslash j$	0	1	2	3	0	1	2	3	4
1	1	1			1	1			
2	1	$1+q$	1		1	$\bar{1}_2$	1		
3	1	$1+q+q^2$	$1+q+q^2$	1	1	$\bar{1}_3$	$\bar{1}_3$	1	
4	1	$1+q+q^2+q^3$	$1+q+2q^2+q^3+q^4$	$1+q+q^2+q^3$	1	$\bar{1}_4$	$\bar{1}_2 2 \bar{1}_2$	1_4	1
5	1	$1+q+\cdots+q^4$	$\dots$		1	$\bar{1}_5$	$\bar{1}_2\bar{2}_3\bar{1}_2$	$\bar{1}_2\bar{2}_3\bar{1}_2$	$\bar{1}_5$

Here $\bar{i}_k$ in $\hat{C}$ is a k-tuple $(i,\dots,i)$ for $i \in \mathbb{Z}$. When $q = \pm 1$, $C^{(1)}$ and $C^{(-1)}$ are Pascal and Pauli Pascal matrices, respectively [4]. A well known Lucas theorem about divisibilities over $C^{(1)}$ says that $e^{(1)}_{p^k,j}$ ($1 \le j < p^k$) is divisible by p if p is prime ([3]). In this work as a generalization of the Lucas theorem, we study divisibilities over $C^{(q)}$ for any q. Since the coefficient matrix $\hat{C}$ is simpler than

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*Corresponding Author.

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$C^{(q)}$, we study divisibilities over $\hat{C}$ and then retrieve our findings to $C^{(q)}$. And the divisibility properties are applied to find 2×2 block matrix forms of $C^{(q)}$. In the work we denote the Pascal matrix $C^{(1)} = [e^{(1)}_{i,j}]$ simply by $P = [e_{i,j}]$.

2. Divisibility in $C^{(q)}$

Over the coefficients matrix $\hat{C} = [\hat{e}_{i,j}]$ of $C^{(q)} = [e^{(q)}_{i,j}]$, operations are naturally determined by, for $(a_i)_{i\geq 0} = (a_0, a_1, \dots)$, $(b_i)_{i\geq 0} = (b_0, b_1, \dots) \in \hat{C}$,

$$(a_i)_{i\geq 0} + (b_i)_{i\geq 0} = (a_i + b_i)_{i\geq 0} \text{ and } (a_i)_{i\geq 0} \cdot (b_i)_{i\geq 0} = (\sum_{j+k=i} a_j b_k)_{i\geq 0}.$$

Indeed, the multiplicative operation arises from

$$((a_i) \cdot (b_i)) \circ (1, q, q^2, \dots) = (a_0 b_0, \sum_{i+j=1} a_i b_j, \sum_{i+j=2} a_i b_j, \dots) \circ (1, q, q^2, \dots).$$

For example $\hat{e}_{4,2} = (1, 1, 2, 1, 1) = (1, 1, 1, 0, 0) + (0, 0, 1, 1, 1) = (1, 1, 1) \cdot (1, 0, 1)$, so it gives a factorization of $e^{(q)}_{4,2}$ in $C^{(q)}$ that

$$e^{(q)}_{4,2} = 1 + q + 2q^2 + q^3 + q^4 = ((1, 1, 1) \cdot (1, 0, 1)) \circ (1, q, q^2) = \sum_{t=0}^{2} q^t (1 + q^2).$$

The next recurrence rules are fundamental in $\hat{C} = [\hat{e}_{i,j}]$.

Theorem 2.1. *Let $\bar{0}_k \hat{e}_{i,j}$ denote $\bar{0}_k$ followed by $\hat{e}_{i,j}$. Then $\hat{e}_{i,j} = \hat{e}_{i-1,j-1} + \bar{0}_j \hat{e}_{i-1,j} = \bar{0}_{i-j} \hat{e}_{i-1,j-1} + \hat{e}_{i-1,j}$ and $\hat{e}_{i,j} = \sum_{t=1}^{i-1} \bar{0}_{j(t-1)} \hat{e}_{i-t,j-1}$ for $0 < j < i$.*

Proof. The expansion $(x + y)^i = \sum_{j=0}^{i} e^{(q)}_{i,j} x^{i-j} y^j$ with $yx = qxy$ yields a recurrence $e^{(q)}_{i,j} = e^{(q)}_{i-1,j-1} + q^j e^{(q)}_{i-1,j} = q^{i-j} e^{(q)}_{i-1,j-1} + e^{(q)}_{i-1,j}$ ([5]). Let $e^{(q)}_{i,j} = \hat{e}_{i,j} \circ (1, q, q^2, \dots)$ be with $\hat{e}_{i,j} = (a_0, a_1, \dots)$ for $a_i \in \mathbb{Z}$. Then

$$q^k e^{(q)}_{i,j} = (a_0, a_1, \dots) \circ (q^k, q^{k+1}, q^{k+2}, \dots)$$
$$= (\underbrace{0, \dots, 0}_{k}, a_0, a_1, \dots) \circ (1, q, q^2, \dots) = \bar{0}_k \hat{e}_{i,j} \circ (1, q, q^2, \dots).$$

Thus

$$\hat{e}_{i,j} \circ (1, q, \dots) = e^{(q)}_{i-1,j-1} + q^j e^{(q)}_{i-1,j} = (\hat{e}_{i-1,j-1} + \bar{0}_j \hat{e}_{i-1,j}) \circ (1, q, \dots)$$

implies $\hat{e}_{i,j} = \hat{e}_{i-1,j-1} + \bar{0}_j \hat{e}_{i-1,j}$. Now if $i = 5$ then

$$\hat{e}_{5,2} = \hat{e}_{4,1} + \bar{0}_2\hat{e}_{4,2} = \hat{e}_{4,1} + \bar{0}_2\hat{e}_{3,1} + \bar{0}_4(\hat{e}_{2,1} + \bar{0}_2\hat{e}_{2,2}) = \sum_{t=1}^{4} \bar{0}_{2(t-1)}\hat{e}_{5-t,1}.$$

So by assuming $\hat{e}_{i,j} = \sum_{t=1}^{i-1} \bar{0}_{j(t-1)}\hat{e}_{i-t,j-1}$, we have

$$\hat{e}_{i+1,j} = \hat{e}_{i,j-1} + \bar{0}_j \sum_{t=1}^{i-1} \bar{0}_{j(t-1)}\hat{e}_{i-t,j-1} = \sum_{t=1}^{i} \bar{0}_{j(t-1)}\hat{e}_{i+1-t,j-1}. \qquad \square$$

The 2nd identity in Theorem 2.1 corresponds to $e^{(q)}_{i,j} = \sum_{t=1}^{i-1} q^{j(t-1)}e_{i-t,j-1}$ in $C^{(q)}$, which is a type of hockey stick formula ([6]).

Theorem 2.2. $(1-q^j)e^{(q)}_{i,j} = (1-q^i)e^{(q)}_{i-1,j-1}$, $(1-q^{i-j})e^{(q)}_{i,j} = (1-q^i)e^{(q)}_{i-1,j}$ *and* $(1-q^j)e^{(q)}_{i,j} = (1-q^{i-j+1})e^{(q)}_{i,j-1}$.

Proof. Clearly $e^{(q)}_{i,j} = e^{(q)}_{i-1,j-1} + q^j(e^{(q)}_{i,j} - q^{i-j}e^{(q)}_{i-1,j-1}) = q^j e^{(q)}_{i,j} + (1-q^i)e^{(q)}_{i-1,j-1}$ and $e^{(q)}_{i,j} = q^{i-j}(e^{(q)}_{i,j} - q^j e^{(q)}_{i-1,j}) + e^{(q)}_{i-1,j} = q^{i-j}e^{(q)}_{i,j} + (1-q^i)e^{(q)}_{i-1,j}$ by Theorem 2.1. So $(1-q^j)e^{(q)}_{i,j} = (1-q^i)e^{(q)}_{i-1,j-1}$ and $(1-q^{i-j})e^{(q)}_{i,j} = (1-q^i)e^{(q)}_{i-1,j}$. $\square$

Lemma 2.3. [1] *Let the length* $len(\hat{e}_{i,j})$ *of* $\hat{e}_{i,j}$ *be the number of digits in* $\hat{e}_{i,j}$. *Then* $len(\hat{e}_{i,j}) = 1 + (i-j)j$ *for any* $0 \le j \le i$.

As a polynomial $e^{(q)}_{i,j}$ in q, Lemma 2.3 shows $\deg(e^{(q)}_{i,j}) = \mathrm{len}(\hat{e}_{i,j}) - 1$.

Lemma 2.4. *Polynomial* $x^u - 1$ *divides* $x^v - 1$ *if and only if* u *divides* v.

Theorem 2.5. *If* p *is a prime then* $e^{(q)}_{p,1} \mid e^{(q)}_{p,j}$ *and* $\hat{e}_{p,1} \mid \hat{e}_{p,j}$ *for* $1 \le j \le p$.

Proof. It is clear if $p = 2$. Let $p = 2k+1$ be an odd prime. Then

$$e^{(q)}_{p,2} = \tfrac{1-q^{2k}}{1-q^2}e^{(q)}_{p,1} = (1 + q^2 + \cdots + q^{2k-2})e^{(q)}_{p,1}$$

by Theorem 2.2, so $e^{(q)}_{p,1}$ divides $e^{(q)}_{p,2}$. Assume $e^{(q)}_{p,1} \mid e^{(q)}_{p,j}$ for some $j \ge 2$, say $e^{(q)}_{p,j} = f(q)e^{(q)}_{p,1}$. Here, $f(q)$ is a polynomial in q with

$$\deg(f(q)) = \deg(e^{(q)}_{p,j}) - \deg(e^{(q)}_{p,1}) = (p-j-1)(j-1)$$

by Lemma 2.3. Then Theorem 2.2 also implies

$$(1-q^{j+1})e^{(q)}_{p,j+1} = (1-q^{p-j})e^{(q)}_{p,j} = (1-q^{p-j})f(q)e^{(q)}_{p,1},$$

so we have $e^{(q)}_{p,1} \mid (1-q^{j+1})e^{(q)}_{p,j+1}$, i.e., $e^{(q)}_{p,1} \mid (1-q)(1+q+\cdots+q^j)e^{(q)}_{p,j+1}$.

But since p is prime, $e^{(q)}_{p,1} = 1+q+\cdots+q^{p-1}$ is irreducible in $\mathbb{Z}[x]$. So $e^{(q)}_{p,1}$ divides either $(1+q+\cdots+q^j)$ or $e^{(q)}_{p,j+1}$. If $j=p-1$ then $e^{(q)}_{p,j} = e^{(q)}_{p,p-1} = e^{(q)}_{p,1}$, while if $j<p-1$ then $e^{(q)}_{p,1} \nmid (1+q+\cdots+q^j)$ so $e^{(q)}_{p,1} \mid e^{(q)}_{p,j+1}$.

Now from $e^{(q)}_{p,j} = f(q)e^{(q)}_{p,1}$, write $f(q) = a_0 + a_1 q + \cdots + a_t q^t$ $(a_i \in \mathbb{Z})$ with $t = \deg(f(q)) = (p-j-1)(j-1)$. Then $t+p-1 = (p-j)j$, so we have

$$\begin{aligned}\hat{e}_{p,j} \circ (1,q,\ldots,q^{(p-j)j}) &= e^{(q)}_{p,j} = f(q)e^{(q)}_{p,1}\\ &= ((a_0,a_1,\ldots,a_t)\circ(1,q,\ldots,q^t))((1,1,\ldots,1)\circ(1,q,\ldots,q^{p-1}))\\ &= \big(\hat{e}_{p,1}\cdot(a_0,a_1,\ldots,a_t)\big)\circ(1,q,\ldots,q^{(p-j)j}).\end{aligned}$$ □

A long polynomial $e^{(q)}_{7,3} = 1+q+2q^2+3q^3+4q^4+4q^5+5q^6+4q^7+4q^8+3q^9+2q^{10}+q^{11}+q^{12}$ was given in [7](p.140). Theorem 2.5 shows $\hat{e}_{7,3} = \bar{1}_2 23\bar{4}_2 5\bar{4}_2 32\bar{1}_2$ is divisible by $\hat{e}_{7,1} = \bar{1}_7$. In fact, $\hat{e}_{7,3} = \bar{1}_7 \cdot (10\bar{1}_3 01)$, which simply provides a factorization $e^{(q)}_{7,3} = \sum_{t=0}^{6} q^t(1+q^2+q^3+q^4+q^6)$.

Theorem 2.6. *If p is a prime then $e^{(q)}_{p,1} \mid e^{(q)}_{p^k,1}$ and $\hat{e}_{p,1} \mid \hat{e}_{p^k,1}$ for $k \geq 1$.*

Proof. Clearly $e^{(q)}_{p^2,1} = \sum_{t=0}^{p^2-1} q^t$ is equal to

$$\sum_{t=0}^{p-1} q^t(1+q^p+\cdots+q^{(p-1)p}) = e^{(q)}_{p,1}(1+q^p+\cdots+q^{(p-1)p}),$$

so we have $\hat{e}_{p^2,1} = \hat{e}_{p,1}\cdot(1,\bar{0}_{p-1},1,\bar{0}_{p-1},\ldots,1,\bar{0}_{p-1},1)$. Similarly

$$e^{(q)}_{p^3,1} = \sum_{t=0}^{p^2-1} q^t(1+q^{p^2}+\cdots+q^{(p-1)p^2}) = e^{(q)}_{p^2,1}(1+q^{p^2}+\cdots+q^{(p-1)p^2}),$$

and $e^{(q)}_{p^k,1} = e^{(q)}_{p^{k-1},1}(1+q^{p^{k-1}}+\cdots+q^{(p-1)p^{k-1}})$ for any $k \geq 1$. Hence $e^{(q)}_{p,1} \mid e^{(q)}_{p^2,1}$, $e^{(q)}_{p^2,1} \mid e^{(q)}_{p^3,1}$ and $e^{(q)}_{p^{k-1},1} \mid e^{(q)}_{p^k,1}$. So we have $\hat{e}_{p,1} \mid \hat{e}_{p^k,1}$ for $k \geq 1$. □

Theorem 2.6 is an extension of the Lucas theorem that if $q=1$ and p is prime then $e_{p,1} = p \mid e_{p^k,j}$ $(1 \leq j < p^k)$ in $P = [e_{i,j}]$. In $C^{(q)}$, every $e^{(q)}_{p,j}$ and $e^{(q)}_{p^k,1}$ are divisible by $e^{(q)}_{p,1}$ by Theorems 2.5 and 2.6. But $e^{(q)}_{p,1}$ may not divide $e^{(q)}_{p^k,j}$ $(j>1)$.

For instance $e_{9,3}^{(q)} = 1+q+2q^2+3q^3+4q^4+5q^5+7q^6+7q^7+8q^8+8q^9+8q^{10}+7q^{11}+7q^{12}+5q^{13}+4q^{14}+3q^{15}+2q^{16}+q^{17}+q^{18} = (1+q^2)(1+q^3+q^6)(1+q^4)\sum_{t=0}^{6} q^t$, which shows $e_{3,1}^{(q)} \nmid e_{9,3}^{(q)}$. But $e_{9,3}$ is still divisible by $(1+q^3+q^6) = 3 = e_{3,1}$ if $q = 1$. Note that $\hat{e}_{6,3} = \bar{1}_2 2\bar{3}_4 2\bar{1}_2 = \bar{1}_6 + \bar{0}_2\bar{1}_6 + \bar{0}_3\bar{1}_6 + \bar{0}_6 1\bar{0}_2 1 = \bar{1}_6 \cdot (10\bar{1}_2) + \bar{0}_6 1\bar{0}_2 1$ is not divisible by $\bar{1}_6 = \hat{e}_{6,1}$. We study divisibilities of $e_{i,j}^{(q)}$ for any i.

Theorem 2.7. *If i is even then $e_{i,1}^{(q)} \nmid e_{i,2}^{(q)}$ but $e_{i-1,1}^{(q)} \mid e_{i,2}^{(q)}$. Indeed $e_{i,2}^{(q)} = e_{i-1,1}^{(q)} \sum_{t=0}^{\frac{i}{2}-1} q^{2t}$. Moreover if $i = p+1$ (p: odd prime) then $e_{i-1,1}^{(q)} \mid e_{i,j}^{(q)}$ for all $1 < j < i-1$.*

Proof. Assume to the contrary that $e_{i,2}^{(q)} = f(q)e_{i,1}^{(q)}$ $(i > 1)$ with a polynomial $f(q)$. Since $e_{i,2}^{(q)} = \frac{1-q^{i-1}}{1-q^2} e_{i,1}^{(q)}$ by Theorem 2.2, $f(q) = \frac{1-q^{i-1}}{1-q^2}$ is of degree $\deg(e_{i,2}^{(q)}) - \deg(e_{i,1}^{(q)}) = i-3$ by Lemma 2.3. But since i is even, $(1-q^2) \nmid (1-q^{i-1})$ by Lemma 2.4, so $e_{i,1}^{(q)}$ does not divide $e_{i,2}^{(q)}$.

Let $i = 2k$. Then $e_{i,1}^{(q)} = 1 + \cdots + q^{i-1} = (1+q)(1+q^2+\cdots+q^{2(k-1)})$, so

$$e_{i,2}^{(q)} = \frac{1-q^{i-1}}{1-q^2} e_{i,1}^{(q)} = \frac{1-q^{2k-1}}{1-q^2}\ (1+q)(1+q^2+q^4+\cdots+q^{2(k-1)})$$
$$= (1+q+q^2+\cdots+q^{2k-2})(1+q^2+q^4+\cdots+q^{2(k-1)}) = e_{i-1,1}^{(q)} \sum_{t=0}^{\frac{i}{2}-1} q^{2t},$$

it shows $\hat{e}_{i,2} = \hat{e}_{i-1,1} \cdot (\underbrace{101010\ldots101}_{i-1})$ in the table $\hat{C}$.

Now if $i = p+1$ (prime $p > 2$) then both $e_{i-1,j-1}^{(q)}$ and $e_{i-1,j}^{(q)}$ are multiples of $e_{i-1,1}^{(q)}$ by Theorem 2.5. So $e_{i,j}^{(q)} = e_{i-1,j-1}^{(q)} + q^j e_{i-1,j}^{(q)}$ implies $e_{i-1,1}^{(q)} \mid e_{i,j}^{(q)}$. □

Corollary 2.8. *In $P = [e_{i,j}]$, if i is even then $i \nmid e_{i,2}$ and $i-1 | e_{i,2}$.*

Indeed for even i, $e_{i,2}^{(q)} = \sum_{t=0}^{i-2} q^t(1+q^2+\cdots+q^{i-2})$ shows $e_{i,2} = (i-1)\frac{i}{2}$ if $q = 1$. Theorem 2.7 gives an explicit factorization of $e_{i,2}^{(q)}$, for instance $\hat{e}_{10,2} = \bar{1}_9 \cdot (101010101) = \bar{1}_9 + \bar{0}_2\bar{1}_9 + \bar{0}_4\bar{1}_9 + \bar{0}_6\bar{1}_9 + \bar{0}_8\bar{1}_9 = \bar{1}_2\bar{2}_2\bar{3}_2\bar{4}_2 5\bar{4}_2\bar{3}_2\bar{2}_2\bar{1}_2$ implies $e_{10,2}^{(q)} = 1+q+2q^2+2q^3+3q^4+3q^5+4q^6+4q^7+5q^8+4q^9+4q^{10}+3q^{11}+3q^{12}+2q^{13}+2q^{14}+q^{15}+q^{16} = \sum_{t=0}^{8} q^t(1+q^2+\cdots+q^8) = e_{9,1}^{(q)}(1+q^2+\cdots+q^8)$.

3. Divisibility in $\hat{C}$

Consider $\hat{e}_{7,2} = 11223332211 = \bar{1}_7 + \bar{0}_2\bar{1}_7 + \bar{0}_4\bar{1}_7 = \bar{1}_7 \cdot (10101)$ and $e_{7,2}^{(q)} = \sum_{t=0}^{6} q^t(1+q^2+q^4)$ by Theorem 2.5. And for $\hat{e}_{6,2} = 112232211 = \bar{1}_6 + \bar{0}_2\bar{1}_6 + \bar{0}_4 10101$, Theorem 2.7 shows

$$\begin{aligned} e_{6,2}^{(q)} &= (1+q+\cdots+q^5) + q^2(1+q+\cdots+q^5) + q^4(1+q^2+q^4) \\ &= (1+q+\cdots+q^4)(1+q^2) + q^4(1+q+\cdots+q^4) = \sum_{t=0}^{4} q^t(1+q^2+q^4), \end{aligned}$$

so $\hat{e}_{6,2} = \bar{1}_5 \cdot (10101)$. Notice that the middle three digits(MTD) of $\hat{e}_{7,2}$ and $\hat{e}_{6,2}$ are 333 and 232. Write $\hat{e}_{7,2} = 1122\underline{333}2211$ and $\hat{e}_{6,2} = 112\underline{232}211$ to show $\mathrm{MTD}(\hat{e}_{7,2}) = 333$, $\mathrm{MTD}(\hat{e}_{6,2}) = 232$. If we subtract $\bar{1}_6$ from $\hat{e}_{6,2}$ repeatedly then $\hat{e}_{6,2} - \bar{1}_6 = 001\underline{121}211$ and $(\hat{e}_{6,2} - \bar{1}_6) - 00\bar{1}_6 = 000\underline{010}101$ show that the MTD end in 010.

Theorem 3.1. *If i is even then $MTD(\hat{e}_{i,2}) = (\frac{i}{2}-1, \frac{i}{2}, \frac{i}{2}-1)$ and $\hat{e}_{i,2}$ is divisible by $\hat{e}_{i-1,1}$ but not by $\hat{e}_{i,1}$.*

Proof. Theorem 2.7 shows $\hat{e}_{8,2} = \bar{1}_7 \cdot (1010101) = 11223\underline{343}32211$. Moreover

$$\hat{e}_{10,2} = \hat{e}_{9,1} + \bar{0}_2\hat{e}_{8,1} + \bar{0}_4\hat{e}_{8,2} = 1122334\underline{454}4332211,$$

$$\hat{e}_{12,2} = \hat{e}_{11,1} + \bar{0}_2\hat{e}_{10,1} + \bar{0}_4\hat{e}_{10,2} = 112233445\underline{565}544332211,$$

imply $\mathrm{MTD}(\hat{e}_{2k,2})$ $(3 \le k \le 6)$ are 232, 343, 454, 565, respectively, i.e., $\mathrm{MTD}(\hat{e}_{2k,2}) = (k-1, k, k-1)$ which are the coefficients of q^{2k-3}, q^{2k-2} and q^{2k-1} in $e_{2k,2}^{(q)}$, for $\deg(e_{2k,2}^{(q)}) = (2k-2)2$ by Lemma 2.3. Now we assume that

$$\hat{e}_{2k,2} = (\underbrace{1,1,2,2,\ldots k-1}_{2k-3}, \underline{k-1,k,k-1}, \underbrace{k-1,\ldots,2,2,1,1}_{2k-3})$$

with $\mathrm{MTD}(\hat{e}_{2k,2}) = (k-1,k,k-1)$ for some k. Then

$$\begin{aligned} \hat{e}_{2(k+1),2} &= \hat{e}_{2k+1,1} + \bar{0}_2\hat{e}_{2k,1} + \bar{0}_4\hat{e}_{2k,2} \\ &= \bar{1}_{2k+1} + \bar{0}_2\bar{1}_{2k} + \bar{0}_4(1,1,2,\ldots k-1, \underline{k-1,k,k-1}, k-1,\ldots,2,1,1) \\ &= \begin{pmatrix} 1\,1\,1\,1\,1\ldots & 1 & 1 & 1 & & \\ \quad 1\,1\,1\ldots & 1 & 1 & 1 & 1 & \\ + \qquad 1\ldots\, k-2, & k-2, & k-1, & \underline{k-1}, & \underline{k}, & \underline{k-1}, \ldots\, 2,1,1 \end{pmatrix} \end{aligned}$$

$$= (1,1,2,2,3,\dots,k,\underline{k,k+1,k},k,k-1,\dots,2,1,1),$$

which shows $\mathrm{MTD}(\hat{e}_{2(k+1),2}) = (k,k+1,k)$.

Note $\mathrm{len}(\hat{e}_{2k,2}) = 1+(2k-2)2$ and $\mathrm{len}(\hat{e}_{2k,1}) = 2k$ by Lemma 2.3. Then

$$\hat{e}_{2k,2} - \hat{e}_{2k,1} = (\underbrace{1,1,2,\dots,k-1}_{2k-3},\underline{k-1,k,k-1},\underbrace{k-1,\dots,2,1,1}_{2k-3}) - \bar{1}_{2k}$$
$$= (0,0,1,1,\dots,k-3,k-2,\underline{k-2,k-1,k-2},k-1,k-2,k-2,\dots,1,1),$$

so $\mathrm{MTD}(\hat{e}_{2k,2} - \hat{e}_{2k,1}) = (k-2,k-1,k-2)$. Now if we subtract $\bar{0}_2\hat{e}_{2k,1}$ again from $\hat{e}_{2k,2} - \hat{e}_{2k,1}$ then $\mathrm{MTD}(\hat{e}_{2k,2} - \hat{e}_{2k,1} - \bar{0}_2\hat{e}_{2k,1}) = (k-3,k-2,k-3)$. So continuing subtraction $\bar{0}_{2t}\hat{e}_{2k,1}$ $(t \geq 0)$ repeatedly from $\hat{e}_{2k,2}$, it finally reaches to $(0,\dots,\underline{0,1,0},1,0,1,0,\dots,1,0,1)$. Thus we have

$$\hat{e}_{2k,2} = (\hat{e}_{2k,1} + \bar{0}_2\hat{e}_{2k,1} + \cdots + \bar{0}_{2k-4}\hat{e}_{2k,1}) + \bar{0}_{2k-2}\,\underbrace{101\dots01}_{2k-1},$$

so $\hat{e}_{2k,2}$ is not divisible by $e^{(q)}_{2k,1}$.

Hence with $i = 2k$ and $e^{(q)}_{i,1} = \sum_{t=0}^{i-1} q^t$, the identity $\hat{e}_{2k,2}$ yields

$$e^{(q)}_{i,2} = (e^{(q)}_{i,1} + q^2 e^{(q)}_{i,1} + \cdots + q^{i-4} e^{(q)}_{i,1}) + q^{i-2}(1+q^2+\cdots+q^{i-2})$$
$$= (1+q^2+\cdots+q^{i-4})\sum_{t=0}^{i-1} q^t + (1+q^2+\cdots+q^{i-2})q^{i-2}$$
$$= \sum_{t=0}^{i-2} q^t(1+q^2+\cdots+q^{i-4}+q^{i-2}) = e^{(q)}_{i-1,1}(1+q^2+\cdots+q^{i-2}). \qquad \square$$

Let us move on $\hat{e}_{i,2}$ with odd i. If i is prime then $\hat{e}_{i,1} \mid \hat{e}_{i,2}$ by Theorem 2.5. The $\hat{e}_{7,2} = 1122\underline{333}2211$ and $\hat{e}_{9,2} = 112233\underline{444}332211$ show MTDs 333, 444 respectively. By subtracting $\hat{e}_{9,1} = \bar{1}_9$ from $\hat{e}_{9,2}$, $\mathrm{MTD}(\hat{e}_{9,2} - \hat{e}_{9,1}) = 333$. In following subtractions, $\mathrm{MTD}(\hat{e}_{9,2} - \hat{e}_{9,1} - \bar{0}_2\hat{e}_{9,1}) = 222$, and $\hat{e}_{9,2} - \hat{e}_{9,1} - \bar{0}_2\hat{e}_{9,1} - \bar{0}_4\hat{e}_{9,1} = \bar{0}_6\hat{e}_{9,1}$, i.e., $\hat{e}_{9,2} = \hat{e}_{9,1}\cdot(1010101)$, a multiple of $\hat{e}_{9,1}$.

Theorem 3.2. *If i is odd then MTD$(\hat{e}_{i,2}) = (\frac{i-1}{2},\frac{i-1}{2},\frac{i-1}{2})$ and $\hat{e}_{i,1} \mid \hat{e}_{i,2}$.*

Proof. $\hat{e}_{i,2}$ $(i = 3,5,7)$ equals $\underline{111}$, $11\underline{222}11$, $1122\underline{333}2211$, and $\hat{e}_{9,2} = \bar{1}_9 + \bar{0}_2\bar{1}_9 + \bar{0}_4\bar{1}_9 + \bar{0}_6\bar{1}_9 = 112233\underline{444}332211 = \bar{1}_9\cdot(1010101)$. For some $i = 2k+1$, we assume $\mathrm{MTD}(\hat{e}_{i,2}) = (k,k,k)$ in $\hat{e}_{i,2} = \underbrace{1122\dots(k-1)}_{2(k-1)}\ \underline{kkk}\ \underbrace{(k-1)\dots2211}_{2(k-1)}$. Then

$$\begin{aligned}\hat{e}_{i+2,2} &= \hat{e}_{2k+2,1} + \bar{0}_2(\hat{e}_{2k+1,1} + \bar{0}_2\hat{e}_{2k+1,2}) \\ &= \bar{1}_{2k+2} + \bar{0}_2\bar{1}_{2k+1} + \bar{0}_4(1122\ldots(k-1)(k-1)\underline{kkk}(k-1)(k-1)\ldots 2211) \\ &= \underbrace{1122\ldots kk}_{2k}\,\underline{(k+1)(k+1)(k+1)}\,\underbrace{kk\ldots 2211}_{2k},\end{aligned}$$

because $\mathrm{len}(\hat{e}_{i,2}) = 1 + (i-2)2$. Thus $\mathrm{MTD}(\hat{e}_{i+2,2}) = (k+1, k+1, k+1)$.

Now subtract $\hat{e}_{i,1}$ from $\hat{e}_{i,2}$ repeatedly. Then $\mathrm{MTD}(\hat{e}_{i,2})$ are changed to (k,k,k), $(k-1,k-1,k-1)$, $(k-2,k-2,k-2)$ and so on, and finally ends up with $(0,0,0)$. This concludes that $\hat{e}_{i,1} = \bar{1}_i$ divides $\hat{e}_{i,2}$ when i is odd. □

The $\left(\begin{array}{l} \qquad \hat{e}_{7,3} \\ \hline 1\,111111 \\ 1111111 \\ 1111111 \\ 1111111 \\ +1111111 \\ \hline 1\,123445443211 \end{array}\right)$ and $\left(\begin{array}{l} \qquad\qquad \hat{e}_{9,3} \\ \hline 1\,11111111 \\ 111111111 \\ 111111111 \\ 111111111 \\ 111111111 \\ 222222222 \\ 111111111 \\ 111111111 \\ +1001001 \\ \hline 1\,12345778887754321 1 \end{array}\right)$ show $\hat{e}_{7,3} = \bar{1}_2 23\bar{4}_2 5\bar{4}_2 32\bar{1}_2$ $= \hat{e}_{7,1} \cdot (1011101)$ and $e^{(q)}_{7,3} = \sum_{t=0}^{6} q^t(1 + q^2 + q^3 + q^4 + q^6)$ is a multiple of $e^{(q)}_{7,1}$. So $\hat{e}_{7,1}$ divides $\hat{e}_{7,j}$ $(1 \le j \le 6)$. But $e^{(q)}_{9,3}$ is not a multiple of $e^{(q)}_{9,1}$, for $\hat{e}_{9,3} = \bar{1}_9 \cdot (10\bar{1}_4 20\bar{1}_2) + \bar{0}_{12}1\bar{0}_2 1\bar{0}_2 1$.

4. Block matrix forms of $C^{(q)}$ and $\hat{C}$

Let $\hat{B}_{\langle n,m\rangle}$ [resp. $B^{(q)}_{\langle n,m\rangle}$] be a block matrix situated from $(n,m)^{\text{th}}$ to $(n+1, m+1)^{\text{th}}$ places in $\hat{C}$ [resp. $C^{(q)}$]. Let $J = \begin{bmatrix} 1\,0 \\ 1\,1 \end{bmatrix}$ and $L = \begin{bmatrix} 1, 1+q \\ 0, q^2 \end{bmatrix}$ for $q \in \mathbb{Z}$. Then $J^q = \begin{bmatrix} 1\,0 \\ q\,1 \end{bmatrix}$, $J^{1+q} = \begin{bmatrix} 1, & 0 \\ 1+q, & 1 \end{bmatrix}$ and $J^{1+q+q^2} = \begin{bmatrix} 1, & 0 \\ 1+q+q^2, & 1 \end{bmatrix}$. By rewriting $J^{1+q} = \begin{bmatrix} 1, & 0 \\ \bar{1}_2 \circ (1,q), & 1 \end{bmatrix}$ and $J^{1+q+q^2} = \begin{bmatrix} 1, & 0 \\ \bar{1}_3 \circ (1,q,q^2), & 1 \end{bmatrix}$, we denote the associated matrices in $\hat{C}$ by $\hat{J}^1 = J$ and $\hat{J}^{\bar{1}_2} = \begin{bmatrix} 1 & 0 \\ \bar{1}_2 & 1 \end{bmatrix}$, so $\hat{J}^{\bar{1}_k} = \begin{bmatrix} 1 & 0 \\ \bar{1}_k & 1 \end{bmatrix}$ in $\hat{C}$ matches to $J^{1+q+\cdots+q^{k-1}}$ in $C^{(q)}$ for $k \ge 0$. Similarly from $L = \begin{bmatrix} 1, \bar{1}_2 \circ (1,q) \\ 0, \bar{0}_2 1 \circ (1,q,q^2) \end{bmatrix}$, $L^2 = \begin{bmatrix} 1, 1+q+q^2+q^3 \\ 0, q^4 \end{bmatrix} = \begin{bmatrix} 1, \bar{1}_4 \circ (1,q,q^2,q^3) \\ 0, \bar{0}_4 1 \circ (1,q,\ldots,q^4) \end{bmatrix}$ and $L^k = \begin{bmatrix} 1, \bar{1}_{2k} \circ (1,q,\ldots,q^{2k-1}) \\ 0, \bar{0}_{2k} 1 \circ (1,q,\ldots,q^{2k}) \end{bmatrix}$, and their corresponding matrices in $\hat{C}$ are $\hat{L} = \begin{bmatrix} 1, \bar{1}_2 \\ 0, \bar{0}_2 1 \end{bmatrix}$, $\hat{L}^2 = \begin{bmatrix} 1, \bar{1}_4 \\ 0, \bar{0}_4 1 \end{bmatrix}$ and $\hat{L}^k = \begin{bmatrix} 1, \bar{1}_{2k} \\ 0, \bar{0}_{2k} 1 \end{bmatrix}$.

Theorem 4.1. *The $\hat{C}$ is partitioned into 2×2 blocks with $\hat{B}_{\langle 2k,0\rangle} = \hat{J}\hat{L}^k$ and $\hat{B}_{\langle 2k,2k\rangle} = \hat{J}^{\bar{1}_{2k+1}}$ for all $k \geq 0$.*

Proof. In $\hat{C}$, $\hat{J}^1 = \begin{bmatrix} 1\,0 \\ 1\,1 \end{bmatrix}$, $\hat{J}^{\bar{1}_3} = \begin{bmatrix} 1, & 0 \\ \bar{1}_3, & 1 \end{bmatrix}$ are blocks $\hat{B}_{\langle 0,0\rangle}$ and $\hat{B}_{\langle 2,2\rangle}$, so $\begin{bmatrix} 1, & 0 \\ \bar{1}_{2k+1}, & 1 \end{bmatrix}$ $= \hat{J}^{\bar{1}_{2k+1}}$ is the block $\hat{B}_{\langle 2k,2k\rangle}$. Moreover $\hat{J}\hat{L} = \begin{bmatrix} 1, \bar{1}_2 \\ 1, \bar{1}_3 \end{bmatrix}$, $\hat{J}\hat{L}^2 = \begin{bmatrix} 1, \bar{1}_4 \\ 1, \bar{1}_5 \end{bmatrix}$ and $\hat{J}\hat{L}^k = \begin{bmatrix} 1\,0 \\ 1\,1 \end{bmatrix} \begin{bmatrix} 1, \bar{1}_{2k} \\ 0, \bar{0}_{2k}1 \end{bmatrix} = \begin{bmatrix} 1, \bar{1}_{2k} \\ 1, \bar{1}_{2k+1} \end{bmatrix}$ are the blocks $\hat{B}_{\langle 2k,0\rangle}$ for all $k \geq 0$. □

Besides the first and last blocks $\hat{J}\hat{L}^k$ and $\hat{J}^{\bar{1}_{2k+1}}$ in the k^{th}row of the block matrix $\hat{C}$, consider the block $\hat{B}_{\langle 4,2\rangle} = \begin{bmatrix} \hat{e}_{4,2}, \hat{e}_{4,3} \\ \hat{e}_{5,2}, \hat{e}_{5,3} \end{bmatrix} = \begin{bmatrix} \bar{1}_2 2\bar{1}_2, & \bar{1}_4 \\ \bar{1}_2\bar{2}_3\bar{1}_2, & \bar{1}_2\bar{2}_3\bar{1}_2 \end{bmatrix}$. Since $e^{(q)}_{i,2}$ is divisible by $1 + q^2 + \cdots + q^{i-2}$ (Theorem 2.7 with even i) or $\sum_{t=0}^{i-1} q^t$ (Theorem 2.5 with prime i), the associated block in $C^{(q)}$ is $B^{(q)}_{\langle 4,2\rangle} = (1+q^2) \begin{bmatrix} \sum_{t=0}^{2} q^t, 1+q \\ \sum_{t=0}^{4} q^t, \sum_{t=0}^{4} q^t \end{bmatrix}$, so

$$\hat{B}_{\langle 4,2\rangle} = (101) \begin{bmatrix} \bar{1}_3, \bar{1}_2 \\ \bar{1}_5, \bar{1}_5 \end{bmatrix} \text{ in } \hat{C}. \tag{1}$$

Consider the blocks $\hat{B}_{\langle 6,j\rangle}$ $(j = 2, 4)$. For $\hat{B}_{\langle 6,2\rangle} = \begin{bmatrix} \bar{1}_2\bar{2}_2 3\bar{2}_2\bar{1}_2, & \bar{1}_2 2\bar{3}_4 2\bar{1}_2 \\ \bar{1}_2\bar{2}_2\bar{3}_3\bar{2}_2\bar{1}_2, & \bar{1}_2 23\bar{4}_2 5\bar{4}_2 32\bar{1}_2 \end{bmatrix}$, the divisibilities in Theorem 2.7 for $i = 6$ and Theorem 2.5 for $i = 7$ yield

$$e^{(q)}_{6,2} = (1-q+q^2) \sum_{t=0}^{2} q^t \sum_{t=0}^{4} q^t, \quad e^{(q)}_{6,3} = (1-q+q^2) \sum_{t=0}^{3} q^t \sum_{t=0}^{4} q^t,$$

and $e^{(q)}_{7,2} = (1-q+q^2) \sum_{t=0}^{2} q^t \sum_{t=0}^{6} q^t$, $e^{(q)}_{7,3} = (1-q+q^2) \sum_{t=0}^{4} q^t \sum_{t=0}^{6} q^t$, so

$$\hat{B}_{\langle 6,2\rangle} = (1,-1,1) \begin{bmatrix} \bar{1}_5 & \\ & \bar{1}_7 \end{bmatrix} \begin{bmatrix} \bar{1}_3, \bar{1}_4 \\ \bar{1}_3, \bar{1}_5 \end{bmatrix} \text{ in } \hat{C}. \tag{2}$$

Similarly for $\hat{B}_{\langle 6,4\rangle} = \begin{bmatrix} \bar{1}_2\bar{2}_2 3\bar{2}_2\bar{1}_2 & \bar{1}_6 \\ \bar{1}_2 23\bar{4}_2 5\bar{4}_2 32\bar{1}_2 & \bar{1}_2\bar{2}_2\bar{3}_3\bar{2}_2\bar{1}_2 \end{bmatrix}$, Theorems 2.7 and 2.5 show the corresponding block $B^{(q)}_{\langle 6,4\rangle} = (1-q+q^2) \begin{bmatrix} \sum_{t=0}^{2} q^t \sum_{t=0}^{4} q^t, & \sum_{t=0}^{1} q^t \sum_{t=0}^{2} q^t \\ \sum_{t=0}^{4} q^t \sum_{t=0}^{6} q^t, & \sum_{t=0}^{2} q^t \sum_{t=0}^{6} q^t \end{bmatrix}$, so

$$\hat{B}_{\langle 6,4\rangle} = (1,-1,1) \begin{bmatrix} \bar{1}_3 & \\ & \bar{1}_7 \end{bmatrix} \begin{bmatrix} \bar{1}_5, \bar{1}_2 \\ \bar{1}_5, \bar{1}_3 \end{bmatrix} \text{ in } \hat{C}. \tag{3}$$

Now for $\hat{B}_{\langle 8,j\rangle}$ $(j = 2, 4, 6)$, Theorem 2.7 for $i = 8$ and Theorem 3.2 for $i = 9$ show $B^{(q)}_{\langle 8,2\rangle} = (1+q^2+q^4+q^6) \begin{bmatrix} \sum_{t=0}^{6} q^t, & (1+q^3) \sum_{t=0}^{6} q^t \\ (1+q^3+q^6) \sum_{t=0}^{2} q^t, & (1+q^3+q^6) \sum_{t=0}^{6} q^t \end{bmatrix}$, so

$$\hat{B}_{\langle 8,2\rangle} = (101)(1\bar{0}_3 1) \begin{bmatrix} \bar{1}_7 & \\ & 1\bar{0}_2 1\bar{0}_2 1 \end{bmatrix} \begin{bmatrix} 1, & 1\bar{0}_2 1 \\ \bar{1}_3, & \bar{1}_7 \end{bmatrix}. \tag{4}$$

And $\hat{B}^{(q)}_{\langle 8,4\rangle} = \begin{bmatrix} \bar{1}_2 23\bar{5}_2\bar{7}_2 8\bar{7}_2\bar{5}_2 32\bar{1}_2, & \bar{1}_2 2345\bar{6}_4 5432\bar{1}_2 \\ \bar{1}_2 235689(\bar{1}1)_2(12)(\bar{1}1)_2 98653 2\bar{1}_2, & \bar{1}_2 235689(\bar{1}1)_2 12(\bar{1}1)_2 98653 2\bar{1}_2 \end{bmatrix}$

yields its matching block $B^{(q)}_{\langle 8,4\rangle} = (1-q+q^2)(1+q^4)\sum_{t=0}^{6} q^t \begin{bmatrix} \sum_{t=0}^{4} q^t, & \sum_{t=0}^{3} q^t \\ \sum_{t=0}^{8} q^t, & \sum_{t=0}^{8} q^t \end{bmatrix}$, so

$$\hat{B}_{\langle 8,4\rangle} = (1,-1,1)(1\bar{0}_3 1)\begin{bmatrix} \bar{1}_7 & \\ & \bar{1}_9 \end{bmatrix}\begin{bmatrix} \bar{1}_5, \bar{1}_4 \\ \bar{1}_7, \bar{1}_7 \end{bmatrix}. \tag{5}$$

Also $B^{(q)}_{\langle 8,6\rangle} = (1+q^2+q^4+q^6)\begin{bmatrix} \sum_{t=0}^{6} q^t, & \sum_{t=0}^{1} q^t \\ (1+q^3+q^6)\sum_{t=0}^{6} q^t, & (1+q^3+q^6)\sum_{t=0}^{2} q^t \end{bmatrix}$ gives

$$\hat{B}_{\langle 8,6\rangle} = (101)(1\bar{0}_3 1)\begin{bmatrix} 1 & \\ & 1\bar{0}_2 1\bar{0}_2 1 \end{bmatrix}\begin{bmatrix} \bar{1}_7, \bar{1}_2 \\ \bar{1}_7, \bar{1}_3 \end{bmatrix}. \tag{6}$$

Moreover Theorems 2.7 and 2.5 show $\hat{B}_{\langle 10,j\rangle}$ $(j = 2,4,6,8)$ in $\hat{C}$ that

$$\begin{aligned} \hat{B}_{\langle 10,2\rangle} &= (1,-1,1,-1,1)\bar{1}_5 \begin{bmatrix} 1 & \\ & \bar{1}_{11} \end{bmatrix}\begin{bmatrix} \bar{1}_9, \bar{1}_8(1\bar{0}_2 1\bar{0}_2 1) \\ 1, \quad 1\bar{0}_2 1\bar{0}_2 1 \end{bmatrix}, \\ \hat{B}_{\langle 10,4\rangle} &= (1,-1,1,-1,1)(1\bar{0}_2 1\bar{0}_2 1)(1\bar{0}_3 1)\begin{bmatrix} \bar{1}_7 & \\ & \bar{1}_{11} \end{bmatrix}\begin{bmatrix} \bar{1}_5, \bar{1}_6 \\ \bar{1}_5\ \bar{1}_7 \end{bmatrix}, \\ \hat{B}_{\langle 10,6\rangle} &= (1,-1,1,-1,1)(1\bar{0}_2 1\bar{0}_2 1)(1\bar{0}_3 1)\begin{bmatrix} \bar{1}_5 & \\ & \bar{1}_{11} \end{bmatrix}\begin{bmatrix} \bar{1}_7, \bar{1}_4 \\ \bar{1}_7, \bar{1}_5 \end{bmatrix}, \\ \hat{B}_{\langle 10,8\rangle} &= (1,-1,1,-1,1)\bar{1}_5 \begin{bmatrix} 1 & \\ & \bar{1}_{11} \end{bmatrix}\begin{bmatrix} \bar{1}_3(1\bar{0}_2 1\bar{0}_2 1), \bar{1}_2 \\ 1\bar{0}_2 1\bar{0}_2 1, \quad 1 \end{bmatrix}. \end{aligned} \tag{7}$$

Thus block matrix $\hat{C} = \begin{bmatrix} \hat{J} \\ \hat{J}\hat{L} & \hat{J}^{\bar{1}_3} \\ \hat{J}\hat{L}^2 & \hat{B}_{\langle 4,2\rangle} & \hat{J}^{\bar{1}_5} \\ \hat{J}\hat{L}^3 & \hat{B}_{\langle 6,2\rangle} & \hat{B}_{\langle 6,4\rangle} & \hat{J}^{\bar{1}_7} \\ \hat{J}\hat{L}^4 & \hat{B}_{\langle 8,2\rangle} & \hat{B}_{\langle 8,4\rangle} & \hat{B}_{\langle 8,6\rangle} & \hat{J}^{\bar{1}_9} \\ \hat{J}\hat{L}^5 & \hat{B}_{\langle 10,2\rangle} & \hat{B}_{\langle 10,4\rangle} & \hat{B}_{\langle 10,6\rangle} & \hat{B}_{\langle 10,8\rangle} & \hat{J}^{\bar{1}_{11}} \end{bmatrix}$ satisfies (1) to (7), so that $C^{(q)}$ is obtained easily. For example when $q = 2$, $\bar{1}_k$ in $\hat{C}$ matches with $1+2+\cdots+2^{k-1} = 2^k - 1$ in $C^{(2)}$. Similarly $(1,-1,1)$ and $(1,-1,1,-1,1)$ in $\hat{C}$ correspond to $1+(-2)+(-2)^2 = 3$ and $1+(-2)+\cdots+(-2)^4 = 11$ in $C^{(2)}$, respectively. Thus $\hat{B}_{\langle 4,2\rangle} = (101)\begin{bmatrix} \bar{1}_3, \bar{1}_2 \\ \bar{1}_5, \bar{1}_5 \end{bmatrix}$ yields $B^{(2)}_{\langle 4,2\rangle} = 5\begin{bmatrix} 2^3-1, 2^2-1 \\ 2^5-1, 2^5-1 \end{bmatrix}$. And $\hat{B}_{\langle 6,2\rangle}$, $\hat{B}_{\langle 6,4\rangle}$ and $\hat{B}_{\langle 8,2\rangle}$ also match with $B^{(2)}_{\langle 6,2\rangle} = 3\begin{bmatrix} 2^5-1 & \\ & 2^7-1 \end{bmatrix}\begin{bmatrix} 2^3-1, 2^4-1 \\ 2^3-1, 2^5-1 \end{bmatrix}$, $B^{(2)}_{\langle 6,4\rangle} = 3\begin{bmatrix} 2^3-1 & \\ & 2^7-1 \end{bmatrix}\begin{bmatrix} 2^5-1, 2^2-1 \\ 2^5-1, 2^3-1 \end{bmatrix}$ and $B^{(2)}_{\langle 8,2\rangle} = 5\cdot 17\begin{bmatrix} 2^7-1 & \\ & 73 \end{bmatrix}\begin{bmatrix} 1, & 9 \\ 2^3-1, & 2^7-1 \end{bmatrix}$,

and so on. So the block matrix is

$$C^{(2)} = \begin{array}{cc|cc|cc|cc|} \multicolumn{2}{c}{0} & \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & 3 & 4 \\ \hline 1 & & & & & & & \\ 1 & 1 & & & & & & \\ \cdots & & & & & & & \\ 1 & 3 & 1 & & & & & \\ 1 & 7 & 7 & 1 & & & & \\ \cdots & & & & & & & \\ 1 & 15 & 35 & 15 & 1 & & & \\ 1 & 31 & 155 & 155 & 31 & 1 & & \\ \cdots & & & & & & & \\ 1 & 63 & 651 & 1395 & 651 & 63 & 1 & \\ 1 & 127 & 2667 & 11811 & 11811 & 2667 & 127 & 1 \\ & & \cdots & & & & & \end{array}$$

In particular the block matrix $C^{(q)}$ with $q = \pm 1$ satisfies the followings.

Theorem 4.2. *(1)* $C^{(-1)} = \begin{bmatrix} J \\ J & J \\ J & 2J & J \\ J & 3J & 3J & J \\ J & 4J & 6J & 4J & J \\ J & 5J & 10J & \dots \end{bmatrix}$ *with the matrix J above.*

(2) In $C^{(1)}$, all integer factors of $B^{(1)}_{\langle n,m\rangle}$ equal the subscripts k of $\bar{1}_k$ in $\hat{B}_{\langle n,m\rangle}$.

Proof. If $q = -1$ then both $L = \begin{bmatrix} 1, 1+q \\ 0, q^2 \end{bmatrix}$ and $\hat{L}$ are identity matrices, and $\bar{1}_k$ in $\hat{C}$ matches with 1 if k is odd, otherwise 0 in $C^{(-1)}$. So $\hat{J}\hat{L}^k = J = \hat{J}^{\bar{1}_{2k+1}}$ for all k. And $(1,-1,1)$ and $(1,-1,1,-1,1)$ in $\hat{C}$ are 3 and 5 in $C^{(-1)}$. So the blocks $\hat{B}_{\langle 4,2\rangle}$ and $\hat{B}_{\langle 6,2\rangle}$ correspond to $B^{(-1)}_{\langle 4,2\rangle} = 2\begin{bmatrix} 1 0 \\ 1 1 \end{bmatrix} = 2J$ and $B^{(-1)}_{\langle 6,2\rangle} = 3\begin{bmatrix} 1 0 \\ 1 1 \end{bmatrix} = 3J$. Similarly $B^{(-1)}_{\langle 6,4\rangle} = 3\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 0 \\ 1 1 \end{bmatrix} = 3J$, $B^{(-1)}_{\langle 8,2\rangle} = 4\begin{bmatrix} 1 0 \\ 1 1 \end{bmatrix} = 4J = B^{(-1)}_{\langle 8,6\rangle}$, $B^{(-1)}_{\langle 8,4\rangle} = 6\begin{bmatrix} 1 0 \\ 1 1 \end{bmatrix} = 6J$. These give the block matrix form of $C^{(-1)}$ as in (1).

Now if $\begin{bmatrix} B^{(-1)}_{\langle i,j\rangle} & B^{(-1)}_{\langle i,j+2\rangle} \\ & B^{(-1)}_{\langle i+2,j+2\rangle} \end{bmatrix} = \left[\begin{array}{cc|cc} a & b & e & f \\ c & d & g & h \\ \hline & & x & y \\ & & z & w \end{array}\right]$ is a part of the block matrix $C^{(-1)}$ then $x = d + (-1)^i g = a + (-1)^{i+1} b + (-1)^i b + e = a + e$, and similarly $y = b + f$, $z = c + g$ and $w = d + h$. Thus the block matrix $C^{(-1)}$ satisfies a recurrence rule $B^{(-1)}_{\langle i+2,j+2\rangle} = B^{(-1)}_{\langle i,j\rangle} + B^{(-1)}_{\langle i,j+2\rangle}$, so it completes the proof (1).

If $q = 1$ then $\bar{1}_k$, $(1,-1,1)$ and $(1,-1,1,-1,1)$ in $\hat{C}$ equal k, 1 and 1 in $C^{(1)}$ respectively, so $L^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$ for $k \geq 0$. Thus $JL^k = \begin{bmatrix} 1 & 2k \\ 1 & 2k+1 \end{bmatrix}$ and $J^{2k+1} = \begin{bmatrix} 1 & 0 \\ 2k+1 & 1 \end{bmatrix}$ are the first and the last blocks in k^{th}row of the block matrix $C^{(1)} = P$. Now the integer subscripts $3, 2, 5$ of $\hat{B}_{\langle 4,2\rangle} = (101)\begin{bmatrix} \bar{1}_3, \bar{1}_2 \\ \bar{1}_5, \bar{1}_5 \end{bmatrix}$ in $\hat{C}$ correspond to integer factors of the block $B^{(1)}_{\langle 4,2\rangle} = 2\begin{bmatrix} 3 2 \\ 5 5 \end{bmatrix}$ in $C^{(1)}$. Similarly the integer subscripts $3, 4, 5$ in $\hat{B}_{\langle 6,2\rangle} = (1,-1,1)\begin{bmatrix} \bar{1}_5 & \\ & \bar{1}_7 \end{bmatrix}\begin{bmatrix} \bar{1}_3, \bar{1}_4 \\ \bar{1}_3, \bar{1}_5 \end{bmatrix}$ and $5, 2, 3$ in $\hat{B}_{\langle 6,4\rangle} = (1,-1,1)\begin{bmatrix} \bar{1}_3 & \\ & \bar{1}_7 \end{bmatrix}\begin{bmatrix} \bar{1}_5, \bar{1}_2 \\ \bar{1}_5, \bar{1}_3 \end{bmatrix}$ are equal to the integer entries in the block $B^{(1)}_{\langle 6,2\rangle} = \begin{bmatrix} 5 & \\ & 7 \end{bmatrix}\begin{bmatrix} 3, 4 \\ 3, 5 \end{bmatrix}$ and $B^{(1)}_{\langle 6,4\rangle} = \begin{bmatrix} 3 & \\ & 7 \end{bmatrix}\begin{bmatrix} 5, 2 \\ 5, 3 \end{bmatrix}$. Moreover all integer entries in blocks $B^{(1)}_{\langle 8,2\rangle} = 4\begin{bmatrix} 7 & \\ & 3 \end{bmatrix}\begin{bmatrix} 1, 2 \\ 3, 7 \end{bmatrix}$, $B^{(1)}_{\langle 8,4\rangle} = 2\begin{bmatrix} 7 & \\ & 9 \end{bmatrix}\begin{bmatrix} 5, 4 \\ 7, 7 \end{bmatrix}$ and $B^{(1)}_{\langle 8,6\rangle} = 4\begin{bmatrix} 1 & \\ & 3 \end{bmatrix}\begin{bmatrix} 7, 2 \\ 7, 3 \end{bmatrix}$ are equal to the subscripts in $\hat{B}_{\langle 8,2\rangle}$, $\hat{B}_{\langle 8,4\rangle}$ and $\hat{B}_{\langle 8,6\rangle}$ respectively (see (4), (5), (6)). Similarly the integer entries in $B^{(1)}_{\langle 10,2\rangle} = 5\begin{bmatrix} 1 & \\ & 11 \end{bmatrix}\begin{bmatrix} 9, 24 \\ 1, 3 \end{bmatrix}$, $B^{(1)}_{\langle 10,4\rangle} = 3 \cdot 2\begin{bmatrix} 7 & \\ & 11 \end{bmatrix}\begin{bmatrix} 5, 6 \\ 5, 7 \end{bmatrix}$, $B^{(1)}_{\langle 10,6\rangle}$

$= 3\cdot 2\begin{bmatrix}5 & \\ & 11\end{bmatrix}\begin{bmatrix}7,4\\7,5\end{bmatrix}$ and $B^{(1)}_{\langle 10,8\rangle} = 5\begin{bmatrix}1 & \\ & 11\end{bmatrix}\begin{bmatrix}9,2\\3,1\end{bmatrix}$ correspond to the subscripts in $\hat{B}_{\langle 10,j\rangle}$ $(j = 2,4,6,8)$ in (7), respectively. □

Theorem 4.3. *The* 2×2 *block matrix* $C^{(1)} = [b_{i,j}]$ *satisfies a recurrence* $b_{s+1,t+1}$ $= b_{s,t}\begin{bmatrix}1 & 0\\ 1 & \frac{2s+1}{2t+2}\end{bmatrix} + b_{s,t+1}\begin{bmatrix}\frac{s+1}{s-t} & 1\\ 0 & 1\end{bmatrix} + \frac{1}{2(s-t)}\begin{bmatrix}\binom{2s}{2t+1} & 0\\ 0 & \binom{2s+1}{2t+2}\end{bmatrix}$.

Proof. Let $\begin{bmatrix}B^{(1)}_{\langle i,j\rangle} & B^{(1)}_{\langle i,j+2\rangle}\\ & B^{(1)}_{\langle i+2,j+2\rangle}\end{bmatrix} = \left[\begin{array}{cc|cc} a & b & e & f\\ c & d & g & h\\ \hline & & x & y\\ & & z & w\end{array}\right]$ be a part of the block matrix $C^{(1)}$. Then

$$\begin{aligned} B^{(1)}_{\langle i+2,j+2\rangle} &= \begin{bmatrix}x\,y\\ z\,w\end{bmatrix} = \begin{bmatrix}a+b, b\frac{i+1}{j+2}\\ c+d, d\frac{i+2}{j+2}\end{bmatrix} + \begin{bmatrix}e\frac{i+1}{i-j-1}, e+f\\ g\frac{i+2}{i-j}, \; g+h\end{bmatrix}\\ &= \begin{bmatrix}a\,b\\ c\,d\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & \frac{i+1}{j+2}\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \binom{i+1}{j+2}\frac{1}{i-j}\end{bmatrix} + \begin{bmatrix}e\,f\\ g\,h\end{bmatrix}\begin{bmatrix}\frac{i+2}{i-j} & 1\\ 0 & 1\end{bmatrix} + \begin{bmatrix}\binom{i}{j+1}\frac{1}{i-j} & 0\\ 0 & 0\end{bmatrix}\\ &= B^{(1)}_{\langle i,j\rangle}\begin{bmatrix}1 & 0\\ 1 & \frac{i+1}{j+2}\end{bmatrix} + B^{(1)}_{\langle i,j+2\rangle}\begin{bmatrix}\frac{i+2}{i-j} & 1\\ 0 & 1\end{bmatrix} + \frac{1}{i-j}\begin{bmatrix}\binom{i}{j+1} & 0\\ 0 & \binom{i+1}{j+2}\end{bmatrix}. \end{aligned}$$

Thus $C^{(1)} = [b_{i,j}] = [B^{(1)}_{\langle 2i,2j\rangle}]$ satisfies

$$b_{s+1,t+1} = b_{s,t}\begin{bmatrix}1 & 0\\ 1 & \frac{2s+1}{2t+2}\end{bmatrix} + b_{s,t+1}\begin{bmatrix}\frac{s+1}{s-t} & 1\\ 0 & 1\end{bmatrix} + \frac{1}{2(s-t)}\begin{bmatrix}\binom{2s}{2t+1} & 0\\ 0 & \binom{2s+1}{2t+2}\end{bmatrix}.$$ □

In fact $B^{(1)}_{\langle 12,2\rangle} = B^{(1)}_{\langle 10,0\rangle}\begin{bmatrix}1 & 0\\ 1 & \frac{11}{2}\end{bmatrix} + B^{(1)}_{\langle 10,2\rangle}\begin{bmatrix}\frac{6}{5} & 1\\ 0 & 1\end{bmatrix} + \frac{1}{10}\begin{bmatrix}\binom{10}{1} & 0\\ 0 & \binom{11}{2}\end{bmatrix} = \begin{bmatrix}66,220\\ 78,286\end{bmatrix}$.

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Eunmi Choi
Dept. of Mathematics, HanNam University, Daejeon
Email address: emc@hnu.kr