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A BLOW-UP RESULT FOR A STOCHASTIC HIGHER-ORDER KIRCHHOFF-TYPE EQUATION WITH NONLINEAR DAMPING AND SOURCE TERMS

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Abstract. In this paper, we consider a stochastic higher-order Kirchhoff-type equation with nonlinear damping and source terms. We prove the blow-up of solution for a stochastic higher-order Kirchhoff-type equation with positive probability or explosive in energy sense.

1. Introduction

In this paper, we are concerned with the following stochastic higher-order Kirchhoff-type equation with nonlinear damping and source terms

$$\begin{aligned} &u_{tt}(t) + (\int_\Omega |D^m u(t)|^2 dx)^q (-\triangle)^m u(t) + |u_t(t)|^r u_t(t) = |u(t)|^p u(t) \\ &\qquad\qquad +\varepsilon\sigma(x,t)\partial_t W(x,t), \ \ \text{in } D\times[0,T], \\ &u(x,t) = 0,\ \frac{\partial^i u}{\partial \nu^i} = 0,\ i = 1,2,\cdots,m-1,\ \text{ in } \partial D\times[0,T], \\ &u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x),\ x\in D, \end{aligned} \tag{1}$$

where $m \geq 1, p, q, r \geq 0$, D is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary ∂D and a unit outer normal ν. Here, $W(x,t)$ is a finite dimensional Wiener process and $\sigma(x,t)$ is $L^2(D)$ valued progressively measurable, and ε is a given positive constant which measures the strength of noise.

For the form of High-order Kirchhoff type. Fucai Li [9] considered the higher order Kirchhoff type equation with nonlinear dissipation as follow

$$u_{tt}(t) + (\int_\Omega |D^m u(t)|^2 dx)^q (-\triangle)^m u(t) + |u_t(t)|^r u_t(t) = |u(t)|^p u(t) \text{ in } D\times(0,\infty).$$

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He obtained that solution exists globally if $p \le r$, while if $p > \max\{r, 2q\}$, the solution with negative initial energy blows up at finite time.
Gao et al.[4] proved that the solution blows up in finite time under suitable conditions on the initial datum and when $p > q \ge 2,\ m, n \ge 1$,

$$u_{tt}(t) + M(||D^m u(t)||_2^2)(-\triangle)^m u(t) + |u_t(t)|^{q-2}u_t(t) = |u(t)|^{p-2}u(t) \text{ in } D \times (0, \infty).$$

Under the consideration of random environment, there are many works on the stochastic wave equation with global existence and invariant measure for linear and nonlinear damping (see reference in [1, 2, 3, 5, 10]). For the nonlinear stochastic viscoelastic wave equation with linear damping, the authors has proved the global solutions and blow-up with positive probability for the stochastic viscoelastic wave equation (see in [2, 5, 7, 12, 13, 14]).

Cheng et al.[2] consider the stochastic viscoelastic wave equation with nonlinear damping and source term

$$\begin{aligned} u_{tt}(t) - \Delta u(t) + \int_0^t h(t-\tau)\Delta u(\tau)d\tau + |u_t(t)|^{q-2}u_t(t) \\ = |u(t)|^{p-2}u(t) + \epsilon\sigma(x,t)\partial_t W(x,t) \ \text{ in } D \times [0, T]. \end{aligned}$$

They studied the local solution of stochastic viscoelastic wave equation and investigated the solution blow-up with positive probability or it is explosive in energy sense in $p > q$. Kim et al. [8] consider the stochastic quasilinear viscoelastic wave equation with nonlinear damping and source terms

$$\begin{aligned} |u(t)|^{\rho}u_{tt}(t) - \Delta u(t) - \Delta u_{tt}(t) + \int_0^t h(t-\tau)\Delta u(\tau)d\tau + |u_t(t)|^{q-2}u_t(t) \\ = |u(t)|^{p-2}u(t) + \epsilon\sigma(x,t)\partial_t W(x,t) \ \text{ in } D \times (0, T). \end{aligned}$$

Authors proved that finite time blow-up is possible under the condition blow if $p > \max\{q, \rho + 2\}$ and the initial data are large enough. Moreover, Rana et al. [13] proved the global existence and finite time blow-up in a class of stochastic nonlinear wave equations form

$$\begin{aligned} \partial_{tt}u(t) - \Delta\partial_t u(t) - div(|\nabla u(t)|^{\alpha-1}\nabla u(t)) - div(|\nabla\partial_t u(t)|^{\beta-2}\nabla\partial_t u(t)) \\ + a|\partial_t u(t)|^{q-2}\partial_t u(t) = b|u(t)|^{p-2}u(t) + \sigma(x,t)\partial_t W(x,t) \ \text{ in } D \times [0, T). \end{aligned}$$

Motivated by previous works, for any $p > \max\{r, 2q\}$, we study the blow-up of solution for stochastic higher-order Kirchhoff-type equation with nonlinear damping and source terms with positive probability or explosive in energy sense.

2. Preliminaries

Let $(X, ||\cdot||_X)$ be a separable Hilbert space with Borel σ-algebra $\mathbf{B}(X)$, and let $(\Omega, \mathfrak{F}, P)$ be a probability space. We set $H = L^2(D)$ with the inner product and norm denoted by $(\cdot,\cdot)$ and $||\cdot||$, respectively. Denote by $||\cdot||_q$ the $L^q(D)$ norm for $1 \le q \le \infty$ and by $||\nabla\cdot||$ the Dirichlet norm in $V = H_0^1(D)$ which is equivalent to $H^1(D)$ norm.

Now, we introduce the following hypotheses:

(H1) We assume that p, q, r satisfy

$$p > max\{r, 2q\} \text{ and } 0 < p \leq \frac{2}{n-2m} \text{ if } n > 2m, \ p > 0 \text{ if } n \leq 2m. \tag{1}$$

(H2) $\sigma(x,t)$ is $H_0^1(D) \cap L^\infty(D)$ valued progressively measurable such that

$$E\int_0^T (||\nabla\sigma(t)||^2 + ||\nabla\sigma(t)||_\infty^2)dt \leq \infty. \tag{2}$$

In this paper, $E(\cdot)$ stands for expectation with respect to probability measure P, and $W(x,t)(t \geq 0)$ is a V-valued Q-Wiener process on the probability space with the covariance operator Q satisfying $Tr(Q) < \infty$. A complete orthonormal system$\{e_k\}_{k=1}^\infty$ in V with $c_0 := \sup_{k\geq 1} ||e_k||_\infty < \infty$, and a bounded sequence of nonnegative real members $\{\lambda_k\}_{k=1}^\infty$ satisfies that

$$Qe_k = \lambda_k e_k, \quad k = 1, 2, \cdots.$$

To simplify the computations, we assume that the covariance operator Q and Laplacian $-\triangle$ with a homogeneous Dirichlet boundary condition have a common set of eigenfunctions, that is

$$-\triangle e_k = \mu_k e_k, \quad x \in D,$$
$$e_k = 0, \quad x \in \partial D,$$

and then, for any $t \in [0,T]$, $W(x,t)$ has an expansion

$$W(x,t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k(t), \tag{3}$$

where $\{\beta_k(t)\}_{k=1}^\infty$ are real valued Brownian motions mutually independent on $(\Omega, \mathfrak{F}, P)$. Let $\mathcal{H}$ be the set of $L_2^0 = L^2(Q^{1/2}V, V)$-valued processes with the norm

$$||\Phi(t)||_{\mathcal{H}} = \left(E\int_0^t ||\Phi(s)||_{L_2^0}^2 ds\right)^{1/2} = \left(E\int_0^t Tr(\Phi(s)Q\Phi^*(s))ds\right)^{1/2} < \infty,$$

where $\Phi^*(s)$ denotes the adjoint operator of $\Phi(s)$. For any $\Phi^*(t) \in \mathcal{H}$, we can define the stochastic integral with respect to the Q-Wiener process as $\int_0^t \Phi(s)dW(s)$, which is martingale. For more details about the infinite dimension Wiener process and the stochastic integral, we refer the readers to [13].

By combining the arguments of [5, 9], we have the following existence theorem.

Definition 1. Assume that $(u_0, u_1) \in (H^{2m}(D) \times H_0^m(D)) \times H_0^m(D)$, and $E(\int_0^T ||\sigma(t)||^2 dt) < \infty$, u is said to be solution of (1.1) on the interval $[0,T)$, if (u, u_t) is $(H^{2m}(D) \times H_0^m(D)) \times H_0^m(D)$-valued progressively measurable, $u \in L^2(\Omega; L^2(0,T; H^{2m}(D) \cap H_0^m(D))) \cap L^2(\Omega; C([0,T); H_0^m(D)))$, $u_t \in L^2(\Omega; L^\infty(0,T; H_0^m(D))) \cap L^2(\Omega; C([0,T); H_0^m(D)))$, and such that (1.1) holds in the sense of distributions over $(0,T) \times D$ for almost all w.

Theorem 2.1. *([5, 6]). Assume that **(H1)** − **(H2)** hold. Then, for the initial data* $(u_0, u_1) \in (H^{2m}(D) \cap H_0^m(D)) \times H_0^m(D)$*, problem (1) has a pointwise unique solution u such that*

$$u \in L^2(\Omega; L^2(0,T; H^{2m}(D) \cap H_0^m(D))) \cap L^2(\Omega; C([0,T); H_0^m(D))),$$

and

$$u_t \in L^2(\Omega; L^\infty(0,T; H_0^m(D))) \cap L^2(\Omega; C([0,T); H_0^m(D))).$$

2.1. Blow-up result

In this section, we prove our main result for $p > \max\{r, 2q\}$. For this purpose, we give defined restrictions on $\sigma(x,t)$ such that

$$E\int_0^\infty \int_D \sigma^2(x,t)dxdt < \infty. \tag{1}$$

Let B be the best constant of the embedding inequality $||u||_{p+2} \leq B||D^m u||$. We set

$$\begin{aligned} \alpha_1 &= B^{-(p+2)/(p-2q)}, \\ E_1 &= (\frac{1}{2(q+1)} - \frac{1}{p+2})\alpha_1^{2(q+1)}, \qquad (2) \\ &\text{and} \\ E(t) &= \frac{1}{2}||u_t||^2 + \frac{1}{2(q+1)}||D^m u||^{2(q+1)} - \frac{1}{p+2}||u||_{p+2}^{p+2}. \qquad (3) \end{aligned}$$

Then we have the following.

Lemma 2.2. *([11]). Let u be solution of (1). Assume that* $E(0) < E_1$ *and* $||D^m u_0||_{\alpha_1} > \alpha_1$*. Then there exists a constant* $\alpha_2 > \alpha_1$ *such that*

$$||D^m u(\cdot,t)|| \geq \alpha_2, \forall t \geq 0, \tag{4}$$

$$||u||_{p+2} \geq B\alpha_2, \forall t \geq 0. \tag{5}$$

For each N, stopping time τ_N is given as

$$\tau_N = \inf\{t > 0 : ||D^m u(t)||^2 \geq N\}, \tag{6}$$

where τ_N is increasing in N, and

$$\tau_\infty = \lim_{N\to+\infty} \tau_N.$$

In order to prove our result, we rewrite (1) as an equivalent Itô's system

$$\begin{aligned} &du = vdt \\ &dv = -\left((\int_\Omega |D^m u|^2 dx)^q (-\Delta)^m u - |v|^r v + |u|^p u \right) dt \\ &\qquad\qquad + \varepsilon\sigma(x,t)dW_t(x,t), \quad (x,t) \in D \times (0,T) \qquad (7) \\ &u(x,t) = 0, \quad (x,t) \in \partial D \times (0,T), \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x) = u_1(x), \quad x \in D, \end{aligned}$$

where $(u_0, u_1) \in (H^{2m}(D) \cap H_0^m(D)) \times H_0^m(D)$. Then the energy function $F(t)$ becomes

$$F(t) = \frac{1}{2}||v(t)||^2 + \frac{1}{2(q+1)}||D^m u(t)||^{2(q+1)} - \frac{1}{p+2}||u(t)||_{p+2}^{p+2}. \quad (8)$$

Next, we give a lemma.

Lemma 2.3. *Let* (u, v) *be a solution of equation(7) with the initial data* $(u_0, v_0) \in (H^{2m}(D) \cap H_0^m(D)) \times H_0^m(D)$. *Then, we have*

$$\frac{d}{dt}E[F(t)] = -E||v(t)||_{r+2}^{r+2} + \frac{\epsilon^2}{2}\Sigma_{j=1}^{\infty} E \int_D \lambda_j e_j^2(x)\sigma^2(x,t)dx, \quad (9)$$

and

$$\begin{aligned} E(u(t), v(t)) &= (u_0, v_0) - \int_0^t E||D^m u(s)||^{2(q+1)} ds \quad (10) \\ &- \int_0^t E(u(s), |v(s)|^r v(s))ds + \int_0^t E||u(s)||_{p+2}^{p+2} ds + \int_0^t ||v(s)||^2 ds. \end{aligned}$$

Proof. Multiplying equation(7) by $v(t)$ and using Itô's formula, then we deduce (9). We also multiplying equation(7) by $u(t)$ and integrating by parts over $(0, t)$, and we arrive at (10) (see [5]). Let

$$G(t) = \frac{\epsilon^2}{2}\Sigma_{j=1}^{\infty} E \int_0^t \int_D \lambda_j e_j^2(x)\sigma^2(x,s)dxds. \quad (11)$$

Due to (1), we derive

$$\begin{aligned} G(\infty) &= \frac{\epsilon^2}{2}\Sigma_{j=1}^{\infty} E \int_0^{\infty} \int_D \lambda_j e_j^2(x)\sigma^2(x,s)dxds \\ &\le \frac{\epsilon^2}{2} Tr(Q) c_0 E \int_0^{\infty} \int_D \sigma^2(x,s)dxds = E_1 < \infty. \quad (12) \end{aligned}$$

We set

$$H(t) = G(t) - E[F(t)].$$

Then, by(9) we get

$$H'(t) = G'(t) - \tfrac{d}{dt}E[F(t)] \ge E||v(t)||_{r+2}^{r+2} \ge 0. \quad (13)$$

□

Lemma 2.4. *([5]). Let* (u, v) *be a solution of (7). Then, there exists a positive constant* C *such that*

$$E||u(t)||_{p+2}^{s+2} \le C[G(t) - H(t) - E||v(t)||^2 + E||u(t)||_{p+2}^{p+2}, \quad 2 \le s \le p. \quad (14)$$

Proof. If $||u||_{p+2} \le 1$, then $||u||_{p+2}^s \le ||u||_{p+2}^2 \le C||D^m u||^2$ by Sobolev embedding. If $||u||_{p+2} \ge 1$, then $||u||_{p+2}^{s+2} \le ||u||_{p+2}^{p+2}$. Thus, there exists a constant $C > 0$ such that $E||u||_{p+2}^{s+2} \le C(E||D^m u||^2 + E||u||_{p+2}^{p+2})$. Therefore, in combination with the definition of energy function, we get (14). □

Theorem 2.5. *Suppose that $p > max\{r, 2q\}$ and*

$$0 < p \le \frac{2}{n-2m} \text{ if } n > 2m, \quad p > 0 \text{ if } n \le 2m. \tag{15}$$

*Assume that **(H1)-(H2)** and (1) hold. Let (u, v) be a solution of equation (7) with initial data $(u_0, v_0) \in (H^{2m}(D) \cap H_0^m(D)) \times H_0^m(D)$ satisfying*

$$F(0) \le -(1+\beta)E_1, \tag{16}$$

where $\beta > 0$ is an arbitrary constant. If $L(0) > 0$, then the solution (u, v) of equation (7) and the lifespan τ_∞ defined above, either
(1) $P(\tau_\infty < \infty) > 0$, that is, $||D^m u(t)||$ blows up in finite time with positive probability, or
(2) there exists a positive time $T^* \in (0, T_0]$ such that

$$\lim_{t \to T^*} E[F(t)] = +\infty, \tag{17}$$

where

$$T_0 = \frac{1-\alpha}{\alpha K L^{-\alpha/(1-\alpha)}(0)}, \tag{18}$$
$$L(0) = H^{1-\alpha}(0) + \delta E(u_0, v_0) > 0,$$

and α, K are given in later.

Proof. For the lifespan τ_∞ of the solution $\{u(t) : t \ge 0\}$ of (7) with $H_0^m(D)$ norm. Firstly, we treat the case when $P(\tau_\infty = +\infty) < 1$. Then, for sufficiently large $T > 0$, by (13) and (16), we have

$$0 < (1+\beta)E_1 \le -F(0) = H(0) \le H(t) \le G(t)$$
$$+ \frac{1}{p+2} E||u(t)||_{p+2}^{p+2} \le E_1 + \frac{1}{p+2} E||u(t)||_{p+2}^{p+2}. \tag{19}$$

Define by

$$L(t) = H^{1-\alpha}(t) + \delta E(u(t), v(t)),$$

where

$$0 < \alpha < \min\{\frac{1}{2}, \frac{p-r}{(p+2)(r+2)}, \frac{p}{2(p+2)}\} \tag{20}$$

and δ is a sufficiently small constant to be determined in later.

Using (8),(10) and (13), we deduce

$$\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \delta[E||D^m u(t)||^{2(q+1)} - E(u(t),|v(t)|^r v(t)) \\
&\quad + E||u(t)||_{p+2}^{p+2} + E||v(t)||^2] \\
&\geq (1-\alpha)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} + 4\delta(q+1)[H(t) - G(t) + EF(t)] \\
&\quad + \delta[E||D^m u(t)||^{2(q+1)} - E(u(t),|v(t)|^r v(t)) + E||u(t)||_{p+2}^{p+2} + E||v(t)||^2] \\
&= (1-\alpha)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} + 2\delta(q+1)H(t) - 2\delta(q+1)G(t) \\
&\quad + 2\delta(q+1)E||v||^2 + 2\delta E||D^m u(t)||^{2(q+1)} - \frac{4\delta(q+1)}{p+2}E||u(t)||_{p+2}^{p+2} \\
&\quad + \delta E||D^m u(t)||^{2(q+1)} - \delta E(u(t),|v(t)|^r v(t)) + \delta E||u(t)||_{p+2}^{p+2} + \delta E||v(t)||^2 \\
&= (1-\alpha)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} + 4\delta(q+1)H(t) - 4\delta(q+1)G(t) \qquad (21) \\
&\quad + \delta(2q+3)E||v||^2 + 2\delta E||D^m u(t)||^{2(q+1)} - \delta E(u(t),|v(t)|^r v(t)) \\
&\quad + \delta(1 - \frac{4(q+1)}{p+2})E||u(t)||_{p+2}^{p+2}.
\end{aligned}$$

For $r < p$ by $E||u(t)||_{r+2}^{r+2} \leq cE||u(t)||_{p+2}^{r+2}$ and Hölder's inequality, we derive the following estimate (see[2]):

$$\begin{aligned}
E(u(t),|v(t)|^r v(t)) &\leq (E||v(t)||_{r+2}^{r+2})^{\frac{r+1}{r+2}}(E||u(t)||_{r+2}^{r+2})^{\frac{1}{r+2}} \\
&\leq c(E||v(t)||_{r+2}^{r+2})^{\frac{r+1}{r+2}}(E||u(t)||_{p+2}^{r+2})^{\frac{1}{r+2}} \\
&\leq c(E||v(t)||_{r+2}^{r+2})^{\frac{r+1}{r+2}}(E||u(t)||_{p+2}^{p+2})^{\frac{1}{p+2}} \\
&\leq c(E||v(t)||_{r+2}^{r+2})^{\frac{r+1}{r+2}}(E||u(t)||_{p+2}^{p+2})^{\frac{1}{r+2}}(E||u(t)||_{p+2}^{p+2})^{\frac{1}{p+2}-\frac{1}{r+2}}, \quad (22)
\end{aligned}$$

and Young's inequality

$$(E||v(t)||_{r+2}^{r+2})^{\frac{r+1}{r+2}}(E||u(t)||_{p+2}^{p+2})^{\frac{1}{r+2}} \leq \frac{r+1}{r+2}\mu E||v(t)||_{r+2}^{r+2} + \frac{\mu^{-(r+1)}}{r+2}E||u(t)||_{p+2}^{p+2}, \quad (23)$$

where μ is a constant to be determined later. In view of (19), we get

$$E||u(t)||_{p+2}^{p+2} \geq (p+2)(H(t) - G(t)) \geq \rho H(t), \quad (24)$$

where $\rho = \frac{(p+2)\beta}{1+\beta}$. With the assumption of $H(0) > 1$, (20), (23) and (24) implies that

$$\begin{aligned}
(E||u(t)||_{p+2}^{p+2})^{\frac{1}{p+2}-\frac{1}{r+2}} &\leq \rho^{\frac{1}{p+2}-\frac{1}{r+2}}H(t)^{\frac{1}{p+2}-\frac{1}{r+2}} \\
&\leq \rho^{\frac{1}{p+2}-\frac{1}{r+2}}H^{-\alpha}(t) \leq \rho^{\frac{1}{p+2}-\frac{1}{r+2}}H^{-\alpha}(0). \quad (25)
\end{aligned}$$

Combining with (22), (23) and (25), we arrive at

$$\begin{aligned}
|E(u(t),|v(t)|^r v(t))| &\leq a_1\frac{r+1}{r+2}\mu E||v(t)||_{r+2}^{r+2}H^{-\alpha}(t) \quad (26) \\
&\quad + a_1\frac{\mu^{-(r+1)}}{r+2}E||u(t)||_{p+2}^{p+2}H^{-\alpha}(0),
\end{aligned}$$

where $a_1 = c\rho^{\frac{1}{p+2}-\frac{1}{r+2}}$. Hence, substituting (26) in (21), we have

$$\begin{aligned} L'(t) &\geq (1-\alpha)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} + 4\delta(q+1)H(t) - 4\delta(q+1)G(t) \\ &+\delta(2q+3)E||v(t)||^2 + 2\delta E||D^m u(t)||^{2(q+1)} \\ &-a_1\frac{r+1}{r+2}\mu\delta E||v(t)||_{r+2}^{r+2}H^{-\alpha}(t) - a_1\frac{\mu^{-(r+1)}\delta}{r+2}E||u(t)||_{p+2}^{p+2}H^{-\alpha}(0) \\ &+\delta(1-\frac{4(q+1)}{p+2})E||u(t)||_{p+2}^{p+2} \\ &\geq (1-\alpha-a_1\frac{r+1}{r+2}\mu\delta)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} \\ &+2\delta(p+1)H(t) + \delta(2q+3)E||v(t)||^2 - 2\delta(p+2)G(t) \\ &+2\delta E||D^m u(t)||^{2(q+1)} - a_1\frac{\mu^{-(r+1)}\delta}{r+2}E||u(t)||_{p+2}^{p+2}H^{-\alpha}(0). \qquad (27) \end{aligned}$$

Using Lemma 2.4 with $s = p$ and (27), we have

$$\begin{aligned} L'(t) &\geq (1-\alpha-a_1\frac{r+1}{r+2}\mu\delta)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} \\ &\quad +2\delta(p+2)H(t) - 2\delta(p+2)G(t) + 2\delta E||D^m u(t)||^{2(q+1)} \\ &\quad +\delta(2q+3)E||v(t)||^2 + 2\delta E||D^m u(t)||^{2(q+1)} \\ &\quad -a_1\frac{\mu^{-(r+1)}\delta H^{-\alpha}(0)C}{r+2}[G(t) - H(t) - E||v(t)||^2 + E||u(t)||_{p+2}^{p+2}] \\ &= (1-\alpha-a_1\frac{r+1}{r+2}\mu\delta)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} \qquad (28) \\ &\quad +\delta[2(p+2) + a_2\mu^{-(r+1)}]H(t) \\ &\quad -\delta[2(p+2) + a_2\mu^{-(r+1)}]G(t) \\ &\quad +\delta[(2q+3) + a_2\mu^{-(r+1)}]E||v(t)||^2 \\ &\quad +2\delta E||D^m u(t)||^{2(q+1)} - \delta a_2\mu^{-(r+1)}E||u(t)||_{p+2}^{p+2}, \end{aligned}$$

where $a_2 = Ca_1H^{-\alpha}(0)/(r+2)$.
Note that

$$H(t) \geq G(t) + \tfrac{1}{p+2}E||u(t)||_{p+2}^{p+2} - \tfrac{1}{2}E||v(t)||^2 - \tfrac{1}{2(q+1)}E||D^m u(t)||^{2(q+1)}.$$

Then estimate (28) yields

$$\begin{aligned} L'(t) \quad &\geq (1-\alpha-a_1\frac{r+1}{r+2}\mu\delta)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} \\ &+\delta[2(p+2)-a_1+a_2\mu^{-(r+1)}]H(t) \\ &-\delta[2(p+2)-a_1+a_2\mu^{-(r+1)}]G(t) \\ &+\delta[(2q+3)-\frac{a_1}{2}+a_2\mu^{-(r+1)}]E||v(t)||^2 \\ &+\delta[2-\frac{a_1}{2(q+1)}]E||D^m u(t)||^{2(q+1)} \\ &+\delta[\frac{a_1}{p+2}-a_2\mu^{-(r+1)}]E||u(t)||_{p+2}^{p+2}. \end{aligned} \tag{29}$$

From (12) and (19), we deduce

$$\begin{aligned} [2(p+2)-a_1+a_2\mu^{-(r+1)}]G(t) \quad &\leq [2(p+2)-a_1+a_2\mu^{-(r+1)}]E_1 \\ &\leq [\frac{2(p+2)-a_1+a_2\mu^{-(r+1)}}{1+\beta}]H(t). \end{aligned} \tag{30}$$

Substituting (30) in (29), we get

$$\begin{aligned} L'(t) \quad &\geq (1-\alpha-a_1\frac{r+1}{r+2}\mu\delta)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} \\ &+\delta[2(p+2)-a_1+a_2\mu^{-(r+1)}]\frac{\beta}{1+\beta}H(t) \\ &+\delta[(2q+3)-\frac{a_1}{2}+a_2\mu^{-(r+1)}]E||v(t)||^2 \\ &+\delta[2-\frac{a_1}{2(q+1)}]E||D^m u(t)||^{2(q+1)} \\ &+\delta[\frac{a_1}{p+2}-a_2\mu^{-(r+1)}]E||u(t)||_{p+2}^{p+2}. \end{aligned} \tag{31}$$

Next, we can choose μ large enough so that (31) becomes

$$\begin{aligned} L'(t) \quad &\geq (1-\alpha-a_1\frac{r+1}{r+2}\mu\delta)H^{-\alpha}(t)E||v(t)||_{r+2}^{r+2} \\ &+\delta\xi(H(t)+E||v(t)||^2+E||D^m u(t)||^{2(q+1)}+E||u(t)||_{p+2}^{p+2}), \end{aligned} \tag{32}$$

where

$$\begin{aligned} \xi = min\{ \quad &(2(p+2)-a_1+a_2\mu^{-(r+1)})\frac{\beta}{1+\beta},\ (2q+3)-\frac{a_1}{2}+a_2\mu^{-(r+1)}, \\ &2-\frac{a_1}{2(q+1)},\ \frac{a_1}{p+2}-a_2\mu^{-(r+1)}\} > 0. \end{aligned}$$

Once μ is fixed we pick δ small enough so that

$$1-\alpha-a_1\frac{r+1}{r+2}\mu\delta > 0.$$

Using this, (32) takes the form

$$L'(t) \geq \delta\xi(H(t) + E||v(t)||^2 + E||D^m u(t)||^{2(q+1)} + E||u(t)||_{p+2}^{p+2}) \geq 0. \quad (33)$$

Thus, we see that

$$L(t) \geq L(0) = H^{1-\alpha}(0) + \delta(u_0, u_1) > 0, \ \forall t \geq 0. \quad (34)$$

Since

$$|E \int_D u(t)v(t)dx| \leq c(E||u(t)||_p^2)^{\frac{1}{2}}(E||v(t)||^2)^{\frac{1}{2}},$$

it implies that

$$|E \int_D u(t)v(t)dx|^{\frac{1}{1-\alpha}} \leq c[(E||u(t)||_{p+2}^2)^{\frac{\kappa}{2(1-\alpha)}} + (E||v(t)||^2)^{\frac{\nu}{2(1-\alpha)}}], \quad (35)$$

for $1/\kappa + 1/\nu = 1$. We choose $\nu = 2(1-\alpha)$, $\kappa = 2(1-\alpha)/(1-2\alpha)$, then $\kappa/2(1-\alpha) = 1/(1-2\alpha) \leq (p+2)/2$, by (20) and (35) becomes

$$|E \int_D u(t)v(t)dx|^{\frac{1}{1-\alpha}} \leq c[E||u(t)||_{p+2}^{\frac{2}{1-2\alpha}} + (E||v(t)||^2]. \quad (36)$$

Using Lemma 2.4 with $s = 2/(1-2\alpha)$, we obtain

$$\begin{aligned} |E \int_D u(t)v(t)dx|^{\frac{1}{1-\alpha}} &\leq c(H(t) + E||v(t)||^2 + E||D^m u(t)||^{2(q+1)} \\ &\quad + E||u(t)||_{p+2}^{p+2}) \ \forall t \geq 0. \end{aligned} \quad (37)$$

Therefore, we have

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &\leq c(H(t) + \delta^{\frac{1}{1-\alpha}}|E \int_D u(t)v(t)dx|^{\frac{1}{1-\alpha}}) \\ &\leq c(H(t) + E||v(t)||^2 + E||D^m u(t)||^{2(q+1)} + E||u(t)||_{p+2}^{p+2}) \ \forall t \geq 0. \end{aligned} \quad (38)$$

Combining (33) and (38), we get

$$L'(t) \geq KL^{\frac{1}{1-\alpha}}(t), \forall t \geq 0,$$

where K is a positive constant depending only on c and $\delta\xi$, then it yields . It follows that

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1-\alpha}{(1-\alpha)L^{-\frac{\alpha}{1-\alpha}}(0) - \alpha K t}.$$

Let

$$T_0 = \frac{1-\alpha}{\alpha K L^{\frac{\alpha}{1-\alpha}}(0)}.$$

Then, $L(t) \to +\infty$ as $t \to T_0$. This means that there exists a positive time $T^* \in (0, T_0]$ such that

$$\lim_{t \to T^*} E[F(t)] = +\infty.$$

As for the case when $P(\tau_\infty = +\infty) < 1$ (i.e.$P(\tau_\infty < +\infty) > 0$), then $E||D^m u(t)||$ blows up in finite time $T^* \in (0, \tau_\infty)$ with positive probability. Thus the proof of Theorem 3.1 is completed. □

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