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ON RECURSIONS FOR MOMENTS OF A COMPOUND RANDOM VARIABLE: AN APPROACH USING AN AUXILIARY COUNTING RANDOM VARIABLE

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Abstract. We present an identity on moments of a compound random variable by using an auxiliary counting random variable. Based on this identity, we develop a new recurrence formula for obtaining the raw and central moments of any order for a given compound random variable.

1. Introduction

Let $\{X_i, i = 1, 2, 3, \dots\}$ be a sequence of independent and identically distributed random variables. We denote by X the generic random variable for X_i. Let N be a non-negative integer-valued random variable. We assume that $\{X_i, i = 1, 2, 3, \dots\}$ and N are independent. A compound random variable, denoted by S_N, is defined as

$$S_N = X_1 + X_2 + \cdots + X_N.$$

In the case where $N = 0$, it is defined as $S_N = 0$ by convention.

Compound random variables are extensions of random sums, which have been a classical focus in probability theory, encompassing fundamental concepts such as the central limit theorem and the law of large numbers [5, 6]. Moreover, compound random variables have gained significant attention in various practical domains, including insurance mathematics, risk management, and reliability (see [16] and the references therein). For example, in collective risk theory, it has been applied in a manner that N counts the number of claims arising from a portfolio during a certain period, X_i measures the amount of the ith of these claims, and S_N then represents the aggregate claims of the portfolio [7]. In accordance with this application, N is often called a *counting* random variable, whereas X a *severity* random variable.

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In this paper, we study the problem of obtaining the moments of a compound random variable. In particular, the purpose of this paper is to develop recurrence formulas for obtaining the moments of any order for a given compound random variable. As a sort of moment, we consider both the raw and the central moments because they can provide detailed information about the shape of a probability distribution. Moreover, we focus on a recurrence form of formulas because it can be efficient from a computational point of view when obtaining higher-order moments. To achieve this objective, we propose an approach that utilizes an auxiliary counting random variable $\tilde{N}$ derived from N. The proposed approach yields a new recurrence formula in a structured form that consists of finite terms, where each term is decomposed into three factors: (i) a constant determined by the moments of the counting random variable N and the severity random variable X; (ii) a binomial coefficient; and (iii) a lower-order moment of a compound random variable. The factorized structure of our recurrence formula can provide the advantage of further reducing computational complexity. In addition, our approach introduces a new method for determining the moments of S_N, contributing to the enrichment and complementation of the existing understanding of compound random variables. For an explicit formula, we can refer to [2], where Grubbström and Tang presented a closed-form formula for the moments of S_N, provided that the severity random variable X is non-negative.

The rest of the paper is organized as follows. In Section 2, we first overview a list of related works and clarify the difference with this work. In Section 3, we then present a main theorem, from which we obtain a recurrence formula for the raw moment of S_N in Section 4 and the one for the central moment in Section 5. The proof of the main theorem is given in Section 6. Finally, we conclude the paper in Section 7.

2. Related Works

There have been extensive studies on the moments of a compound random variable. A variety of analytic formulas have been presented in a closed or recurrent form. In deriving recurrence formulas, existing works often assume that the counting random variable N belongs to a special family. De Pril [1] and Sundt [15] considered a class of counting random variables satisfying

$$\mathbb{P}(N=n)=\left(a+\frac{b}{n}\right)\mathbb{P}(N=n-1),\qquad n=1,2,3,\ldots, \tag{1}$$

for some constants $a<1$ and b such that $a+b\geq 0$. This family of counting distributions is referred to as Panjer's class [10]. Murat and Szynal [9] and Murat [7] considered a more broadened Panjer's class in which N satisfies

$$\mathbb{P}(N=n)=\left(a+\frac{b}{n+c}\right)\mathbb{P}(N=n-1),\qquad n=1,2,3,\ldots, \tag{2}$$

for some constants a, b, and c. Hesselager [3] generalized the Panjer's class to the one in which the ratio between successive probabilities on N can be written as

$$\mathbb{P}(N=n)=\frac{\sum_{i=0}^{k}a_i n^i}{\sum_{i=0}^{k}b_i n^i}\,\mathbb{P}(N=n-1),\qquad n=1,2,3,\ldots, \tag{3}$$

for some integer k and constants a_i and b_i $(i=0,1,\ldots,k)$.

The equations (1), (2), and (3) are of a first-order recursion. Schröter [12] considered a second-order recursion given by

(4)

$$\mathbb{P}(N=n)=\left(a+\frac{b}{n}\right)\mathbb{P}(N=n-1)+\frac{c}{n}\,\mathbb{P}(N=n-2),\qquad n=1,2,3,\ldots,$$

for some constants $a<1$, b, and c with $\mathbb{P}(N=-1)=0$. Sundt [14], Murat and Szynal [8], and Murat [7] generalized the second-order recursion in (4) to a kth-order recursion given by

$$\mathbb{P}(N=n)=\sum_{i=1}^{k}\left(a_i+\frac{b_i}{n}\right)\mathbb{P}(N=n-i),\qquad n=1,2,3,\ldots, \tag{5}$$

for some integer k and constants a_i and b_i $(i=1,2,\ldots,k)$ with $\mathbb{P}(N=n)=0$ for $n<0$. Sundt [15] further extended the kth-order recursion in (5) as

$$\mathbb{P}(N=n)=\sum_{i=1}^{k}\left(a_i+\frac{b_i}{n}\right)\mathbb{P}(N=n-i),\qquad n=r+1,r+2,r+3,\ldots,$$

for some positive integer r.

As such, existing works are often based on the assumption that the counting random variable N of S_N belongs to a specific class. In this paper, we propose a different approach by introducing an auxiliary random variable $\tilde{N}$ derived from N. Our work is motivated by [11] where a secondary random variable M, called a size-biased version of N, is exploited to obtain a recurrence formula for the probability mass function of S_N, provided that the severity random variable X takes on positive integer values.

Recently, Kim and Kim [4] developed a recurrence formula for higher-order moments of a compound random variable S_N when N follows a Binomial distribution. Seong [13] extended the result in [4] by including it as a special case. In this paper, we further extend the result in [13] by providing a more computationally efficient result that uses the sum of finite terms, where each term is decomposed into three factors under a regular structure. In addition, we also provide a recurrence formula for the moments of N as a by-product of our main result.

3. Preliminary Analysis

In this section, we perform a preliminary analysis with the aim of developing recurrence formulas for the moments of S_N. To begin, let $\tilde{N}$ be a random variable that takes on non-negative integer values. We assume that the distribution of $\tilde{N}$ is determined by that of N as follows:

$$\mathbb{P}(\tilde{N} = n) = \frac{(n+1)\mathbb{P}(N = n+1)}{\mathbb{E}[N]}, \qquad n = 0, 1, 2, \ldots. \tag{6}$$

Below we give examples of $\tilde{N}$ for well-known counting random variables N [11, 13].

Example 3.1. Let N be a Poisson random variable with parameter λ. Since $\mathbb{P}(N = n) = e^{-\lambda}\lambda^n/n!\,(n = 0, 1, 2, \ldots)$ and $\mathbb{E}[N] = \lambda$, the relation (6) gives rise to the probability mass function of $\tilde{N}$ as

$$\begin{aligned}\mathbb{P}(\tilde{N} = n) &= \frac{n+1}{\lambda} \cdot \frac{e^{-\lambda}\lambda^{n+1}}{(n+1)!} \\ &= \frac{e^{-\lambda}\lambda^n}{n!}, \qquad n = 0, 1, 2, \ldots.\end{aligned}$$

That is, $\tilde{N}$ also follows a Poisson distribution with parameter λ.

Example 3.2. Let N be a Binomial random variable with parameters (m, p). Since $\mathbb{P}(N = n) = \binom{m}{n}p^n(1-p)^{m-n}\,(n = 0, 1, \ldots, m)$ and $\mathbb{E}[N] = mp$, the relation (6) gives rise to the probability mass function of $\tilde{N}$ as

$$\begin{aligned}\mathbb{P}(\tilde{N} = n) &= \frac{n+1}{mp} \cdot \binom{m}{n+1}p^{n+1}(1-p)^{m-n-1} \\ &= \frac{n+1}{mp} \cdot \frac{m!}{(n+1)!(m-n-1)!}p^{n+1}(1-p)^{m-n-1} \\ &= \frac{(m-1)!}{n!(m-n-1)!}p^n(1-p)^{m-n-1} \\ &= \binom{m-1}{n}p^n(1-p)^{m-1-n}, \qquad n = 0, 1, \ldots, m-1.\end{aligned}$$

That is, $\tilde{N}$ follows a Binomial distribution with parameters $(m-1, p)$.

Example 3.3. Let N be a negative Binomial random variable with parameters (m, p). Since $\mathbb{P}(N = n) = \binom{n+m-1}{n}p^m(1-p)^n\,(n = 0, 1, 2, \ldots)$ and $\mathbb{E}[N] =$

$m(1-p)/p$, the relation (6) gives rise to the probability mass function of $\tilde{N}$ as

$$\begin{aligned}\mathbb{P}(\tilde{N}=n) &= \frac{n+1}{m(1-p)/p}\cdot\binom{n+m}{n+1}p^m(1-p)^{n+1}\\ &= \frac{n+1}{m(1-p)/p}\cdot\frac{(n+m)!}{(n+1)!(m-1)!}p^m(1-p)^{n+1}\\ &= \frac{(n+m)!}{n!m!}p^{m+1}(1-p)^n\\ &= \binom{n+m}{n}p^{m+1}(1-p)^n, \qquad n=0,1,2,\ldots.\end{aligned}$$

That is, $\tilde{N}$ follows a negative Binomial distribution with parameters $(m+1,p)$.

We assume that the random variable $\tilde{N}$ is independent of $\{X_i, i=1,2,3,\ldots\}$. Then, the sum $S_{\tilde{N}} = X_1 + X_2 + \cdots + X_{\tilde{N}}$ forms another compound random variable which is independent of S_N. In the following theorem, we present a relation between the moments of S_N and $S_{\tilde{N}}$ about points c and $\tilde{c}$, respectively.

Theorem 3.1. *For any $c, \tilde{c} \in \mathbb{R}$, we have*

(7)

$$\mathbb{E}\big[(S_N-c)^{k+1}\big] = \sum_{i=0}^{k} c_i\binom{k}{i}\mathbb{E}\big[(S_{\tilde{N}}-\tilde{c})^k\big]-c\,\mathbb{E}\big[(S_N-c)^k\big], \qquad k=0,1,2,\ldots,$$

where c_i is a constant that is determined by the moments of the counting random variable N and the severity random variable X as

$$c_i = \mathbb{E}[N]\sum_{j=0}^{i}\binom{i}{j}(\tilde{c}-c)^j\,\mathbb{E}[X^{i+1-j}]. \tag{8}$$

Proof. The proof is given in Section 6. □

4. Raw Moments

In this section, we derive a recurrence formula for the raw moments of S_N. We then give examples of how our formula can be used.

Theorem 4.1. *The raw moments of S_N can be obtained by*

$$\mathbb{E}\big[(S_N)^{k+1}\big] = \sum_{i=0}^{k} a_i\binom{k}{i}\mathbb{E}\big[(S_{\tilde{N}})^{k-i}\big], \qquad k=0,1,2,\ldots,$$

where a_i is given by

$$a_i = \mathbb{E}[N]\,\mathbb{E}[X^{i+1}].$$

Proof. Substituting $c=\tilde{c}=0$ in Theorem 3.1 gives Theorem 4.1. □

We note that Theorem 4.1 corresponds to Theorem 1 of [13] in its fundamental aspects. However, the distinction lies in the structure of the recurrence formula expressed in Theorem 4.1, which decomposes each term into three regular factors: (i) a constant a_i determined by the moments of the counting random variable N and the severity random variable X; (ii) a binomial coefficient $\binom{k}{i}$; and (iii) a lower-order raw moment of a compound random variable, $\mathbb{E}[(S_{\tilde{N}})^{k-i}]$. The difference and benefit of this factorization become more apparent in Theorem 5.1, where we derive a recurrence formula for the central moments of S_N.

Example 4.1. Suppose that N follows a Poisson distribution with parameter λ. Then, $\tilde{N}$ also follows a Poisson distribution with parameter λ as illustrated in Example 3.1. Accordingly, to simplify notation, we denote by μ_k the kth raw moment of the compound Poisson random variables S_N and $S_{\tilde{N}}$, i.e.,

$$\mu_k = \mathbb{E}\big[(S_N)^k\big] = \mathbb{E}\big[(S_{\tilde{N}})^k\big].$$

From Theorem 4.1, we obtain

$$\mu_{k+1} = \sum_{i=0}^{k} a_i \binom{k}{i} \mu_{k-i}, \tag{9}$$

where a_i is given by

$$a_i = \lambda \mathbb{E}[X^{i+1}]. \tag{10}$$

With the initial value $\mu_0 = 1$, we can find μ_k $(k = 1, 2, 3, \dots)$ sequentially using (9) and (10). For example, the first three raw moments of S_N are

$$\begin{aligned}
\mu_1 &= a_0\binom{0}{0}\mu_0 = \lambda\mathbb{E}[X],\\
\mu_2 &= a_0\binom{1}{0}\mu_1 + a_1\binom{1}{1}\mu_0 = \lambda\mathbb{E}[X^2] + \lambda^2\mathbb{E}[X]^2,\\
\mu_3 &= a_0\binom{2}{0}\mu_2 + a_1\binom{2}{1}\mu_1 + a_2\binom{2}{2}\mu_0 = \lambda\mathbb{E}[X^3] + 3\lambda^2\mathbb{E}[X]\mathbb{E}[X^2] + \lambda^3\mathbb{E}[X]^3.
\end{aligned}$$

Example 4.2. Suppose that N follows a Binomial distribution with parameters (m, p). Then, $\tilde{N}$ also follows a Binomial distribution, but with parameters $(m-1, p)$ as shown in Example 3.2. Accordingly, we denote by $\mu_k(m, p)$ and $\mu_k(m-1, p)$ the kth raw moments of the compound Binomial random variables S_N and $S_{\tilde{N}}$, respectively, i.e.,

$$\begin{aligned}
\mu_k(m, p) &= \mathbb{E}[(S_N)^k],\\
\mu_k(m-1, p) &= \mathbb{E}[(S_{\tilde{N}})^k].
\end{aligned}$$

From Theorem 4.1, we obtain

$$\mu_{k+1}(m, p) = \sum_{i=0}^{k} a_i \binom{k}{i} \mu_{k-i}(m-1, p), \tag{11}$$

where a_i is given by

$$a_i = mp\mathbb{E}[X^{i+1}]. \tag{12}$$

With the initial value $\mu_0(m,p) = 1$, we can find $\mu_k(m,p)\,(k = 1,2,3,\ldots)$ sequentially using (11) and (12). For example, the first three raw moments of S_N are

$$\begin{aligned}
\mu_1(m,p) &= a_0\binom{0}{0}\underbrace{\mu_0(m-1,p)}_{=1} = mp\mathbb{E}[X],\\
\mu_2(m,p) &= a_0\binom{1}{0}\underbrace{\mu_1(m-1,p)}_{=(m-1)p\mathbb{E}[X]} + a_1\binom{1}{1}\underbrace{\mu_0(m-1,p)}_{=1}\\
&= mp\mathbb{E}[X^2] + m(m-1)p^2\mathbb{E}[X]^2,\\
\mu_3(m,p) &= a_0\binom{2}{0}\underbrace{\mu_2(m-1,p)}_{\substack{=(m-1)p\mathbb{E}[X^2]\\ +(m-1)(m-2)p^2\mathbb{E}[X]^2}} + a_1\binom{2}{1}\underbrace{\mu_1(m-1,p)}_{=(m-1)p\mathbb{E}[X]} + a_2\binom{2}{2}\underbrace{\mu_0(m-1,p)}_{=1}\\
&= mp\mathbb{E}[X^3] + 3m(m-1)p^2\mathbb{E}[X]\mathbb{E}[X^2] + m(m-1)(m-2)p^3\mathbb{E}[X]^3.
\end{aligned}$$

Example 4.3. Suppose that N follows a negative Binomial distribution with parameters (m,p). Then, $\tilde{N}$ also follows a negative Binomial distribution, but with parameters $(m+1,p)$ as shown in Example 3.3. Accordingly, we denote by $\mu_k(m,p)$ and $\mu_k(m+1,p)$ the kth raw moments of the compound negative Binomial random variables S_N and $S_{\tilde{N}}$, respectively, i.e.,

$$\begin{aligned}
\mu_k(m,p) &= \mathbb{E}[(S_N)^k],\\
\mu_k(m+1,p) &= \mathbb{E}[(S_{\tilde{N}})^k].
\end{aligned}$$

From Theorem 4.1, we obtain

$$\mu_{k+1}(m,p) = \sum_{i=0}^{k} a_i\binom{k}{i}\mu_{k-i}(m+1,p), \tag{13}$$

where a_i is given by

$$a_i = m\left(\frac{1-p}{p}\right)\mathbb{E}[X^{i+1}]. \tag{14}$$

With the initial value $\mu_0(m,p) = 1$, we can find $\mu_k(m,p)\,(k = 1,2,3,\ldots)$ sequentially using (13) and (14). For example, the first three raw moments of S_N

are

$$\mu_1(m,p) = a_0\binom{0}{0}\underbrace{\mu_0(m+1,p)}_{=1} = m\left(\frac{1-p}{p}\right)\mathbb{E}[X],$$

$$\begin{aligned}\mu_2(m,p) &= a_0\binom{1}{0}\underbrace{\mu_1(m+1,p)}_{=(m+1)\left(\frac{1-p}{p}\right)\mathbb{E}[X]}+a_1\binom{1}{1}\underbrace{\mu_0(m+1,p)}_{=1}\\ &= m\left(\frac{1-p}{p}\right)\mathbb{E}[X^2]+m(m+1)\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]^2,\end{aligned}$$

$$\begin{aligned}\mu_3(m,p) &= a_0\binom{2}{0}\underbrace{\mu_2(m+1,p)}_{\substack{=(m+1)\left(\frac{1-p}{p}\right)\mathbb{E}[X^2]\\ +(m+1)(m+2)\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]^2}}+a_1\binom{2}{1}\underbrace{\mu_1(m+1,p)}_{=(m+1)\left(\frac{1-p}{p}\right)\mathbb{E}[X]}+a_2\binom{2}{2}\underbrace{\mu_0(m+1,p)}_{=1}\\ &= m\left(\frac{1-p}{p}\right)\mathbb{E}[X^3]+3m(m+1)\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]\mathbb{E}[X^2]\\ &\quad + m(m+1)(m+2)\left(\frac{1-p}{p}\right)^3\mathbb{E}[X]^3.\end{aligned}$$

If the severity random variable X is degenerate with $\mathbb{P}(X=1)=1$, then we have $S_N = N$ and $S_{\tilde{N}} = \tilde{N}$. Hence, from Theorem 4.1, we can obtain the raw moments of N as in the following corollary.

Corollary 4.2. *The raw moments of N can be obtained by*

$$\mathbb{E}\big[N^{k+1}\big] = \mathbb{E}[N]\sum_{i=0}^{k}\binom{k}{i}\mathbb{E}\big[\tilde{N}^{k-i}\big], \qquad k=0,1,2,\ldots.$$

Proof. We apply Theorem 4.1 for X such that $\mathbb{P}(X=1)=1$. In this case, we have $a_i = \mathbb{E}[N]\,\mathbb{E}[X^{i+1}] = \mathbb{E}[N]$ for all i. Hence, by substituting $a_i = \mathbb{E}[N]$ and replacing S_N and $S_{\tilde{N}}$ by N and $\tilde{N}$, respectively, we have Corollary 4.2. □

5. Central Moments

In this section, we derive a recurrence formula for the central moments of S_N. We then give examples of how our formula can be used.

Theorem 5.1. *Let $\mu = \mathbb{E}[S_N]$ and $\tilde{\mu} = \mathbb{E}[S_{\tilde{N}}]$ denote the expectations of S_N and $S_{\tilde{N}}$, respectively. Then, the central moments of S_N can be obtained by*

$$\mathbb{E}\big[(S_N-\mu)^{k+1}\big] = \sum_{i=0}^{k} b_i\binom{k}{i}\mathbb{E}\big[(S_{\tilde{N}}-\tilde{\mu})^{k-i}\big] - b_0\mathbb{E}\big[(S_N-\mu)^{k}\big], \quad k=0,1,2,\ldots,$$

where b_i is given by

$$b_i = \mathbb{E}[N] \sum_{j=0}^{i} \binom{i}{j} \left(d_N - 1\right)^j \mathbb{E}[X]^j \, \mathbb{E}[X^{i+1-j}],$$

where $d_N = \frac{Var(N)}{\mathbb{E}[N]}$ is the dispersion index of N.

Proof. We substitute $c = \mu\,(= \mathbb{E}[S_N])$ and $\tilde{c} = \tilde{\mu}\,(= \mathbb{E}[S_{\tilde{N}}])$ in Theorem 3.1. We then denote by b_i the resulting c_i in (8), i.e.,

$$b_i = \mathbb{E}[N] \sum_{j=0}^{i} \binom{i}{j} \left(\tilde{\mu} - \mu\right)^j \mathbb{E}[X^{i+1-j}].$$

Here, the difference $\tilde{\mu} - \mu$ is obtained by using Wald's equality as

$$\begin{aligned} \tilde{\mu} - \mu &= \mathbb{E}[S_{\tilde{N}}] - \mathbb{E}[S_N] \\ &= \mathbb{E}[\tilde{N}]\mathbb{E}[X] - \mathbb{E}[N]\mathbb{E}[X] \\ &= (\mathbb{E}[\tilde{N}] - \mathbb{E}[N])\mathbb{E}[X], \end{aligned}$$

in which $\mathbb{E}[\tilde{N}] - \mathbb{E}[N]$ can be found by subtracting 1 from the dispersion index d_N of N as follows:

$$\begin{aligned} \mathbb{E}[\tilde{N}] - \mathbb{E}[N] &= \sum_{n=0}^{\infty} n \cdot \mathbb{P}(\tilde{N} = n) - \mathbb{E}[N] \\ &= \sum_{n=0}^{\infty} \frac{n(n+1)}{\mathbb{E}[N]} \cdot \mathbb{P}(N = n+1) - \mathbb{E}[N] \\ &= \sum_{k=1}^{\infty} \frac{(k-1)k}{\mathbb{E}[N]} \cdot \mathbb{P}(N = k) - \mathbb{E}[N] \\ &= \frac{1}{\mathbb{E}[N]} \left(\sum_{k=0}^{\infty} k^2 \cdot \mathbb{P}(N = k) - \sum_{k=0}^{\infty} k \cdot \mathbb{P}(N = k) \right) - \mathbb{E}[N] \\ &= \frac{1}{\mathbb{E}[N]} \Big(\mathbb{E}[N^2] - \mathbb{E}[N] \Big) - \mathbb{E}[N] \\ &= \frac{\mathrm{Var}(N)}{\mathbb{E}[N]} - 1 \\ &= d_N - 1. \end{aligned}$$

Hence, b_i is determined by the moments of N and X as

$$b_i = \mathbb{E}[N] \sum_{j=0}^{i} \binom{i}{j} \left(d_N - 1\right)^j \mathbb{E}[X]^j \mathbb{E}[X^{i+1-j}].$$

Since $\mu = \mathbb{E}[S_N] = \mathbb{E}[N]\,\mathbb{E}[X] = b_0$, for a coherent formulation, we express the second term on the right-hand side of (7) as

$$\mu\,\mathbb{E}\big[(S_N - \mu)^k\big] = b_0\,\mathbb{E}\big[(S_N - \mu)^k\big].$$

This completes the proof of Theorem 5.1. □

We note that Theorem 5.1, similar to Theorem 4.1, presents the recurrence formula in a structured form by decomposing each term into three regular factors: (i) a constant b_i determined by the moments of the counting random variable N and the severity random variable X; (ii) a binomial coefficient $\binom{k}{i}$; and (iii) lower-order central moments of compound random variables, $\mathbb{E}[(S_{\tilde{N}} - \tilde{\mu})^{k-i}]$ and $\mathbb{E}[(S_N - \mu)^k]$. The factorized structure of our recurrence formula can simplify the computation of the central moments, as demonstrated in the following examples.

Example 5.1. Suppose that N follows a Poisson distribution with parameter λ. Then, $\tilde{N}$ also follows a Poisson distribution with parameter λ as illustrated in Example 3.1. Accordingly, to simplify notation, we denote by η_k the kth central moment of the compound Poisson random variables S_N and $S_{\tilde{N}}$, i.e.,

$$\eta_k = \mathbb{E}\big[(S_N - \mathbb{E}[S_N])^k\big] = \mathbb{E}\big[(S_{\tilde{N}} - \mathbb{E}[S_{\tilde{N}}])^k\big].$$

From Theorem 5.1, we obtain

$$\eta_{k+1} = \sum_{i=0}^{k} b_i \binom{k}{i} \eta_{k-i} - b_0\,\eta_k. \tag{15}$$

The dispersion index of N is $d_N = \frac{\mathrm{Var}(N)}{\mathbb{E}[N]} = \frac{\lambda}{\lambda} = 1$. Hence, b_i becomes

$$b_i = \lambda \mathbb{E}[X^{i+1}]. \tag{16}$$

With the initial value $\eta_0 = 1$, we can find η_k ($k = 1, 2, 3, \ldots$) sequentially using (15) and (16). For example, the first three central moments of S_N are

$$\begin{aligned}
\eta_1 &= b_0 \binom{0}{0} \eta_0 - b_0 \eta_0 = 0,\\
\eta_2 &= b_0 \binom{1}{0} \eta_1 + b_1 \binom{1}{1} \eta_0 - b_0 \eta_1 = \lambda \mathbb{E}[X^2],\\
\eta_3 &= b_0 \binom{2}{0} \eta_2 + b_1 \binom{2}{1} \eta_1 + b_2 \binom{2}{2} \eta_0 - b_0 \eta_2 = \lambda \mathbb{E}[X^3].
\end{aligned}$$

Example 5.2. Suppose that N follows a Binomial distribution with parameters (m, p). Then, $\tilde{N}$ follows a Binomial distribution with parameters $(m-1, p)$ as shown in Example 3.2. Accordingly, we denote by $\eta_k(m, p)$ and $\eta_k(m-1, p)$ the kth central moments of the compound Binomial random variables S_N and $S_{\tilde{N}}$, respectively, i.e.,

$$\eta_k(m,p) = \mathbb{E}\big[(S_N - \mathbb{E}[S_N])^k\big],$$
$$\eta_k(m-1,p) = \mathbb{E}\big[(S_{\tilde{N}} - \mathbb{E}[S_{\tilde{N}}])^k\big].$$

From Theorem 5.1, we obtain

$$\eta_{k+1}(m,p) = \sum_{i=0}^{k} b_i \binom{k}{i} \eta_{k-i}(m-1,p) - b_0\, \eta_k(m,p). \tag{17}$$

The dispersion index of N is $d_N = \frac{\mathrm{Var}(N)}{\mathbb{E}[N]} = \frac{mp(1-p)}{mp} = 1-p$. Hence, b_i becomes

$$b_i = mp \sum_{j=0}^{i} \binom{i}{j} (-p)^j\, \mathbb{E}[X]^j \mathbb{E}[X^{i+1-j}]. \tag{18}$$

With the initial value $\eta_0(m,p) = 1$, we can find $\eta_k(m,p)\,(k = 1,2,3,\dots)$ sequentially using (17) and (18). For example, if we want to find the first three central moments of S_N, we compute *a priori* b_0, b_1, b_2 using (18):

$$\begin{aligned} b_0 &= mp\mathbb{E}[X], \\ b_1 &= mp\mathbb{E}[X^2] - mp^2\mathbb{E}[X]^2, \\ b_2 &= mp\mathbb{E}[X^3] - 2mp^2\mathbb{E}[X]\mathbb{E}[X^2] + mp^3\mathbb{E}[X]^3. \end{aligned}$$

We then use (17) in a recursive manner as follows:

$$\begin{aligned} \eta_1(m,p) &= b_0 \binom{0}{0} \underbrace{\eta_0(m-1,p)}_{=1} - b_0 \underbrace{\eta_0(m,p)}_{=1} = 0, \\ \eta_2(m,p) &= b_0 \binom{1}{0} \underbrace{\eta_1(m-1,p)}_{=0} + b_1 \binom{1}{1} \underbrace{\eta_0(m-1,p)}_{=1} - b_0 \underbrace{\eta_1(m,p)}_{=0} \\ &= mp\mathbb{E}[X^2] - mp^2\mathbb{E}[X]^2, \\ \eta_3(m,p) &= b_0 \binom{2}{0} \underbrace{\eta_2(m-1,p)}_{\substack{=(m-1)p\mathbb{E}[X^2] \\ -(m-1)p^2\mathbb{E}[X]^2}} + b_1 \binom{2}{1} \underbrace{\eta_1(m-1,p)}_{=0} + b_2 \binom{2}{2} \underbrace{\eta_0(m-1,p)}_{=1} - b_0 \underbrace{\eta_2(m,p)}_{\substack{=mp\mathbb{E}[X^2] \\ -mp^2\mathbb{E}[X]^2}} \\ &= mp\mathbb{E}[X^3] - 3mp^2\mathbb{E}[X]\mathbb{E}[X^2] + 2mp^3\mathbb{E}[X]^3. \end{aligned}$$

Example 5.3. Suppose that N follows a negative Binomial distribution with parameters (m,p). Then, $\tilde{N}$ follows a negative Binomial distribution with parameters $(m+1,p)$ as shown in Example 3.3. Accordingly, we denote by $\eta_k(m,p)$ and $\eta_k(m+1,p)$ the kth central moments of the compound negative Binomial random variables S_N and $S_{\tilde{N}}$, respectively, i.e.,

$$\eta_k(m,p) = \mathbb{E}\big[(S_N - \mathbb{E}[S_N])^k\big],$$
$$\eta_k(m+1,p) = \mathbb{E}\big[(S_{\tilde{N}} - \mathbb{E}[S_{\tilde{N}}])^k\big].$$

From Theorem 5.1, we obtain

$$\eta_{k+1}(m,p) = \sum_{i=0}^{k} b_i \binom{k}{i} \eta_{k-i}(m+1,p) - b_0\, \eta_k(m,p). \tag{19}$$

The dispersion index of N is $d_N = \frac{\mathrm{Var}(N)}{\mathbb{E}[N]} = \frac{m(1-p)/p^2}{m(1-p)/p} = \frac{1}{p}$. Hence, b_i becomes

$$b_i = m \sum_{j=0}^{i} \binom{i}{j} \left(\frac{1-p}{p}\right)^{j+1} \mathbb{E}[X]^j \mathbb{E}[X^{i+1-j}]. \tag{20}$$

With the initial value $\eta_0(m,p) = 1$, we can find $\eta_k(m,p)\,(k = 1,2,3,\ldots)$ sequentially using (19) and (20). For example, if we want to find the first three central moments of S_N, we compute *a priori* b_0, b_1, b_2 using (20):

$$\begin{aligned}
b_0 &= m\left(\frac{1-p}{p}\right)\mathbb{E}[X],\\
b_1 &= m\left(\frac{1-p}{p}\right)\mathbb{E}[X^2] + m\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]^2,\\
b_2 &= m\left(\frac{1-p}{p}\right)\mathbb{E}[X^3] + 2m\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]\mathbb{E}[X^2] + m\left(\frac{1-p}{p}\right)^3\mathbb{E}[X]^3.
\end{aligned}$$

We then use (19) in a recursive manner as follows:

$$\begin{aligned}
\eta_1(m,p) &= b_0\binom{0}{0}\underbrace{\eta_0(m+1,p)}_{=1} - b_0\underbrace{\eta_0(m,p)}_{=1} = 0,\\
\eta_2(m,p) &= b_0\binom{1}{0}\underbrace{\eta_1(m+1,p)}_{=0} + b_1\binom{1}{1}\underbrace{\eta_0(m+1,p)}_{=1} - b_0\underbrace{\eta_1(m,p)}_{=0}\\
&= m\left(\frac{1-p}{p}\right)\mathbb{E}[X^2] + m\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]^2,\\
\eta_3(m,p) &= b_0\binom{2}{0}\underbrace{\eta_2(m+1,p)}_{\substack{=(m+1)\left(\frac{1-p}{p}\right)\mathbb{E}[X^2]\\ +(m+1)\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]^2}} + b_1\binom{2}{1}\underbrace{\eta_1(m+1,p)}_{=0} + b_2\binom{2}{2}\underbrace{\eta_0(m+1,p)}_{=1} - b_0\underbrace{\eta_2(m,p)}_{\substack{=m\left(\frac{1-p}{p}\right)\mathbb{E}[X^2]\\ +m\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]^2}}\\
&= m\left(\frac{1-p}{p}\right)\mathbb{E}[X^3] + 3m\left(\frac{1-p}{p}\right)^2\mathbb{E}[X]\mathbb{E}[X^2] + 2m\left(\frac{1-p}{p}\right)^3\mathbb{E}[X]^3.
\end{aligned}$$

As noted in Section 4, we have $S_N = N$ and $S_{\tilde{N}} = \tilde{N}$ if the severity random variable X is degenerate with $\mathbb{P}(X = 1) = 1$. Hence, similarly to Corollary 4.2, we can obtain the central moments of N as follows.

Corollary 5.2. *Let $\rho = \mathbb{E}[N]$ and $\tilde{\rho} = \mathbb{E}[\tilde{N}]$ denote the expectations of N and $\tilde{N}$, respectively. Then, the central moments of N can be obtained by*

$$\begin{aligned}\mathbb{E}\big[(N-\rho)^{k+1}\big] = \mathbb{E}[N]\sum_{i=0}^{k}\big(d_N\big)^i\binom{k}{i}\mathbb{E}\big[(\tilde{N}-\tilde{\rho})^{k-i}\big]\\ - \mathbb{E}[N]\,\mathbb{E}\big[(N-\rho)^k\big], \qquad k=0,1,2,\ldots,\end{aligned}$$

where $d_N = \frac{Var(N)}{\mathbb{E}[N]}$ is the dispersion index of N.

Proof. We apply Theorem 5.1 for X such that $\mathbb{P}(X=1)=1$. In this case, we have $\mathbb{E}[X]^j = \mathbb{E}[X^{i+1-j}] = 1$ for all i, j. Hence, b_i in Theorem 5.1 reduces to

$$\begin{aligned} b_i &= \mathbb{E}[N]\sum_{j=0}^{i}\binom{i}{j}\big(d_N-1\big)^j \\ &= \mathbb{E}[N]\big(d_N-1+1\big)^i \\ &= \mathbb{E}[N]\big(d_N\big)^i. \end{aligned}$$

Hence, by substituting $b_i = \mathbb{E}[N](d_N)^i$ and replacing (S_N, μ) and $(S_{\tilde{N}}, \tilde{\mu})$ by (N, ρ) and $(\tilde{N}, \tilde{\rho})$, respectively, we have Corollary 5.2. □

6. Proof of Theorem 3.1

By the linearity of expectation, we have

$$\begin{aligned}\mathbb{E}\big[(S_N-c)^{k+1}\big] &= \mathbb{E}\big[(S_N-c)(S_N-c)^k\big]\\ &= \mathbb{E}\big[S_N(S_N-c)^k\big] - c\,\mathbb{E}\big[(S_N-c)^k\big]. \end{aligned} \tag{21}$$

In the following, we complete the proof in two steps. First, we show that the first term on the right-hand side of (21) can be written as

$$\mathbb{E}\big[S_N(S_N-c)^k\big] = \mathbb{E}[N]\,\mathbb{E}\big[X(S_{\tilde{N}}+X-c)^k\big], \tag{22}$$

where X is the generic random variable for X_i and is independent of $S_{\tilde{N}}$. Second, we show that the term on the right-hand side of (22) can be further written as

$$\mathbb{E}[N]\,\mathbb{E}\big[X(S_{\tilde{N}}+X-c)^k\big] = \sum_{i=0}^{k} c_i\binom{k}{i}\mathbb{E}\big[(S_{\tilde{N}}-\tilde{c})^k\big], \tag{23}$$

where c_i is given in (8). Combining (21), (22) and (23) then yields the theorem.

To begin, we evaluate the expectation $\mathbb{E}\big[S_N(S_N-c)^k\big]$ by conditioning on N:

$$\begin{aligned}\mathbb{E}\big[S_N(S_N-c)^k\big] &= \sum_{n=0}^{\infty}\mathbb{P}(N=n)\,\mathbb{E}\big[S_N(S_N-c)^k \mid N=n\big]\\ &= \sum_{n=1}^{\infty}\mathbb{P}(N=n)\,\mathbb{E}\big[S_N(S_N-c)^k \mid N=n\big]\\ &= \sum_{n=1}^{\infty}\mathbb{P}(N=n)\,\mathbb{E}\big[S_n(S_n-c)^k\big],\end{aligned}$$

where the second equality follows because we have defined $S_0=0$ by convention, and the third one follows from the independence of N and $\{X_i, i=1,2,3,\dots\}$. Since $X_1, X_2, X_3, \dots$ are independent and identically distributed, we have

$$X_1\bigg(\sum_{j=1}^{n}X_j-c\bigg) \overset{\mathrm{d}}{=} X_2\bigg(\sum_{j=1}^{n}X_j-c\bigg) \overset{\mathrm{d}}{=} \cdots \overset{\mathrm{d}}{=} X_n\bigg(\sum_{j=1}^{n}X_j-c\bigg),$$

where the symbol $\overset{\mathrm{d}}{=}$ denotes *equal in distribution*. It then follows that

$$\begin{aligned}\mathbb{E}\big[S_n(S_n-c)^k\big] &= \mathbb{E}\left[\bigg(\sum_{i=1}^{n}X_i\bigg)\cdot\bigg(\sum_{j=1}^{n}X_j-c\bigg)^k\right]\\ &= \sum_{i=1}^{n}\mathbb{E}\left[X_i\bigg(\sum_{j=1}^{n}X_j-c\bigg)^k\right]\\ &= n\,\mathbb{E}\big[X_n\,(S_n-c)^k\big],\end{aligned}$$

which, in turn, leads to

$$\mathbb{E}\big[S_N(S_N-c)^k\big] = \sum_{n=1}^{\infty}\mathbb{P}(N=n)\,n\,\mathbb{E}\big[X_n\,(S_n-c)^k\big].$$

From (6), we have $\mathbb{P}(N=n)\,n = \mathbb{E}[N]\mathbb{P}(\tilde{N}=n-1)$ for $n=1,2,3,\dots$. We use this relation and then make the change of variable $l=n-1$ to obtain

$$\begin{aligned}\mathbb{E}\big[S_N(S_N-c)^k\big] &= \mathbb{E}[N]\sum_{n=1}^{\infty}\mathbb{P}(\tilde{N}=n-1)\,\mathbb{E}\big[X_n(S_n-c)^k\big]\\ &= \mathbb{E}[N]\sum_{l=0}^{\infty}\mathbb{P}(\tilde{N}=l)\,\mathbb{E}\big[X_{l+1}(S_{l+1}-c)^k\big].\end{aligned} \tag{24}$$

Here, the expectation $\mathbb{E}\big[X_{l+1}(S_{l+1}-c)^k\big]$ can be expressed as

$$\begin{aligned}\mathbb{E}\big[X_{l+1}(S_{l+1}-c)^k\big] &= \mathbb{E}\big[X_{l+1}(S_l+X_{l+1}-c)^k\big]\\ &= \mathbb{E}\big[X_{l+1}(S_l+X_{l+1}-c)^k \mid \tilde{N}=l\big]\\ &= \mathbb{E}\big[X_{\tilde{N}+1}(S_{\tilde{N}}+X_{\tilde{N}+1}-c)^k \mid \tilde{N}=l\big]\\ &= \mathbb{E}\big[X(S_{\tilde{N}}+X-c)^k \mid \tilde{N}=l\big], \end{aligned} \tag{25}$$

where, in the second equality, the independence of $\tilde{N}$ and $\{X_i, i=1,2,3,\dots\}$ is used. Substituting (25) into the right-hand side of (24), we have

$$\begin{aligned}\mathbb{E}\big[S_N(S_N-c)^k\big] &= \mathbb{E}[N]\sum_{l=0}^{\infty}\mathbb{P}(\tilde{N}=l)\,\mathbb{E}\big[X(S_{\tilde{N}}+X-c)^k \mid \tilde{N}=l\big]\\ &= \mathbb{E}[N]\,\mathbb{E}\Big[\mathbb{E}\big[X(S_{\tilde{N}}+X-c)^k \mid \tilde{N}\big]\Big]\\ &= \mathbb{E}[N]\,\mathbb{E}\big[X(S_{\tilde{N}}+X-c)^k\big],\end{aligned}$$

which shows (22).

Now we apply the Binomial expansion to the term on the right-hand side of (22). Then, we have

$$\begin{aligned}\mathbb{E}[N]\,\mathbb{E}\big[X(S_{\tilde{N}}+X-c)^k\big] &= \mathbb{E}[N]\,\mathbb{E}\Big[X\big(\{S_{\tilde{N}}-\tilde{c}\}+\{X+\tilde{c}-c\}\big)^k\Big]\\ &= \mathbb{E}[N]\,\mathbb{E}\Big[X\sum_{i=0}^{k}\binom{k}{i}\big(S_{\tilde{N}}-\tilde{c}\big)^i\big(X+\tilde{c}-c\big)^{k-i}\Big]\\ &= \mathbb{E}[N]\sum_{i=0}^{k}\binom{k}{i}\mathbb{E}\big[(S_{\tilde{N}}-\tilde{c})^i\big]\,\mathbb{E}\big[X(X+\tilde{c}-c)^{k-i}\big],\end{aligned}$$

where the last equality comes from the independence of X and $S_{\tilde{N}}$. Applying the Binomial expansion again to the factor $\mathbb{E}\big[X(X+\tilde{c}-c)^{k-i}\big]$, we have

$$\begin{aligned}\mathbb{E}\big[X(X+\tilde{c}-c)^{k-i}\big] &= \mathbb{E}\Big[X\sum_{j=0}^{k-i}\binom{k-i}{j}\big(\tilde{c}-c\big)^j X^{k-i-j}\Big]\\ &= \sum_{j=0}^{k-i}\binom{k-i}{j}\big(\tilde{c}-c\big)^j\mathbb{E}\big[X^{k+1-i-j}\big].\end{aligned}$$

Hence, the term on the right-hand side of (22) can be expressed as

$$\mathbb{E}[N]\,\mathbb{E}\big[X(S_{\tilde{N}}+X-c)^k\big] = \sum_{i=0}^{k} d_{k,i}\,\mathbb{E}\big[(S_{\tilde{N}}-\tilde{c})^i\big], \tag{26}$$

where $d_{k,i}$ is given by

$$d_{k,i} = \mathbb{E}[N]\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\big(\tilde{c}-c\big)^j\mathbb{E}\big[X^{k+1-i-j}\big].$$

In order to reduce the computational complexity involved in (26), we seek to find a simpler expression for $d_{k,i}$. We note that

$$\begin{aligned}
d_{k+1,i+1} &= \mathbb{E}[N] \sum_{j=0}^{(k+1)-(i+1)} \binom{k+1}{i+1} \binom{(k+1)-(i+1)}{j} (\tilde{c}-c)^j \mathbb{E}\big[X^{(k+1)+1-(i+1)-j}\big] \\
&= \mathbb{E}[N] \sum_{j=0}^{k-i} \binom{k+1}{i+1} \binom{k-i}{j} (\tilde{c}-c)^j \mathbb{E}\big[X^{k+1-i-j}\big] \\
&= \mathbb{E}[N] \sum_{j=0}^{k-i} \frac{k+1}{i+1} \binom{k}{i} \binom{k-i}{j} (\tilde{c}-c)^j \mathbb{E}\big[X^{k+1-i-j}\big] \\
&= \frac{k+1}{i+1} \cdot d_{k,i}.
\end{aligned}$$

That is, the following relation holds for any successive terms $d_{k,i}$ and $d_{k+1,i+1}$:

$$d_{k+1,i+1} = \frac{k+1}{i+1} \cdot d_{k,i}, \qquad i = 0, 1, \ldots, k.$$

By induction, we have

$$\begin{aligned}
d_{k,i} &= \frac{k}{i} \cdot d_{k-1,i-1} \\
&= \frac{k}{i} \cdot \frac{k-1}{i-1} \cdot d_{k-2,i-2} \\
&= \frac{k}{i} \cdot \frac{k-1}{i-1} \cdot \frac{k-2}{i-2} \cdot d_{k-3,i-3} \\
&\;\;\vdots \\
&= \frac{k(k-1)(k-2)\cdots(k-i+1)}{i(i-1)(i-2)\cdots(1)} \cdot d_{k-i,0} \\
&= \binom{k}{i} d_{k-i,0}.
\end{aligned}$$

Furthermore, with $d_{i,0} = c_i$, we can simplify the expression in (26) as

$$\begin{aligned}
\mathbb{E}[N]\,\mathbb{E}\big[X(S_{\tilde{N}} + X - c)^k\big] &= \sum_{i=0}^{k} d_{k-i,0} \binom{k}{i} \mathbb{E}\big[(S_{\tilde{N}} - \tilde{c})^i\big] \\
&= \sum_{i=0}^{k} d_{i,0} \binom{k}{i} \mathbb{E}\big[(S_{\tilde{N}} - \tilde{c})^{k-i}\big] \\
&= \sum_{i=0}^{k} c_i \binom{k}{i} \mathbb{E}\big[(S_{\tilde{N}} - \tilde{c})^{k-i}\big],
\end{aligned}$$

which shows (23). This completes the proof of Theorem 3.1.

7. Conclusion

In this paper, we have derived a formula for the raw and central moments of a compound random variable by utilizing an auxiliary counting random variable. This utilization allows us to derive a recurrence formula in a structured form having three regular factors, which provides the advantage of reducing the computational complexity involved in obtaining higher-order moments. A potential future direction is to identify the class of counting random variables to which our formula can be effectively applied.

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