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REMARKS ON GROUP EQUATIONS AND ZERO DIVISORS OF TOPOLOGICAL STRUCTURES

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Abstract. The motivation in this paper comes from the recent results about Bell inequalities and topological insulators from group theory. Symmetries which are interested in group theory could be mainly used to find material structures. In this point of views, we study group extending by adding one relator which is easily called an equation. So a relative group extension by a adding relator is aspherical if the natural injection is one-to-one and the group ring has no zero divisor. One of concepts of asphericity means that a new group by a adding relator is well extended. Also, we consider that several equations and relative presentations over torsion-free groups are related to zero divisors.

1. INTRODUCTION

Complex structures, which are occasionally reported in several places of the nature, are related to exotic physical phenomena. For instance, topological materials with complex electronic structures display insulator in their interior. The surface containing with them would be presented in virus, pattern detection and compute vision. Therefore, geometrical understanding of complex structures is one of interesting issues in pure mathematics, physics, material science as well as engineering [4, 6, 9].

Group theory, which is associated with various studies for invariant structures under transformations in terms of geometry, is a starting point for geometrical understanding of them and have given solutions of problems in several areas, where the understanding of symmetry is crucial. In particular, we use group action of a single generator group to study a method for solving Bell inequalities. Also groups are used to classify insulators by generalized symmetries that combine space-time transformations. Sometimes, the cohomology of topological structures provides a symmetry-based classification of quasi-momentum manifolds.

In this point of views, we study the connection between asphericity of group presentations and making new groups. To generate new groups, we use certain

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equations over groups. It means that the new group is really bigger than a given group. Also, we discuss about the relation of asphericity and zero divisor. Finally, we would describe certain equations of the form $x^{m_1}h_1x^{m_2}h_2\cdots x^{m_k}h_k$ which is related to zero divisors of group rings.

2. Equations over groups and zero divisors

As we know, a group with an operation is a set which satisfies the associativity, the existence of identity and the existence of inverse element for each element. For example, we consider the set G of all one-to-one correspondences from a given set to itself. Here, the operation is the composition of functions. An element in this group is called a symmetry.

Suppose that we are given a group H and a set of words $r \in \mathbf{r}$, involving a set of indeterminates $x \in \mathbf{x}$ the data $P =< H, \mathbf{x} : \mathbf{r} >$ is called a *relative presentation* with *coefficient group* H; the group defined by this data is the quotient group $G = \frac{H*F(\mathbf{x})}{<<\mathbf{r}(\mathbf{x})>>}$, where $H * F(\mathbf{x})$ is the free product of H with the free group $F(\mathbf{x})$ having basis $\mathbf{x}$ and $<< \mathbf{r}(\mathbf{x}) >>$ is the smallest normal subgroup of the free product containing the word $r(\mathbf{x})$, $r \in \mathbf{r}$. The concept of *asphericity* for relative presentations was introduced for the purpose of studying how the cohomology and finite subgroups of G are related to those of H. In this paper, we consider that one indeterminate x and one word r. Simply speaking, we consider that x is a variable and r is an equation as the following form.

$$x^{m_1}h_1x^{m_2}h_2\cdots x^{m_k}h_k$$

where m_i is an integral number and h_i is an element of H.

Asphericity of the relative presentations has been addressed by several authors including Bogley and Pride, [2, 3, 7, 8]. Scanning these results, one finds that for $k \leq 6$, these presentations are always aspherical when H is torsion free. It is then a consequence of the theory of aspherical relative presentations that if $k \leq 6$ and H is torsion-free, then the new group G is also torsion-free as long as the relator is not a proper power in $H * F(x)$. The proof of this question depends on the *asphericity* of a relative presentation in Bogley and Pride paper and Serre theorem in J. Huebschmann, also see the paper of Bogley [2]. It may well be that the relative presentation is always aspherical when H is torsion-free. This in turn would imply that the group defined by the relative presentation is torsion -free whenever H is torsion-free and the relator is not a proper power.

The connection between asphericity and torsion-freeness goes deeper. The famous *zero-divisor conjecture* of Kaplansky asserts that the integral group ring $\mathbb{Z}G$ of a torsion-free group G is a domain. G. Higman showed that the conjecture is true for locally indicable groups which means any finitely generated subgroup has the integer group as a homomorphic image. Obviously, locally indicable groups are torsion free by considering the subgroup generated by any single element. A recent result of Ivanov [7] implies that a non-aspherical relative presentation that defines a torsion-free group G would provide a counterexample to Kaplansky's conjecture. The some results of this paper, the proof of which employs all of the tools used in the properties of zero divisors. Next we study zero divisors of group rings and certain equations over groups. In particular, some relative presentation is aspherical if H is torsion free and the relator is not a proper power. This is indeed motivated by a recent discovery of the connection between the question of general asphericity of 2-complexes and the question of zero divisor conjecture, i.e., the Kaplansky conjecture such that the integral group ring of a torsion free group has no zero divisors [1, 7]. G. Higman showed that locally indicable groups have no zero divisors.

Let $P =< H, \mathbf{x} : \mathbf{r} >$ be a relative group presentation and let P define a group G and let i be the inclusion homomorphism from H to G. Choose a group presentation $Q =< \mathbf{a} : \mathbf{u} >$. As a routine, let L and K be corresponding standard models for G and H, respectively. As before, we use the single 0-cell as a basepoint of homotopy groups.

Definition 1. (See [1, 2]**)** The relative presentation $P =< H, \mathbf{x} : \mathbf{r} >$ is aspherical if the second homotopy group $\pi_2(L, K)$ is trivial.

Now, consider a covering map $p : \widetilde{L} \to L$, where $\tilde{L}$ is the universal cover of L, i.e., simply connected topological space. Then $\widetilde{L}$ is actually a 2-complex and G acts on $\widetilde{L}$ by permuting cells. The action has the following properties. (1) G acts freely on $\widetilde{L}$ in the sense that $g \cdot \tilde{x} \neq \tilde{x}$ for all $\tilde{x} \in \widetilde{L}$ and $g \neq 1$ in G. (2) G acts transitively on $p^{-1}(x)$ for all $x \in L$, i.e., for any two distinct points y_1 and y_2 in $p^{-1}(x)$, there is an element $g \in G$ such that $g \cdot y_1 = y_2$. For more details, see [2]. Let $\overline{K} = p^{-1}(K)$ and K_0 be a connected component of K. Choose a basepoint in the preimage of the basepoint of K, then we will use it as a basepoint of $\widetilde{L}$, K and K_0 simultaneously. Then the group action gives us two properties as follows.

- $G/i(H)$ have the same cardinality as connected components of $\overline{K}$.
- $\pi_1(\overline{K_0})$ is isomorphic to the kernel of $i : H \to G$.

For any simply connected 2-complex K, the second homotopy group $\pi_2(K)$ is isomorphic to the second homology group $H_2(K)$. Then we have a following lemma.

Lemma 2.1. *Suppose that the natural map $i : H \to G$ is injective. Then $P =< H, \mathbf{x} : \mathbf{r} >$ is aspherical if and only if the inclusion induced homomorphism $i_* : H_2(\overline{K}) \to H_2(\widetilde{L})$ is surjective.*

Proof. Suppose that $i : H \to G$ is injective. Define $L, \widetilde{L}, K, \overline{K}$, and $\overline{K_0}$ as above. We already know that $H_2(\widetilde{L})$, $\pi_2(\widetilde{L})$ and $\pi_2(L)$ are all isomorphic. And $H_2(\overline{K})$ is isomorphic to the sum of $H_2(\overline{K_0})$, where the sum is taken over the number of the $G/i(H)$. It is isomorphic to $\mathbb{Z}G \otimes_{\mathbb{Z}H} H_2(\overline{K_0})$. Since $\overline{K_0}$ is the universal cover of K, we have that $H_2(\overline{K_0})$ and $\pi_2(K)$ are isomorphic. Finally, this implies that $H_2(\overline{K})$ and $\mathbb{Z}G \otimes_{\mathbb{Z}H} \pi_2(K)$ are isomorphic. Consequently, $i_* : H_2(\overline{K}) \to H_2(\widetilde{L})$ is surjective if and only if the map $\phi : \mathbb{Z}G \otimes_{\mathbb{Z}H} \pi_2(K) \to \pi_2(L)$ given by

$\phi(g \otimes [f]) = g \cdot i_{\#}([f])$ is surjective. Equivalently, $\pi_2(L)$ is generated as a $\mathbb{Z}G$-module by the image of the inclusion induced homomorphism $i_{\#} : \pi_2(K) \to \pi_2(L)$, that is to say, P is aspherical. This completes our result. $\square$

Therefore, one can naturally ask when the map $i_* : H_2\overline{K} \to H_2\widetilde{L}$ is surjective. To do this we should think of the homology exact sequence of $(\widetilde{L}, \overline{K})$.

$$\cdots \to H_3(\widetilde{L}, \overline{K}) \to H_2\overline{K} \xrightarrow{i_*} H_2\widetilde{L} \to H_2(\widetilde{L}, \overline{K}) \to H_1\overline{K} \to \cdots$$

In this sequence, we have $H_3(\widetilde{L}, \overline{K}) = 0$ because L is a 2-dimensional complex and so there are no 3-cells, and $H_1\overline{K} = 0$ because $\overline{K}$ is a disjoint union of simply connected spaces. So we get a short exact sequence as follows

$$0 \to H_2\overline{K} \xrightarrow{i_*} H_2\widetilde{L} \to H_2(\widetilde{L}, \overline{K}) \to 0$$

Then one can say that i_* is surjective if and only if $H_2(\widetilde{L}, \overline{K}) = 0$. By the definition of homology groups, we have the following sequence

$$0 \to H_2(\widetilde{L}, \overline{K}) \to C_2(\widetilde{L}, \overline{K}) \xrightarrow{\partial} C_1(\widetilde{L}, \overline{K}) \to \cdots$$

where $C_1(\widetilde{L}, \overline{K}) = \bigoplus_{x \in \mathbf{x}} \mathbb{Z}G \cdot e^1_x$ and $C_2(\widetilde{L}, \overline{K}) = \bigoplus_{r \in \mathbf{r}} \mathbb{Z}G \cdot e^2_r$ are free $\mathbb{Z}G$-modules generated by $\{e^1_x : x \in \mathbf{x}\}$ and $\{e^2_r : r \in \mathbf{r}\}$, respectively. And the $\mathbb{Z}G$-module homomorphism ∂ is called *Fox-derivative* given by as follows.

$$\partial(e^2_r) = \sum_{x \in \mathbf{x}} \Big(\frac{\partial r}{\partial x}\Big) e^1_x$$

where $\partial x'/\partial x = \delta_{x,x'}$, the Kronecker delta and if $r = uv$ then

$$\frac{\partial r}{\partial x} = \frac{\partial u}{\partial x} + u\Big(\frac{\partial v}{\partial x}\Big).$$

Lemma 2.2. *Suppose that* $\mathbf{x} = \{x\}$ *and* $\mathbf{r} = \{r\}$ *are singletons and exponent sum of* x *in* r *is non zero. A relative group presentation* $P =< H, x : r >$ *for a group* G *is aspherical if the natural map from* H *to* G *is injective and the group ring* $\mathbb{Z}G$ *has no zero divisors.*

Proof. It is sufficient to show that ∂ is injective. In the case, $H_2(\widetilde{L}, \overline{K}) = 0$ and so $i_* : H_2(\overline{K}) \to H_2(\widetilde{L})$ is surjective. By lemma 2.1, P is aspherical. Now, suppose that the Fox-derivative ∂ is not injective. Then there is a non-zero element $w = \sum_i n_i g_i$ in $\mathbb{Z}G$ such that $\partial(w \cdot e^2_r) = 0$. On the other hand,

$$\partial(w \cdot e^2_r) = w\partial(e^2_r) = w\frac{\partial r}{\partial x} \cdot e^1_x$$

Then $w\frac{\partial r}{\partial x} = 0$. But since exponent sum of x in r is non zero, the difference of terms with positive sign and terms with negative sign in $\frac{\partial r}{\partial x}$ is not zero so that $\frac{\partial r}{\partial x}$ is non zero in $\mathbb{Z}G$. This implies that w and $\frac{\partial r}{\partial x}$ are zero divisors. This leads to a contradiction. $\square$

3. REMARKS ON ASPHERICAL PRESENTATIONS AND ZERO DIVISORS

Next, we will obtain a family of aspherical relative presentations by applying zero divisor test. Let G be the group defined by $P =< H, x : xh_1xh_2 \cdots xh_k >$. Each element T in the group ring $\mathbb{Z}G$ can be written as a finite sum of the form kw where k is an integer and w is a reduced word in $H * F(x)$. An element $T = \sum n_j w_j$ in the group ring $\mathbb{Z}G$ is called *homogeneous* of degree i if the exponent sum of x in each w_j is i.

Let T_i and T_j are homogeneous of degree i and j, respectively. Then $T_i \neq T_j$ in the group ring $\mathbb{Z}G$ when $i \neq j$ and the order of x in G is of infinite. Let f be a homomorphism from G to the infinite cyclic group C such that $f(h) = 1$ for any $h \in H$ and $f(x) = c$, where c is a generator of C. One can naturally extend f to a ring homomorphism f' from the group ring $\mathbb{Z}G$ to the group ring $\mathbb{Z}C$. Then the image of an element with exponent i modulo n under f' is c^i. Since $i \neq j$, $c^i \neq c^j$ in C. The order of x in G is of infinite if the product of coefficients in the equation is not the identity, i.e., $h_1h_2 \cdots h_k \neq 1$.

Recall that f is the homomorphism from G to $\prod_{i=1}^{k} H \rtimes C_k$ and $f(x) = (h_1^{-1}, h_2^{-1}, \cdots, h_k^{-1}) \cdot c^{-1}$, where C_k is the cyclic group of order m generated by c. Since c has order m, $f(x^i)$ is not the identity for $1 \leq i \leq k-1$. Furthermore, one can find that $f(x^k)$ is not the identity by the assumption.

Theorem 3.1. [1] *Consider a relative presentation*

$$P =< H, x : x^{m_1}h_1 \cdots x^{m_k}h_k > .$$

Suppose that the kernel of the natural homomorphism H to G is acyclic and that the exponent sum $\sum_i m_i$ is non zero. If P is non-weakly aspherical , then the group ring $\mathbb{Z}G$ contains a non-trivial zero divisor.

Recently, a representation of a single-generator group acting on a tensor-product Hilbert space is one of tools for studying Bell inequalities. Symmetry groups and group representations are closely associated with the thermodynamic covariance principle. Also, cohomology is one of main objects to study the topological structure. So in this paper, we studied groups by adding one relator and zero divisors. As we know that the extension of the real numbers is the set of complex numbers by solving the equation $x^2 + 1 = 0$. By the similar method, the equation $x^{m_1}h_1 \cdots x^{m_k}h_k = 1$ over a group is important to extend the given group, [6]. On the other hand, this is closely related to the asphericity of relative presentations [5, 8].

Remark 1. The problem about extensions of groups is still important and not solved in many cases. To find the solution of this problem, we really research various methods including zero divisors as one of main tools.

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