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NOTES ON SAWYER TYPE ESTIMATES FOR THE DYADIC PARAPRODUCT

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Abstract. In this paper, we provide Sawyer type conditions on a pair of weights so that the dyadic paraproduct π_b is bounded from $L^2(u)$ into $L^2(v)$. The conditions can be obtained by checking the boundedness of the dyadic paraproduct on a collection of test functions.

1. Introduction

We are interested in finding necessary and sufficient conditions on a triple of (u, v, b) such that the dyadic paraproduct π_b defined by

$$\pi_b f(x) = \sum_{I\in\mathcal{D}} \langle f\rangle_I \langle b, h_I\rangle h_I(x)$$

is bounded from $L^2(u)$ to $L^2(v)$, where $\mathcal{D}$ denotes the dyadic intervals on $\mathbb{R}$, h_I are the Haar functions associated with the interval I, $\langle f\rangle_I$ stands for the average of the functions f on the interval I, and $\langle f, g\rangle$ denotes the scalar product on $L^2(\mathbb{R})$. Paraproducts first appeared in Bony's work on nonlinear partial differential equations [2] and have since played an important role in harmonic analysis. The generalized paraproduct is the key to the class of singular integral operators with standard kernels. According to the famous $T(1)$ theorem of David and Journé [3], the Calderón-Zygmund singular integral operator T can be represented as $T = L + \pi_{T(1)} + \pi^*_{T^*(1)}$ where L is an almost translation-invariant operator. A dyadic version of this theorem can be found in [9]. Thus, when looking for bounds on the norm of some reasonably large class of singular integral operators, it is natural to start with the (dyadic) paraproduct.

Eric Sawer presented in [10] that a necessary and sufficient condition for the boundedness of the maximal function M from $L^2(u)$ into $L^2(v)$ is given by the following test conditions for the weights u and v: there exists a constant C such that

$$\|M(\mathbb{1}_I u^{-1})\|_{L^2(v)} \le C\sqrt{u^{-1}(I)} \text{ and } \|M(\mathbb{1}_I v)\|_{L^2(u^{-1})} \le C\sqrt{v(I)}\,,$$

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for all intervals I. These types of conditions are now called Sawyer's testing conditions because of his pioneering work. Sawyer also characterized certain positive operators, the fractional and Poisson integrals [11]. Sawyer's type estimates for square functions[5], Hilbert transform[4, 6], and positive and well localized dyadic operators [12, 8] etc. are known.

The dyadic paraproduct is a well-localized operator in the sense of [8], and therefore its theory can be applied. In this paper, however, we present the simple proof of Sawyer's type estimate for the dyadic paraproduct. The paper is organized as follows. In Section 2, we introduce basic notations and definitions and discuss weighted norm estimates in the general context. We also present the main results of the paper. For the proof of the main theorem, we introduce the auxiliary paraproducts and lemmas in Section 3. Finally, the proof of the main theorem is given in Section 4.

2. Preliminaries

In this section, we introduce basic notations and definitions, and present two weight estimates in general setting.

For any measurable set $E \subset \mathbb{R}$ and a given weight u, $|E|$ stands for its Lebesgue measure and $u(E) = \int_E u(x)dx$. The collection of all dyadic intervals and the collection of all dyadic subintervals of J are denoted by $\mathcal{D}$ and $\mathcal{D}(J)$ respectively. The Haar function associated with an interval $I \subset \mathbb{R}$ is defined as

$$h_I(x) := \frac{1}{\sqrt{|I|}} \left(\mathbb{1}_{I_+}(x) - \mathbb{1}_{I_-}(x) \right) ,$$

where $I_\pm$ are the right and left halves of I, respectively, and the characteristic function $\mathbb{1}_I(x) = 1$ if $x \in I$, zero otherwise. For a given weight u, the weighted Haar function h_I^u is given by

$$h_I^u(x) := \frac{1}{\sqrt{u(I)}} \left(\sqrt{\frac{u(I_-)}{u(I_+)}} \mathbb{1}_{I_+} - \sqrt{\frac{u(I_+)}{u(I_-)}} \mathbb{1}_{I_-} \right) .$$

It is a well known fact that the Haar system $\{h_I\}_{I\in\mathcal{D}}$ and the weighted Haar system $\{h_I^u\}_{I\in\mathcal{D}}$ are orthonormal systems in L^2 and $L^2(u)$, respectively. The norm of $f \in L^2(u)$ is defined by

$$\|f\|_{L^2(u)} := \left(\int_{\mathbb{R}} |f(x)|^2 u(x) dx \right)^2 .$$

Let T be a given operator and M_ϕ be the multiplication operator of ϕ. Then T is a bounded operator from $L^2(u)$ into $L^2(v)$ i.e.

$$\int |Tf|^2 v dx \leq C \int |f|^2 u dx \tag{1}$$

is equivalent to the following:

(i) For $f \in L^2(u)$ and $g \in L^2(v)$, there exists a constant C such that

$$\langle Tf, g\rangle_v \le C\|f\|_{L^2(u)}\|g\|_{L^2(v)}\,.$$

(ii) For $f, g \in L^2$, there exists a constant C such that

$$\langle M_{v^{1/2}}TM_{u^{-1/2}}f, g\rangle = \langle v^{1/2}T(fu^{-1/2}), g\rangle \le C\|f\|_{L^2}\|g\|_{L^2}\,.$$

(iii) For $f \in L^2(u^{-1})$ and $g \in L^2(v)$, there exists a constant C such that

$$\langle TM_{u^{-1}}(f), g\rangle_v = \langle T(u^{-1}f), gv\rangle \le C\|f\|_{L^2(u^{-1})}\|g\|_{L^2(v)}\,.$$

One can easily see that the statement (i) is due to the duality argument, i.e. replacing g by Tf. In the equality (ii) replacing $f = u^{1/2}F$ and $g = v^{1/2}G$ with $F \in L^2(u)$ and $G \in L^2(v)$, we get the statement (ii). Again, replacing $f = u^{1/2}F$ and $g = v^{-1/2}G$ in the inequality of (iii) with $F, G \in L^2$ gives the statement (ii). For compact notation, let us denote a pair of weights (u^{-1}, v) by (μ, ν). Then the inequality in (iii) is equivalent to

$$\int |T_\mu f|^2 \nu dx \le C\int |f|^2 \mu dx\,, \tag{2}$$

where $T_u = TM_u$. Therefore, in this paper, we mainly discuss the Sawyer type estimate for the dyadic paraproduct of the form (2). For the dyadic paraproduct π_b, let $\pi_b^\mu = \pi_b M_\mu$ i.e.

$$\pi_b^\mu(f) := \pi_b M_\mu(f) = \sum_{I\in\mathcal{D}} \langle f\mu\rangle_I \langle b, h_I\rangle h_I\,.$$

Formally adjoint of π_b^μ can be defined by

$$\pi_b^{\nu*}(f) := \pi_b^* M_\nu(f) = \sum_{I\in\mathcal{D}} \langle b, h_I\rangle\langle f\nu, h_I\rangle \frac{\mathbb{1}_I}{|I|}\,.$$

Then we now state the main theorem of this paper as follows. We will prove Theorem 2.1 in the next section.

Theorem 2.1. *Let π_b be the dyadic paraproduct and $\pi_b^\mu = \pi_b M_\mu$. Then π_b^μ is a bounded operator from $L^2(\mu)$ into $L^2(\nu)$ if and only if π_b^μ and its formal adjoint $\pi_b^{\nu*}$ are uniformly bounded on characteristic functions of intervals, i.e., for all intervals $I \in \mathcal{D}$ there exists a constant such that*

$$\|\pi_b^\mu \mathbb{1}_I\|_{L^2(\nu)} \le C\|\mathbb{1}_I\|_{L^2(u)} = C\sqrt{u(I)}\,, \tag{3}$$

$$\|\pi_b^{\nu*} \mathbb{1}_I\|_{L^2(u)} \le C\|\mathbb{1}_I\|_{L^2(\nu)} = C\sqrt{\nu(I)}\,. \tag{4}$$

Let us denote $\langle f\rangle_{I,\mu} = \mu(I)^{-1}\int_I f\mu dx$ the weighted average of f over I and $\mathbb{E}_k^\mu$ the averaging operator in $L^2(\mu)$ over dyadic intervals of the length of the side 2^k, namely $\mathbb{E}_k^\mu f(x) = \langle f\rangle_{I,\mu}\mathbb{1}_I(x)$ where I is a dyadic interval of size 2^k containing x. Let us also define $\Delta_k^\mu = \mathbb{E}_{k-1}^\mu - \mathbb{E}_k^\mu$. If I is a dyadic interval of size 2^k, we denote by $\mathbb{E}_I^\mu f$ and $\Delta_I^\mu u$ the restriction of $\mathbb{E}_k^\mu f$ and $\Delta_k^m u$ to I, respectively. It is clear that for any $f \in L^2(u)$ the functions $\Delta_I^\mu f$, $I \in \mathcal{D}$, are orthogonal to each other, and that for any fixed n we have the orthogonal decomposition

$$f = \sum_{I\in\mathcal{D},|I|\le 2^n} \Delta_I^\mu f + \sum_{I\in\mathcal{D},|I|=2^n} \mathbb{E}_I^\mu f\,,$$

$$\|f\|^2_{L^2(u)} = \sum_{I\in\mathcal{D},|I|\le 2^n} \|\Delta_I^\mu f\|^2_{L^2(u)} + \sum_{I\in\mathcal{D},|I|=2^n} \|\mathbb{E}_I^\mu f\|^2_{L^2(u)}\,. \tag{5}$$

3. Auxiliary paraproduct and useful lemma

Let us introduce the auxiliary paraproduct Π_b^μ, which acts formally from $L^2(\mu)$ into $L^2(\nu)$ by

$$\Pi_b^\mu f := \sum_{I\in\mathcal{D}} \mathbb{E}_I^\mu f \sum_{J\in\mathcal{D}(I),|I|=|J|} \Delta_J^\nu \pi_b \mathbb{1}_I = \sum_{I\in\mathcal{D}} \mathbb{E}_I^\mu f\, \Delta_I^\nu \pi_b \mathbb{1}_I$$

to use in the proof of our main theorem. The auxiliary paraproduct Π_b^ν is also defined similarly,

$$\Pi_b^\nu f := \sum_{I\in\mathcal{D}} \mathbb{E}_I^\nu f\, \Delta_I^\mu \pi_b^* \mathbb{1}_I\,.$$

We now introduce a useful lemma that will be used to see that auxiliary paraproducts Π_b^μ are bounded. The following theorem was first stated in [7].

Theorem 3.1 (Weighted Carleson Embedding Theorem)**.** *Let α_I be a nonnegative sequence such that for all dyadic intervals J,*

$$\sum_{I\in\mathcal{D}(J)} \alpha_I \le \mu(J) \tag{6}$$

Then for all $f\in L^2(\nu)$,

$$\sum_{I\in\mathcal{D}} \alpha_I \langle f\rangle^2_{I,u} \le 4\|f\|^2_{L^2(\mu)}\,.$$

We close this section by observing why Π_b^μ is bounded. Since Δ_I^ν are mutually orthogonal, the $L^2(\nu)$ norm of $\Pi_b^\mu f$ is

$$\|\Pi_b^\mu f\|^2_{L^2(\nu)} = \sum_{I\in\mathcal{D}} \|\mathbb{E}_I^\mu f\,\Delta_I^\nu \pi_b(\mathbb{1}_I)\|^2_{L^2(\nu)} = \sum_{I\in\mathcal{D}} |\langle f\rangle_{I,u}|^2 \|\Delta_I^\nu \pi_b(\mathbb{1}_I)\|^2_{L^2(\nu)}\,.$$

So by Theorem 3.1, we need to show that

$$\sum_{I\in\mathcal{D}(J)} \|\Delta_I^\nu \pi_b(\mathbb{1}_I)\|^2_{L^2(\nu)} \le Cu(J)\,.$$

Using the condition (3) in Theorem 2.1 we get the following estimate:

$$\sum_{I\in\mathcal{D}(J)} \|\Delta_I^\nu \pi_b(\mathbb{1}_I)\|^2_{L^2(\nu)} \le \sum_{I\in\mathcal{D}(J)} \|\Delta_I^\nu \pi_b(\mathbb{1}_J)\|^2_{L^2(\nu)} \le \|\mathbb{1}_J \pi_b(\mathbb{1}_J)\|^2_{L^2(\nu)} \le Cu(J)\,.$$

Thus the sequence $\alpha_I = \|\Delta_I^\nu \pi_b(\mathbb{1}_I)\|^2_{L^2(\nu)}$, $I\in\mathcal{D}$ satisfies the embedding condition (6) and this implies the desired estimate for the boundedness of Π_b^μ.

We need the following lemma which describes the basic properties of Π_b^μ with respect to the weighted Haar systems in $L^2(\mu)$ and $L^2(\nu)$.

Lemma 3.2. *Let I, J be dyadic intervals, then the following holds for the auxiliary paraproduct Π_b^μ.*

(1) *If $|J| \geq |I|$ then $\langle \Pi_b^\mu h_I^\mu, h_J^\nu \rangle_\nu = 0$ for all weighted Haar functions h_I^μ and h_J^ν.*
(2) *If $J \not\subset I$, then $\langle \Pi_b^\mu h_I^\mu, h_J^\nu \rangle_\nu = 0$ for all weighted Haar functions h_I^μ and h_J^ν.*
(3) *If $|J| < |I|$ then for all weighted Haar functions h_I^μ and h_J^ν,*

$$\langle \Pi_b^\mu h_I^\mu, h_J^\nu \rangle_\nu = \langle \pi_b h_I^\mu, h_J^\nu \rangle_\nu \,.$$

Proof. Since h_J^ν is orthogonal to the ranges of all projections Δ_K^ν except Δ_J^ν, we get

$$\begin{aligned} \langle \Pi_b^\mu h_I^\mu, h_J^\nu \rangle_\nu &= \left\langle \sum_{K \in \mathcal{D}} \mathbb{E}_K^\mu h_I^\mu \, \Delta_K^\nu \pi_b \mathbb{1}_K, h_J^\nu \right\rangle_J \\ &= \langle \mathbb{E}_J^\mu h_I^\mu \, \Delta_J^\nu \pi_b \mathbb{1}_J, h_J^\nu \rangle_\nu \\ &= \mathbb{E}_J^\mu h_I^\mu \, \langle \pi_b \mathbb{1}_J, h_J^\nu \rangle_\nu \,. \end{aligned} \tag{7}$$

We can also easily see $\mathbb{E}_J^\mu h_I^\mu = 0$ for $J \not\subset I$ as follows.

$$\begin{aligned} \mathbb{E}_J^\mu h_I^\mu &= \left[\frac{1}{u(J)} \int_J \frac{1}{\sqrt{u(I)}} \left(\sqrt{\frac{u(I_-)}{u(I_+)}} \mathbb{1}_{I_+} - \sqrt{\frac{u(I_+)}{u(I_-)}} \mathbb{1}_{I_-} \right) u dx \right] \mathbb{1}_J \\ &= \begin{cases} \sqrt{\frac{u(I_-)}{u(I)u(I_+)}} \mathbb{1}_J & (J \subset I_+) \\ -\sqrt{\frac{u(I_+)}{u(I)u(I_-)}} \mathbb{1}_J & (J \subset I_-) \\ \frac{1}{u(J)} \left(\sqrt{\frac{u(I_-)u(I_+)}{u(I)}} - \sqrt{\frac{u(I_+)u(I_-)}{u(I)}} \right) \mathbb{1}_I = 0 & (J \not\subset I \text{ or } J = I) \,. \end{cases} \end{aligned}$$

Therefore, we have

$$\langle \Pi_b^\mu h_I^\mu, h_J^\nu \rangle_\nu = A \, \langle \pi_b \mathbb{1}_J, h_J^\nu \rangle_\nu \,,$$

where A is the value of $E_J^\mu h_I^\mu$, when $J \subsetneq I$, which is not necessarily zero. For $J \subsetneq I$,

$$\begin{aligned} \langle \pi_b h_I^\mu, h_J^\nu \rangle_\nu &= \left\langle \sum_{K \in \mathcal{D}} \langle h_I^\mu \rangle_K \langle b, h_K \rangle h_K, h_J^\nu \right\rangle_\nu \\ &= \sum_{K \in \mathcal{D}} \langle h_I^\mu \rangle_K \langle b, h_K \rangle \langle h_K, h_J^\nu \rangle_\nu \,. \end{aligned}$$

Note also that $\langle h_K, h_J^\nu \rangle_\nu$ can only be non-zero if $K \subseteq J$. For $K \subseteq J \subsetneq I$,

$$\langle h_I^\mu \rangle_K = \frac{1}{|K|} \int_K \frac{1}{\sqrt{u(I)}} \left(\sqrt{\frac{u(I_-)}{u(I_+)}} \mathbb{1}_{I_+} - \sqrt{\frac{u(I_+)}{u(I_-)}} \mathbb{1}_{I_-} \right) dx = A \langle \mathbb{1}_J \rangle_K \,,$$

Therefore,

$$\begin{aligned}
\langle \pi_b h_I^\mu, h_J^\nu\rangle_\nu &= \sum_{K\in\mathcal{D}(J)} \langle h_I^\mu\rangle_K \langle b, h_K\rangle \langle h_K, h_J^\nu\rangle_\nu \\
&= \sum_{K\in\mathcal{D}(J)} A\langle \mathbb{1}_J\rangle_K \langle b, h_K\rangle \langle h_K, h_J^\nu\rangle_\nu \\
&= A\langle \pi_b \mathbb{1}_J, h_J^\nu\rangle_\nu \,.
\end{aligned}$$

□

4. Proof of the Main Result

In order to see the main result, by duality argument, we need to estimate

$$\left|\langle \pi_b^\mu f, g\rangle_\nu\right| \le C\|f\|_{L^2(\mu)}\|g\|_{L^2(\nu)} \tag{8}$$

for $f \in L^2(\mu)$ and $g \in L^2(\nu)$. It is also sufficient to prove the estimate on a dense set of compactly supported functions. Every compact subset of $\mathbb{R}$ is contained in the dyadic interval J with size at most 2^j. We now decompose f, g using the following orthogonal decomposition,

$$f = \mathbb{E}_J^\mu f + \sum_{I\in\mathcal{D}(J)} \Delta_I^\mu f, \quad g = \mathbb{E}_J^\nu g + \sum_{I\in\mathcal{D}(J)} \Delta_I^\nu g\,.$$

Using the decomposition we split the left-hand-side of the inequality (8) as follows.

$$\begin{aligned}
\langle \pi_b^\mu f, g\rangle_\nu &= \left\langle \pi_b^\mu \left(\mathbb{E}_J^\mu f + \sum_{I\in\mathcal{D}(J)} \Delta_I^\mu f\right), \mathbb{E}_J^\nu g + \sum_{I\in\mathcal{D}(J)} \Delta_I^\nu g\right\rangle_\nu \\
&= \langle \pi_b^\mu \left(\mathbb{E}_J^\mu f\right), \mathbb{E}_J^\nu g\rangle_\nu + \left\langle \pi_b^\mu \left(\sum_{I\in\mathcal{D}(J)} \Delta_I^\mu f\right), \mathbb{E}_J^\nu g\right\rangle_\nu \\
&\quad + \left\langle \pi_b^\mu \left(\mathbb{E}_J^\mu f\right), \sum_{I\in\mathcal{D}(J)} \Delta_I^\nu g\right\rangle_\nu + \left\langle \pi_b^\mu \left(\sum_{I\in\mathcal{D}(J)} \Delta_I^\mu f\right), \sum_{I\in\mathcal{D}(J)} \Delta_I^\nu g\right\rangle_\nu .
\end{aligned}$$

$$\begin{aligned}
\langle \pi_b^\mu \left(\mathbb{E}_J^\mu f\right), \mathbb{E}_J^\nu g\rangle_\nu &= \langle f\rangle_{J,\mu}\langle g\rangle_{J,\nu}\langle \pi_b^\mu(\mathbb{1}_J), \mathbb{1}_J\rangle_\nu \\
&= \frac{1}{\mu(J)^{1/2}}\left(\int_J |f|^2\mu\right)^{1/2}\left(\int_J |g|^2\nu\right)^{1/2}\left(\int_J |\pi_b^\mu(\mathbb{1}_J)|^2\nu\right)^{1/2} \\
&\le \|f\|_{L^2(\mu)}\|g\|_{L^2(\nu)}\,.
\end{aligned}$$

For fixed J, by the assumption of Theorem 2.1

$$\begin{aligned}\left\langle \pi_b^\mu\left(\mathbb{E}_J^\mu f\right), \sum_{I\in\mathcal{D}(J)} \Delta_I^\nu g\right\rangle_\nu &\le |\mathbb{E}_J^\mu f| \|\pi_b^\mu \mathbb{1}_J\|_{L^2(\nu)} \|g\|_{L^2(\nu)} \\ &\le \frac{1}{\mu(J)^{1/2}}\left(\int_J |f|^2\mu\right)^{1/2} \mu(J)^{1/2}\|g\|_{L^2(\nu)} \\ &\le \|f\|_{L^2(\mu)}\|g\|_{L^2(\nu)}\,.\end{aligned}$$

The term $\left\langle \pi_b^\mu\left(\sum_{I\in\mathcal{D}(J)}\Delta_I^\mu f\right), \mathbb{E}_J^\nu g\right\rangle_\nu$ can be estimated similarly. We now estimate the remaining term $\left\langle \pi_b^\mu\left(\sum_{I\in\mathcal{D}(J)}\Delta_I^\mu f\right), \sum_{I\in\mathcal{D}(J)}\Delta_I^\nu g\right\rangle_\nu$. Since $\Delta_I^\mu f$ and $\Delta_I^\nu g$ are μ- and ν-Haar functions on the interval I, we have the following equality by lemma 3.2.

$$\left\langle \pi_b^\mu\left(\sum_{I\in\mathcal{D}(J)}\Delta_I^\mu f\right), \sum_{I\in\mathcal{D}(J)}\Delta_I^\nu g\right\rangle_\nu = \langle \Pi_b^\mu f, g\rangle_\nu + \langle f, \Pi_b^\nu g\rangle_\nu + \sum_{I\in\mathcal{D}(J)} \langle \pi_b^\mu \Delta_I^\mu f, \Delta_I^\nu g\rangle_\nu$$

Since we have already observed that the auxiliary paraproducts Π_b^μ and Π_b^ν are bounded, the last term can be estimated as follows

$$\sum_{I\in\mathcal{D}(J)} \langle \pi_b^\mu \Delta_I^\mu f, \Delta_I^\nu g\rangle_\nu \le \sum_{I\in\mathcal{D}(J)} \|\Delta_I^\mu f\|_{L^2(\mu)}\|\Delta_I^\nu g\|_{L^2(\nu)}\,.$$

Then by the Cauchy-Schwarz inequality and (5) we can get the desired estimate.

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