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GENERALIZED SASAKIAN SPACE FORMS ON W_0-CURVATURE TENSOR

Tuğba Mert* and Mehmet Atçeken

Abstract. In this article, generalized Sasakian space forms are investigated on W_0 -curvature tensor. Characterizations of generalized Sasakian space forms are obtained on W_0-curvature tensor. Special curvature conditions established with the help of Riemann, Ricci, concircular, projective curvature tensors are discussed on W_0-curvature tensor. With the help of these curvature conditions, important characterizations of generalized Sasakian space forms are obtained. In addition, the concepts of W_0-pseudosymmetry and W_0 -Ricci pseudosymmetry are defined and the behavior according to these concepts for the generalized Sasakian space form is examined.

1. Introduction

Let $M(\phi,\xi,\eta,g)$ be the almost contact metric manifold. If there are functions F_1, F_2, F_3 on M such that

$$
\begin{aligned}
R(V_1,V_2)V_3 &= F_1\left[g(V_2,V_3)V_1 - g(V_1,V_3)V_2\right] \\
&+F_2\left[g(V_1,\phi V_3)\phi V_2 - g(V_2,\phi V_3)\phi V_1\right. \\
&\left.+2g(V_1,\phi V_2)\phi V_3\right] + F_3\left[\eta(V_1)\eta(V_3)V_2\right. \\
&-\eta(V_2)\eta(V_3)V_1 + g(V_1,V_3)\eta(V_2)\xi \\
&\left.-g(V_2,V_3)\eta(V_1)\xi\right],
\end{aligned} \tag{1}
$$

$M = M(\phi,\xi,\eta,g)$ is called a generalized Sasakian space form and such a manifold is denoted by $M^{2n+1}(F_1,F_2,F_3)$. Such manifolds were introduced by P. Alegre et al [1]. P. Alegre, D. Blair and A. Carriazo calculated the Riemann curvature tensor of a generalized Sasakian space form. In [5], generalized Sasakian space forms are studied under some conditions related to projective

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*Corresponding author

curvature. In this work, U.C. De and A. Sarkar obtained the necessary and sufficient conditions for generalized Sasakian space forms satisfying $P \cdot S = 0$ and $P \cdot R = 0$. Again, in [6], the same authors studied quasi conformal flat, Ricci symmetric and Ricci semi-symmetric generalized Sasakian space forms. In [9], the curvatures of para-Sasakian manifolds are studied and in this study C. Özgür and M.M. Tiripathi found necessary and sufficient conditions for the curvatures of para-Sasakian manifolds. M. Atçeken studied and classified generalized Sasakian space forms for some curvature conditions related to concircular, Riemann, Ricci and projective curvature tensors in [3]. Again, many authors have worked on generalized Sasakian space forms ([10]-[7]) and have studied the curvature conditions for different manifolds on some special curvature tensors ([8],[12]).

In this article, generalized Sasakian space forms are investigated on W_0 -curvature tensor. Characterizations of generalized Sasakian space forms are obtained on W_0-curvature tensor. Special curvature conditions established with the help of Riemann, Ricci, concircular, projective curvature tensors are discussed on W_0-curvature tensor. With the help of these curvature conditions, important characterizations of generalized Sasakian space forms are obtained. In addition, the concepts of W_0-pseudosymmetry and W_0-Ricci pseudosymmetry are defined and the behavior according to these concepts for the generalized Sasakian space form is examined.

2. Preliminary

Let's take an $(2n+1)-$dimensional differentiable manifold M. If it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$$\phi^2 V_1 = -V_1 + \eta(V_1)\xi \text{ and } \eta(\xi) = 1,$$

then this structure (ϕ, ξ, η) is called an almost contact structure, and the (M, ϕ, ξ, η) is called an almost contact manifold. If there is a metric g that satisfies the condition

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2) \text{ and } g(V_1, \xi) = \eta(V_1),$$

for all $V_1, V_2 \in \chi(M)$, the structure (ϕ, ξ, η, g) is called almost contact metric structure and (M, ϕ, ξ, η, g) is called almost contact metric manifold. On the $(2n+1)$ dimensional manifold M,

$$g(\phi V_1, V_2) = -g(V_1, \phi V_2)$$

for all $V_1, V_2 \in \chi(M)$, that is, ϕ is an anti-symmetric tensor field according to the metric g. The transformation Φ defined as

$$\Phi(V_1, V_2) = g(V_1, \phi V_2)$$

for all $V_1, V_2 \in \chi(M)$, is called the fundamental 2-form of the almost contact metric structure (ϕ, ξ, η, g), where

$$\eta \wedge \Phi^n \neq 0.$$

Sasakian space forms are very important for contact metric geometry. The curvature tensor for the Sasakian space form is defined as

$$R(V_1, V_2) V_3 = \left(\tfrac{c+3}{4}\right) [g(V_2, V_3) V_1 - g(V_1, V_3) V_2]$$

$$+ \left(\tfrac{c-1}{4}\right) [g(V_1, \phi V_3) \phi V_2 - g(V_2, \phi V_3) \phi V_1$$

$$+2g(V_1, \phi V_2) \phi V_3 + \eta(V_1) \eta(V_3) V_2$$

$$-\eta(V_2) \eta(V_3) V_1 + g(V_1, V_3) \eta(V_2) \xi$$

$$-g(V_2, V_3) \eta(V_1) \xi].$$

If we choose $V_1 = \xi, V_2 = \xi$ and $V_3 = \xi$ respectively in (1), we obtain

$$R(\xi, V_2) V_3 = (F_1 - F_3) [g(V_2, V_3) \xi - \eta(V_3) V_2], \tag{2}$$

$$R(V_1, \xi) V_3 = (F_1 - F_3) [-g(V_1, V_3) \xi + \eta(V_3) V_1], \tag{3}$$

$$R(V_1, V_2) \xi = (F_1 - F_3) [\eta(V_2) V_1 - \eta(V_1) V_2]. \tag{4}$$

Also, if we take inner product of both sides of (1) by $\xi \in \chi\left(M^{2n+1}\right)$, we get

$$\eta(R(V_1, V_2) V_3) = (F_1 - F_3) [g(V_2, V_3) \eta(V_1) - g(V_1, V_3) \eta(V_2)].$$

Lemma 2.1. *For a $(2n+1)$ −dimensional generalized Sasakian space form $M^{2n+1}(F_1, F_2, F_3)$ the following equations are provided* [5].

$$S(V_1, V_2) = [2nF_1 + 3F_2 - F_3] g(V_1, V_2) - [3F_2 + (2n-1) F_3] \eta(V_1) \eta(V_2),$$

$$S(V_1, \xi) = 2n(F_1 - F_3) \eta(V_1), \tag{5}$$

$$QV_1 = [2nF_1 + 3F_2 - F_3] V_1 - [3F_2 + (2n-1) F_3] \eta(V_1) \xi,$$

$$Q\xi = 2n(F_1 - F_3) \xi,$$
$$r = 2n(2n+1) F_1 + 6nF_2 - 4nF_3,$$

for all $V_1, V_2 \in \chi\left(M^{2n+1}\right)$, where Q,S and r are the Ricci operator, Ricci tensor and scalar curvature of manifold $M^{2n+1}(F_1, F_2, F_3)$, respectively.

M. Tripathi and P. Gunam described a $\tau-$curvature tensors of the $(1,3)$ type in an n-dimensional semi-Riemann manifold (M,g) [11]. One of these tensors is defined as follows.

Definition 2.2. *Let M be a $(2n+1)-$dimensional semi-Riemannian manifold. The curvature tensor defined as*

$$T(V_1,V_2)V_3 = R(V_1,V_2)V_3 - \frac{1}{2n}\left[S(V_2,V_3)V_1 - g(V_1,V_3)QV_2\right] \tag{6}$$

is called the W_0-curvature tensor.

For the $(2n+1)-$dimensional generalized Sasakian space form, if we choose $V_1=\xi, V_2=\xi, V_3=\xi$ respectively in (6), then we get

$$T(\xi,V_2)V_3 = \frac{(1-2n)\digamma_3 - 3\digamma_2}{2n}\left[g(V_2,V_3)\xi - \eta(V_3)V_2\right], \tag{7}$$

$$T(V_1,\xi)V_3 = 0, \tag{8}$$

$$T(V_1,V_2)\xi = \frac{(1-2n)\digamma_3 - 3\digamma_2}{2n}\left[-\eta(V_1)V_2 + \eta(V_1)\eta(V_2)\xi\right]. \tag{9}$$

Definition 2.3. *Let M be a paracontact manifold. If its Ricci tensor S of type $(0,2)$ is of the form*

$$S(V_1,V_2) = ag(V_1,V_2) + b\eta(V_1)\eta(V_2),$$

where a,b are smooth functions on M, then M is called $\eta-$Einstein manifold. Also, if $b=0$, then the manifold is called Einstein.

Definition 2.4. *Let (M,g) be a semi-Riemannian manifold and the two-dimensional subspace Π of the tangent space $T_p(M)$. If $K(V_{1p},V_{2p})$ is constant for each $p\in M$ and $V_{1p},V_{2p}\in T_p(M)$, then M is called a real space form, where $K(V_{1p},V_{2p})$ is the section curvature of the plane Π.*

3. Generalized Sasakian Space Forms On W_0- Curvature Tensor

In this section, the characterization of generalized Sasakian space forms under special curvature conditions created by W_0-curvature tensor with Riemann, Ricci, concircular and projective curvature tensors will be given. Let us state and prove the following theorems.

Theorem 3.1. *Let $M^{2n+1}(\digamma_1,\digamma_2,\digamma_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If $M^{2n+1}(\digamma_1,\digamma_2,\digamma_3)$ satisfies the curvature condition $T(V_1,V_2)R=0$, then $M^{2n+1}(\digamma_1,\digamma_2,\digamma_3)$ is either the real space form or $3\digamma_2=(1-2n)\digamma_3$.*

Proof. Let's assume that

$$(T(V_1, V_2) R)(V_3, V_5, V_4) = 0$$

for every $V_1, V_2, V_3, V_4, V_5 \in \chi(M^{2n+1})$. So, we can write

$$\begin{aligned} & T(V_1, V_2) R(V_3, V_5) V_4 - R(T(V_1, V_2) V_3, V_5) V_4 \\ & -R(V_3, T(V_1, V_2) V_5) V_4 - R(V_3, V_5) T(V_1, V_2) V_4 = 0. \end{aligned} \tag{10}$$

If we choose $V_1 = \xi$ in (10) and make use of (7), we get

$$\begin{aligned} & \tfrac{(1-2n)F_3 - 3F_2}{2n} \{g(V_2, R(V_3, V_5) V_4) \xi \\ & -\eta(R(V_3, V_5) V_4) V_2 - g(V_2, V_3) R(\xi, V_5) V_4 \\ & +\eta(V_3) R(V_2, V_5) V_4 - g(V_2, V_5) R(V_3, \xi) V_4 \\ & +\eta(V_5) R(V_3, V_2) V_4 - g(V_2, V_4) R(V_3, V_5) \xi \\ & +\eta(V_4) R(V_3, V_5) V_2\} = 0. \end{aligned} \tag{11}$$

If we use $(2), (3), (4)$ in (11), we obtain

$$\begin{aligned} & \tfrac{(1-2n)F_3 - 3F_2}{2n} \{g(V_2, R(V_3, V_5) V_4) \xi \\ & -\eta(R(V_3, V_5) V_4) V_2 + \eta(V_3) R(V_2, V_5) V_4 \\ & +\eta(V_5) R(V_3, V_2) V_4 + \eta(V_4) R(V_3, V_5) V_2 \\ & -(F_1 - F_3)[g(V_2, V_3) g(V_5, V_4) \xi \\ & -g(V_2, V_3) \eta(V_4) V_5 - g(V_2, V_5) g(V_3, V_4) \xi \\ & +g(V_2, V_5) \eta(V_4) V_3 + g(V_2, V_4) \eta(V_5) V_3 \\ & -g(V_2, V_4) \eta(V_3) V_5]\} = 0. \end{aligned} \tag{12}$$

If we choose $V_3 = \xi$ in (12) and make the necessary adjustments using (2), we get

$$\begin{aligned} & \tfrac{(1-2n)F_3 - 3F_2}{2n} \{R(V_2, V_5) V_4 - (F_1 - F_3) \\ & [g(V_5, V_4) V_2 - g(V_2, V_4) V_5]\} = 0. \end{aligned} \tag{13}$$

Therefore, the proof of the theorem is completed. □

Theorem 3.2. *Let $M^{2n+1}(F_1, F_2, F_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If $M^{2n+1}(F_1, F_2, F_3)$ satisfies the curvature condition $T(V_1, V_2)T = 0$, then $M^{2n+1}(F_1, F_2, F_3)$ is either an $\eta-$Einstein manifold provided $F_3 \neq 2nF_1 + 3F_2$ and $3F_2 \neq (1-2n)F_3$ or $3F_2 = (1-2n)F_3$.*

Proof. Let's assume that

$$(T(V_1, V_2)T)(V_3, V_5, V_4) = 0$$

for every $V_1, V_2, V_3, V_4, V_5 \in \chi\left(M^{2n+1}\right)$. So, we can write

$$\begin{aligned} &T(V_1, V_2)T(V_3, V_5)V_4 - T(T(V_1, V_2)V_3, V_5)V_4 \\ &-T(V_3, T(V_1, V_2)V_5)V_4 - T(V_3, V_5)T(V_1, V_2)V_4 = 0. \end{aligned} \tag{14}$$

If we choose $V_1 = \xi$ in (14) and make use of (7), we get

$$\begin{aligned} &\tfrac{(1-2n)F_3 - 3F_2}{2n}\{g(V_2, T(V_3, V_5)V_4)\xi \\ &-\eta(T(V_3, V_5)V_4)V_2 - g(V_2, V_3)T(\xi, V_5)V_4 \\ &+\eta(V_3)T(V_2, V_5)V_4 - g(V_2, V_5)T(V_3, \xi)V_4 \\ &+\eta(V_5)T(V_3, V_2)V_4 - g(V_2, V_4)T(V_3, V_5)\xi \\ &+\eta(V_4)T(V_3, V_5)V_2\} = 0. \end{aligned} \tag{15}$$

If we use $(7), (8), (9)$ in (15), we obtain

$$\begin{aligned} &\tfrac{(1-2n)F_3 - 3F_2}{2n}\{g(V_2, T(V_3, V_5)V_4)\xi \\ &-\eta(T(V_3, V_5)V_4)V_2 + \eta(V_3)T(V_2, V_5)V_4 \\ &+\eta(V_5)T(V_3, V_2)V_4 + \eta(V_4)T(V_3, V_5)V_2 \\ &-\tfrac{(1-2n)F_3 - 3F_2}{2n}[g(V_2, V_3)g(V_5, V_4)\xi \\ &-g(V_2, V_3)\eta(V_4)V_5 - g(V_2, V_4)\eta(V_3)V_5 \\ &+g(V_2, V_4)\eta(V_3)\eta(V_5)\xi]\} = 0. \end{aligned} \tag{16}$$

If we choose $V_3 = \xi$ in (16) and make the necessary adjustments using (7), we get

$$\begin{aligned} &\tfrac{(1-2n)F_3 - 3F_2}{2n}\Big\{T(V_2, V_5)V_4 - \tfrac{(1-2n)F_3 - 3F_2}{2n} \\ &[g(V_5, V_4)V_2 - g(V_2, V_4)V_5]\} = 0. \end{aligned} \tag{17}$$

If (6) is written in (17), we obtain

$$\frac{(1-2n)F_3-3F_2}{2n}\left\{R(V_2,V_5)V_4-\frac{1}{2n}S(V_5,V_4)V_2\right.$$

$$(18)\qquad +\frac{1}{2n}g(V_2,V_4)QV_5-\frac{(1-2n)F_3-3F_2}{2n}\left[g(V_5,V_4)V_2\right.$$

$$\left.\left.-g(V_2,V_4)V_5\right]\right\}=0.$$

If we choose $V_4=\xi$ in (18) and make use of (4) and (5), we get

$$\frac{(1-2n)F_3-3F_2}{2n}\left\{-\frac{(1-2n)F_3-3F_2}{2n}\eta(V_5)V_2\right.$$

$$\left.+\frac{1}{2n}\eta(V_2)QV_5-\frac{2nF_1+3F_2-F_3}{2n}\eta(V_2)V_5\right\}=0.$$

In the last equation, if we choose $V_2=\xi$, and then we take inner product both sides of the equation by $V_4\in\chi(M)$, we have

$$\frac{(1-2n)F_3-3F_2}{2n}\left\{\frac{(1-2n)F_3-3F_2}{2n}\eta(V_5)\eta(V_4)\right.$$

$$\left.+\frac{1}{2n}S(V_5,V_4)-\frac{2nF_1+3F_2-F_3}{2n}g(V_5,V_4)\right\}=0.$$

This completes the proof. □

Corollary 3.3. *Let $M^{2n+1}(F_1,F_2,F_3)$ be the $(2n+1)-$dimensional generalized Sasakian space form. If $M^{2n+1}(F_1,F_2,F_3)$ satisfies the curvature condition*
$T(V_1,V_2)T=0$, then $M^{2n+1}(F_1,F_2,F_3)$ is an Einstein manifold if and only if $2nF_1+3F_2\neq F_3$ and $3F_2=(1-2n)F_3$ relations are provided.

Let us now prepare to examine the curvature condition associated with the concircular curvature tensor.

Definition 3.4. *Let M be a $(2n+1)-$dimensional Riemannian manifold. The curvature tensor defined as*

$$(19)\quad \tilde{Z}(V_1,V_2)V_3=R(V_1,V_2)V_3-\frac{r}{2n(2n+1)}\left[g(V_2,V_3)V_1-g(V_1,V_3)V_2\right]$$

is called the concircular curvature tensor.

For the $(2n+1)-$dimensional generalized Saskian space form, if we choose $V_1=\xi,V_2=\xi,V_3=\xi$ respectively in (19), then we get

$$(20)\qquad \tilde{Z}(\xi,V_2)V_3=\left[(F_1-F_3)-\frac{r}{2n(2n+1)}\right]\left[g(V_2,V_3)\xi-\eta(V_3)V_2\right],$$

$$(21)\qquad \tilde{Z}(V_1,\xi)V_3=\left[(F_1-F_3)-\frac{r}{2n(2n+1)}\right]\left[-g(V_1,V_3)\xi+\eta(V_3)V_1\right],$$

$$(22)\qquad \tilde{Z}(V_1,V_2)\xi=\left[(F_1-F_3)-\frac{r}{2n(2n+1)}\right]\left[\eta(V_2)V_1-\eta(V_1)V_2\right].$$

Theorem 3.5. *Let $M^{2n+1}(\digamma_1,\digamma_2,\digamma_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If $M^{2n+1}(\digamma_1,\digamma_2,\digamma_3)$ satisfies the curvature condition $T(V_1,V_2)\tilde{Z}=0$, then $M^{2n+1}(\digamma_1,\digamma_2,\digamma_3)$ is either the real space form or $3\digamma_2=(1-2n)\digamma_3$.*

Proof. Let's assume that

$$\left(T(V_1,V_2)\tilde{Z}\right)(V_3,V_5,V_4)=0$$

for every $V_1,V_2,V_3,V_4,V_5\in\chi\left(M^{2n+1}\right)$. So, we can write

$$\begin{aligned}&T(V_1,V_2)\tilde{Z}(V_3,V_5)V_4-\tilde{Z}(T(V_1,V_2)V_3,V_5)V_4\\&-\tilde{Z}(V_3,T(V_1,V_2)V_5)V_4-\tilde{Z}(V_3,V_5)T(V_1,V_2)V_4=0.\end{aligned}\tag{23}$$

If we choose $V_1=\xi$ in (23) and make use of (7), we get

$$\begin{aligned}&\tfrac{(1-2n)\digamma_3-3\digamma_2}{2n}\Big\{g\left(V_2,\tilde{Z}(V_3,V_5)V_4\right)\xi\\&--\eta\left(\tilde{Z}(V_3,V_5)V_4\right)V_2g(V_2,V_3)\tilde{Z}(\xi,V_5)V_4\\&+\eta(V_3)\tilde{Z}(V_2,V_5)V_4-g(V_2,V_5)\tilde{Z}(V_3,\xi)V_4\\&+\eta(V_5)\tilde{Z}(V_3,V_2)V_4-g(V_2,V_4)\tilde{Z}(V_3,V_5)\xi\\&+\eta(V_4)\tilde{Z}(V_3,V_5)V_2\Big\}=0.\end{aligned}\tag{24}$$

If we use $(20),(21),(22)$ in (24), we obtain

$$\begin{aligned}&\tfrac{(1-2n)\digamma_3-3\digamma_2}{2n}\Big\{g\left(V_2,\tilde{Z}(V_3,V_5)V_4\right)\xi\\&-\eta\left(\tilde{Z}(V_3,V_5)V_4\right)V_2+\eta(V_3)\tilde{Z}(V_2,V_5)V_4\\&+\eta(V_5)\tilde{Z}(V_3,V_2)V_4+\eta(V_4)\tilde{Z}(V_3,V_5)V_2\\&-A\left[g(V_2,V_3)g(V_5,V_4)\xi\right.\\&-g(V_2,V_3)\eta(V_4)V_5-g(V_2,V_5)g(V_3,V_4)\xi\\&+g(V_2,V_5)\eta(V_4)V_3+g(V_2,V_4)\eta(V_5)V_3\\&\left.-g(V_2,V_4)\eta(V_3)V_5\right]\Big\}=0,\end{aligned}\tag{25}$$

where $A = \left[(F_1 - F_3) - \frac{r}{2n(2n+1)} \right]$. If we choose $V_3 = \xi$ in (25) and make the necessary adjustments using (20), we get

$$\frac{(1-2n)F_3-3F_2}{2n} \Big\{ \tilde{Z}(V_2, V_5) V_4 - A\left[g(V_5, V_4) V_2 - g(V_2, V_4) V_5 \right] \Big\} = 0. \tag{26}$$

If we substitute the (19) in (26) and we make the necessary arrangements, we obtain

$$\frac{(1-2n)F_3-3F_2}{2n} \{ R(V_2, V_5) V_4 - (F_1 - F_3) [g(V_5, V_4) V_2 - g(V_2, V_4) V_5] \} = 0.$$

This completes the proof. □

Now let's characterize the $M^{2n+1}(F_1, F_2, F_3)$ with the help of the special curvature condition established between the W_0-curvature tensor and the Ricci curvature tensor for the generalized Sasakian space form.

Theorem 3.6. *Let $M^{2n+1}(F_1, F_2, F_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If $M^{2n+1}(F_1, F_2, F_3)$ satisfies the curvature condition $T(V_1, V_2) Q = 0$, then $M^{2n+1}(F_1, F_2, F_3)$ is either an Einstein manifold or $3F_2 = (1-2n) F_3$.*

Proof. Let's assume that

$$T(V_1, V_2) Q = 0.$$

From here it is clear that

$$(T(V_1, V_2) S)(V_3, V_5) = 0,$$

for every $V_1, V_2, V_3, V_5 \in \chi\left(M^{2n+1}\right)$. So, we can write

$$S(T(V_1, V_2) V_3, V_5) + S(V_3, T(V_1, V_2) V_5) = 0. \tag{27}$$

If we choose $V_1 = \xi$ in (27) and make use of (7), we get

$$\frac{(1-2n)F_3-3F_2}{2n} \{ g(V_2, V_3) S(\xi, V_5) - \eta(V_3) S(V_2, V_5) + g(V_2, V_5) S(\xi, V_3) - \eta(V_5) S(V_3, V_2) \} = 0. \tag{28}$$

If we choose $V_3 = \xi$ in (28) and make use of (5), we have

$$\frac{(1-2n)F_3 - 3F_2}{2n} \{ -S(V_2, V_5) + 2n(F_1 - F_3) g(V_2, V_5) \} = 0.$$

This completes the proof. □

Let us now prepare to examine the curvature condition associated with the projective curvature tensor.

Definition 3.7. *Let M be a $(2n+1)-$dimensional Riemannian manifold. The curvature tensor defined as*

$$P(V_1,V_2)V_3 = R(V_1,V_2)V_3 - \frac{1}{2n}\left[S(V_2,V_3)V_1 - S(V_1,V_3)V_2\right] \tag{29}$$

is called the projective curvature tensor.

For the $(2n+1)-$dimensional generalized Saskian space form, if we choose $V_1=\xi, V_2=\xi, V_3=\xi$ respectively in (29), then we get

$$P(\xi,V_2)V_3 = \frac{3F_2+(2n-1)F_3}{2n}\left[-g(V_2,V_3)\xi + \eta(V_2)\eta(V_3)\xi\right], \tag{30}$$

$$P(V_1,\xi)V_3 = \frac{3F_2+(2n-1)F_3}{2n}\left[g(V_1,V_3)\xi - \eta(V_1)\eta(V_3)\xi\right], \tag{31}$$

$$P(V_1,V_2)\xi = 0. \tag{32}$$

Theorem 3.8. *Let $M^{2n+1}(F_1,F_2,F_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If $M^{2n+1}(F_1,F_2,F_3)$ satisfies the curvature condition $T(V_1,V_2)P=0$, then $M^{2n+1}(F_1,F_2,F_3)$ provides the relation $3F_2=(1-2n)F_3$.*

Proof. Let's assume that

$$(T(V_1,V_2)P)(V_3,V_5,V_4)=0$$

for every $V_1,V_2,V_3,V_4,V_5 \in \chi\left(M^{2n+1}\right)$. So, we can write

$$\begin{aligned} &T(V_1,V_2)P(V_3,V_5)V_4 - P(T(V_1,V_2)V_3,V_5)V_4 \\ &-P(V_3,T(V_1,V_2)V_5)V_4 - P(V_3,V_5)T(V_1,V_2)V_4 = 0. \end{aligned} \tag{33}$$

If we choose $V_1=\xi$ in (33) and make use of (7), we get

$$\begin{aligned} &\tfrac{(1-2n)F_3-3F_2}{2n}\{g(V_2,P(V_3,V_5)V_4)\xi \\ &-\eta(P(V_3,V_5)V_4)V_2 - g(V_2,V_3)P(\xi,V_5)V_4 \\ &+\eta(V_3)P(V_2,V_5)V_4 - g(V_2,V_5)P(V_3,\xi)V_4 \\ &+\eta(V_5)P(V_3,V_2)V_4 - g(V_2,V_4)P(V_3,V_5)\xi \\ &+\eta(V_4)P(V_3,V_5)V_2\} = 0. \end{aligned} \tag{34}$$

If we use $(30), (31), (32)$ in (34), we obtain

$$\begin{aligned}
&\tfrac{(1-2n)F_3-3F_2}{2n}\{g(V_2,P(V_3,V_5)V_4)\xi \\
&-\eta(P(V_3,V_5)V_4)V_2+\eta(V_3)P(V_2,V_5)V_4 \\
&+\eta(V_5)P(V_3,V_2)V_4+\eta(V_4)P(V_3,V_5)V_2 \\
&+B[g(V_2,V_3)g(V_5,V_4)\xi-g(V_2,V_3)\eta(V_4)\eta(V_5)\xi \\
&-g(V_2,V_5)g(V_3,V_4)\xi+g(V_2,V_5)\eta(V_4)\eta(V_3)\xi]\}=0,
\end{aligned} \tag{35}$$

where $B=\frac{3F_2+(2n-1)F_3}{2n}$. If we choose $V_3=\xi$ in (35) and make the necessary adjustments using (30), we get

$$\begin{aligned}
&\tfrac{(1-2n)F_3-3F_2}{2n}\{P(V_2,V_5)V_4+B[g(V_5,V_4)V_2 \\
&-\eta(V_5)\eta(V_4)V_2-g(V_2,V_4)\eta(V_5)\xi \\
&-g(V_2,V_5)\eta(V_4)\xi+2\eta(V_4)\eta(V_5)\eta(V_2)\xi]\}=0.
\end{aligned} \tag{36}$$

If we choose $V_4=\xi$ in (36) and then we take inner product of both sides of the equation by $\xi\in\chi(M^{2n+1})$, we obtain

$$B^2[g(V_2,V_5)-\eta(V_2)\eta(V_5)]=0.$$

This completes the proof. □

4. W_0-Pseudosymmetric And W_0-Ricci Pseudosymmetric Generalized Sasakian Space Form

Let us now investigate the concepts of W_0-pseudosymmetry and W_0-Ricci pseudosymmetry for the generalized Sasakian space form.

Definition 4.1. *Let $M^{2n+1}(F_1,F_2,F_3)$ be a $(2n+1)$ dimensional generalized Sasakian space form and R be the Riemann curvature tensor of $M^{2n+1}(F_1,F_2,F_3)$. If the pair $R.T$ and $Q(g,T)$ are linearly dependent, that is, if a λ_1 function can be found on the set $M_1=\{V_1\in M^{2n+1}|\, g(V_1)\neq T(V_1)\}$ such that*

$$R\cdot T=\lambda_1 Q(g,T),$$

the $M^{2n+1}(F_1,F_2,F_3)$ manifold is called a W_0-pseudosymmetric manifold. Particularly, if $\lambda_1=0$, then this manifold is said to be semi-symmetric.

Let us state and prove the following theorem.

Theorem 4.2. *Let $M^{2n+1}(F_1,F_2,F_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If M^{2n+1} is W_0-pseudosymmetric, then $M^{2n+1}(F_1,F_2,F_3)$ is either an $\eta-$Einstein manifold provided $3F_2\neq 2nF_1+F_3$ and $(1-2n)F_3\neq 3F_2$ or $\lambda_1=F_1-F_3$.*

Proof. Let's assume $M^{2n+1}(\digamma_1, \digamma_2, \digamma_3)$ is W_0-pseudosymmetric. So, we can write

$$(R(V_1,V_2)T)(V_4,V_5,V_3) = \lambda_1 Q(g,T)(V_4,V_5,V_3;V_1,V_2),$$

that is

$$\begin{aligned}
&R(V_1,V_2)T(V_4,V_5)V_3 - T(R(V_1,V_2)V_4,V_5)V_3\\
&-T(V_4,R(V_1,V_2)V_5)V_3 - T(V_4,V_5)R(V_1,V_2)V_3\\
&= -\lambda_1\{g(V_2,V_4)T(V_1,V_5)V_3 - g(V_1,V_4)T(V_2,V_5)V_3\\
&+g(V_2,V_5)T(V_4,V_1)V_3 - g(V_1,V_5)T(V_4,V_2)V_3\\
&+g(V_2,V_3)T(V_4,V_5)V_1 - g(V_1,V_3)T(V_4,V_5)V_2\},
\end{aligned} \tag{37}$$

for every $V_1,V_2,V_3,V_4,V_5 \in \chi(M^{2n+1})$. If we choose $V_1 = \xi$ in (37) and use (2),(7),(8),(9) , we get

$$\begin{aligned}
&(\digamma_1 - \digamma_3)\{g(V_2,T(V_4,V_5)V_3)\xi - \eta(T(V_4,V_5)V_3)V_2\\
&+\eta(V_4)T(V_2,V_5)V_3 + \eta(V_5)T(V_4,V_2)V_3\\
&+\eta(V_3)T(V_4,V_5)V_2 - C[g(V_2,V_4)g(V_5,V_3)\xi\\
&-g(V_2,V_4)\eta(V_3)V_5 - g(V_2,V_3)\eta(V_4)V_5\\
&+g(V_2,V_3)\eta(V_4)\eta(V_5)\xi]\}\\
&= -\lambda_1\{-\eta(V_4)T(V_2,V_5)V_3 - \eta(V_5)T(V_4,V_2)V_3\\
&-\eta(V_3)T(V_4,V_5)V_2 + C[g(V_2,V_4)g(V_5,V_3)\xi\\
&-g(V_2,V_4)\eta(V_3)V_5 - g(V_2,V_3)\eta(V_4)V_5\\
&+g(V_2,V_3)\eta(V_4)\eta(V_5)\xi]\},
\end{aligned} \tag{38}$$

where $C = \frac{(1-2n)\digamma_3 - 3\digamma_2}{2n}$. If we choose $V_4 = \xi$ in (38) and make use of (7) , we obtain

$$\begin{aligned}
&(\digamma_1 - \digamma_3)\{T(V_2,V_5)V_3 - C[g(V_5,V_3)V_2 - g(V_2,V_3)V_5]\}\\
&= -\lambda_1\{-T(V_2,V_5)V_3 + C[g(V_5,V_3)\eta(V_2)\xi\\
&+\eta(V_5)\eta(V_3)V_2 - g(V_2,V_3)V_5 - g(V_5,V_2)\eta(V_3)\xi]\}.
\end{aligned} \tag{39}$$

If we substitute (6) in (39) and choose $V_3 = \xi$, we have

$$
\begin{aligned}
&(F_1 - F_3)\left\{\tfrac{1}{2n}\eta(V_2)QV_5 - (F_1 - F_3)\eta(V_2)V_5\right.\\
&\left.+C[\eta(V_2)V_5 - \eta(V_5)V_2]\right\}\\
&= -\lambda_1\left\{-\tfrac{1}{2n}\eta(V_2)QV_5 + (F_1 - F_3)\eta(V_2)V_5\right.\\
&+C[\eta(V_5)\eta(V_2)\xi + \eta(V_5)V_2 - \eta(V_2)V_5\\
&\left.-g(V_5,V_2)\xi]\right\}.
\end{aligned} \tag{40}
$$

If we choose $V_2 = \xi$ and then we take inner product of both sides of the equation by $V_3 \in \chi\left(M^{2n+1}\right)$, we have

$$
\begin{aligned}
&[\lambda_1 - (F_1 - F_3)]\,[S(V_5,V_3) + (2nF_1 - 3F_2 + F_3)\,g(V_5,V_3)\\
&-((1-2n)F_3 - 3F_2)\,\eta(V_5)\,\eta(V_3)] = 0.
\end{aligned}
$$

This completes the proof. □

Corollary 4.3. *Let $M^{2n+1}(F_1, F_2, F_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If $M^{2n+1}(F_1, F_2, F_3)$ is W_0-pseudosymmetric, then $M^{2n+1}(F_1, F_2, F_3)$ is Einstein manifold if and only if $3F_2 = (1-2n)F_3$ and $3F_2 \neq 2nF_1 + F_3$ relations are provided.*

Definition 4.4. *Let $M^{2n+1}(F_1, F_2, F_3)$ be a $(2n+1)$ dimensional generalized Sasakian space form, R and S be the Riemann and Ricci curvature tensor of $M^{2n+1}(F_1, F_2, F_3)$, respectively. If the pair $R.T$ and $Q(S,T)$ are linearly dependent, that is, if a λ_2 function can be found on the set $M_2 = \left\{V_1 \in M^{2n+1} \middle| S(V_1) \neq T(V_1)\right\}$ such that*

$$R \cdot T = \lambda_2 Q(S,T)$$

the $M^{2n+1}(F_1, F_2, F_3)$ manifold is called a W_0-Ricci pseudosymmetric manifold.

Let us state and prove the following theorem.

Theorem 4.5. *Let $M^{2n+1}(f_1, f_2, f_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If M^{2n+1} is W_0-Ricci pseudosymmetric, then $M^{2n+1}(f_1, f_2, f_3)$ is either an $\eta-$Einstein manifold provided $f_3 \neq 2nf_1 + 3f_2$ and $(1-2n)f_3 \neq 3f_2$ or $\lambda_2 = \frac{1}{2n}$.*

Proof. Let's assume $M^{2n+1}(f_1, f_2, f_3)$ is W_0-Ricci pseudosymmetric. So, we can write

$$(R(V_1,V_2)T)(V_4,V_5,V_3) = \lambda_2 Q(S,T)(V_4,V_5,V_3;V_1,V_2),$$

that is

$$
\begin{aligned}
& R(V_1,V_2)T(V_4,V_5)V_3 - T(R(V_1,V_2)V_4,V_5)V_3 \\
& -T(V_4,R(V_1,V_2)V_5)V_3 - T(V_4,V_5)R(V_1,V_2)V_3 \\
& = -\lambda_2\{S(V_2,V_4)T(V_1,V_5)V_3 - S(V_1,V_4)T(V_2,V_5)V_3 \\
& +S(V_2,V_5)T(V_4,V_1)V_3 - S(V_1,V_5)T(V_4,V_2)V_3 \\
& +S(V_2,V_3)T(V_4,V_5)V_1 - S(V_1,V_3)T(V_4,V_5)V_2\},
\end{aligned} \tag{41}
$$

for every $V_1,V_2,V_3,V_4,V_5 \in \chi\left(M^{2n+1}\right)$. If we choose $V_1=\xi$ in (41) and use (2),(5) , (7),(8),(9) , we get

$$
\begin{aligned}
& (f_1-f_3)\{g(V_2,T(V_4,V_5)V_3)\xi - \eta(T(V_4,V_5)V_3)V_2 \\
& +\eta(V_4)T(V_2,V_5)V_3 + \eta(V_5)T(V_4,V_2)V_3 \\
& +\eta(V_3)T(V_4,V_5)V_2 - C[g(V_2,V_4)g(V_5,V_3)\xi \\
& -g(V_2,V_4)\eta(V_3)V_5 - g(V_2,V_3)\eta(V_4)V_5 \\
& +g(V_2,V_3)\eta(V_4)\eta(V_5)\xi]\} \\
& = -\lambda_2\{-2n(f_1-f_3)\eta(V_4)T(V_2,V_5)V_3 \\
& -2n(f_1-f_3)\eta(V_5)T(V_4,V_2)V_3 \\
& -2n(f_1-f_3)\eta(V_3)T(V_4,V_5)V_2 \\
& +C[S(V_2,V_4)g(V_5,V_3)\xi - S(V_2,V_4)\eta(V_3)V_5 \\
& -S(V_2,V_3)\eta(V_4)V_5 + S(V_2,V_3)\eta(V_4)\eta(V_5)\xi]\},
\end{aligned} \tag{42}
$$

where $C = \frac{(1-2n)f_3-3f_2}{2n}$. If we choose $V_4 = \xi$ in (42) and make use of (5) and (7), we obtain

$$\begin{aligned}
&(f_1 - f_3)\{T(V_2, V_5)V_3 - C[g(V_5, V_3)V_2 - g(V_2, V_3)V_5]\} \\
&= -\lambda_2\{-2n(f_1 - f_3)T(V_2, V_5)V_3 \\
&+2n(f_1 - f_3)C[g(V_5, V_3)\eta(V_2)\xi \\
&+\eta(V_5)\eta(V_3)V_2 - g(V_2, V_3)\eta(V_5)\xi \\
&-g(V_5, V_2)\eta(V_3)\xi] \\
&-C[S(V_2, V_3)V_5 - S(V_2, V_3)\eta(V_5)\xi]\}.
\end{aligned} \tag{43}$$

If we choose $V_3 = \xi$ in (43), we have

$$\begin{aligned}
&(f_1 - f_3)\left\{\tfrac{1}{2n}\eta(V_2)QV_5 - (f_1 - f_3)\eta(V_2)V_5\right. \\
&+C[\eta(V_2)V_5 - \eta(V_5)V_2]\} = -\lambda_2\{-(f_1 - f_3)\eta(V_2)QV_5 \\
&+2n(f_1 - f_3)^2\eta(V_2)V_5 + 2n(f_1 - f_3)C[\eta(V_5)\eta(V_2)\xi \\
&+\eta(V_5)V_2 - \eta(V_2)V_5 - g(V_5, V_2)\xi]\}.
\end{aligned} \tag{44}$$

If we choose $V_2 = \xi$ in (44)and then we take inner product of both sides of (44) by $V_3 \in \chi\left(M^{2n+1}\right)$, we have

$$\begin{aligned}
&(f_1 - f_3)(1 - 2n\lambda_2)[S(V_5, V_3) + (f_3 - 2nf_1 - 3f_2)g(V_5, V_3) \\
&-((1 - 2n)f_3 - 3f_2)\eta(V_5)\eta(V_3)] = 0.
\end{aligned}$$

This completes the proof. □

Corollary 4.6. *Let $M^{2n+1}(f_1, f_2, f_3)$ be the $(2n+1)$-dimensional generalized Sasakian space form. If $M^{2n+1}(f_1, f_2, f_3)$ is W_0-Ricci pseudosymmetric, then $M^{2n+1}(f_1, f_2, f_3)$ is Einstein manifold if and only if $3f_2 = (1-2n)f_3$ and $f_3 \neq 2nf_1 + 3f_2$ relations are provided.*

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Tuğba Mert
Department of Mathematics, University of Sivas Cumhuriyet,
58140, Sivas, Turkey.
E-mail: tmert@cumhuriyet.edu.tr

Mehmet Atçeken
Department of Mathematics, University of Aksaray,
68100, Aksaray, Turkey.
E-mail: mehmet.atceken382@gmail.com