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A NEW MODELLING OF TIMELIKE Q-HELICES

Yasin Ünlütürk*, Cumali Ekici, and Doğan Ünal

Abstract. In this study, we mean that timelike q-helices are curves whose q-frame fields make a constant angle with a non-zero fixed axis. We present the necessary and sufficient conditions for timelike curves via the q-frame to be q-helices in Lorentz-Minkowski 3-space. Then we find some results of the relations between q-helices and Darboux q-helices. Furthermore, we portray Darboux q-helices as special curves whose Darboux vector makes a constant angle with a non-zero fixed axis by choosing the curve as one of the types of q-helices, and also the general case.

1. Introduction

There are different approaches to frame a curve such as parallel transport frame, Frenet frame, and etc. in differential geometry of curves [2, 3, 4, 22]. The way to establish the quasi-frame has been firstly paved with introducing the quasi normal vector of a space curve by Coquillart [3]. Then Shin et al. has defined the quasi-normal vector for each point of the curve which lies in the plane perpendicular to the tangent of the curve at this point [17]. The local theory of space curves via q-frame has also been studied by Dede in [4, 19, 20].

Slant helices as a kind of helices have been conceptualized and characterized by some researchers such as Izumiya and Takeuchi [6], Kula and Yayli [7], Kula et al. [8]. The notion "k-type slant helices" is related to the class of curves having a property that the scalar product of frame's vector field and a fixed axis is constant [5]. For example, general helices are type-0 helices, and also type-1 slant helix is one whose normal vector field makes a constant angle with a non zero fixed axis.

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*Corresponding author

Researches are constantly increasing on k-type slant helices with their various aspects [10, 11, 14, 15, 18]. For instance, this topic has been studied and developed in different types of spaces such as Euclidean, Galilean, and Lorentzian spaces [1, 10, 12, 16]. Another classification called as "k-type Darboux slant helices" is based on the idea that Darboux vector, obtained by the frame fields in which curves' behaviour is taken into consideration, makes a constant angle with a non-zero fixed axis [10, 14, 15, 21].

In this work, we take timelike q-helices into consideration. By q-helices, we mean curves due to the quasi-frame (abbv. q-frame) whose vector fields' inner product with a non-zero fixed axis is constant. One by one, all types of these q-helices we study in the work are therefore classified in three dimensional Lorentz-Minkowski space. Additionally, we study Darboux q-helices by using Darboux vector obtained with respect to q-frames fields of a timelike curve. For a curve enclosed with q-frame as a general case, we reach some results for Darboux q-helices.

2. Preliminaries

The three dimensional Lorentz-Minkowski space $\mathbb{E}_1^3$ is the real vector space $\mathbb{R}^3$ equipped with

$$g = -dx_1^2 + dx_2^2 + dx_3^3,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$ [13].

Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a timelike space curve with a non-vanishing second derivative. The Frenet formula for the unit timelike curve $\gamma(t)$ is given

$$\begin{aligned}
&\mathbf{T}' = \kappa\mathbf{N}, \\
&\mathbf{N}' = \kappa\mathbf{T} + \tau\mathbf{B}, \\
&\mathbf{B}' = -\tau\mathbf{N},
\end{aligned}$$

where κ, and τ are the curvature and the torsion functions of the curve γ which are defined as $\kappa = \|\mathbf{T}'\|$ and $\tau = \langle N', B\rangle$, respectively [9].

The quasi-frame (abbv. q-frame) as an alternative frame to Frenet trihedron has been introduced as follows: Given a space curve $\gamma(t)$, the q-frame composes of three orthonormal vectors, these vectors are, respectively, the unit tangent vector $\mathbf{T}$, the quasi-normal $\mathbf{N}_q$ and the quasi-binormal vector $\mathbf{B}_q$. The q-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{k}\}$ is given by

$$\mathbf{T} = \frac{\gamma'}{\|\gamma'\|}, \mathbf{N}_q = \frac{\mathbf{T}\wedge_L \mathbf{k}}{\|\mathbf{T}\wedge_L \mathbf{k}\|}, \mathbf{B}_q = \mathbf{T} \wedge_L \mathbf{N}_q,$$

where $\mathbf{k}$ is the projection vector. For clarity, the projection vector $\mathbf{k}$ has been chosen as $\mathbf{k} = (0, 0, 1)$ along with the paper. Nevertheless, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ become parallel. Hence, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$ [20].

Let $\gamma(s)$ be a timelike curve that is parameterized by arc-length s. The variation equations of the q-frame for a timelike curve when tangent vector (timelike), projection vector $\mathbf{k} = (0, 1, 0)$ (spacelike), quasi-normal vector (spacelike) and quasi-binormal vector (spacelike), are given ([20]) by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_q' \\ \mathbf{B}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix},$$

where the q-curvatures are

$$k_1 = \langle \mathbf{T}', \mathbf{N}_q \rangle_L, \quad k_2 = \langle \mathbf{T}', \mathbf{B}_q \rangle_L, \quad k_3 = \left\langle \mathbf{N}_q', \mathbf{B}_q \right\rangle_L.$$

3. The timelike q-helices

In this section, we study different types of q-helices which means k-type slant helices of curves via quasi frame (abbv. q-frame) in Lorentz-Minkowski 3-space $\mathbb{E}_1^3$. By q-helices, we intend the curves whose quasi-frame vector fields' dot product with a non-zero fixed axis is constant. These types of helices within the q-frame are enclosed as depending on the inner product between the tangent vector field $\mathbf{T}$ and the fixed vector field $\mathbf{U}$, the quasi-normal vector field $\mathbf{N}_q$ and the fixed vector field $\mathbf{U}$, and the quasi-binormal vector field $\mathbf{B}_q$ and the fixed vector field $\mathbf{U}$ become constant.

Definition 3.1. A timelike curve γ in $\mathbb{E}_1^3$ given by the q-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$ is called a slant helix of type-0, a slant helix of type-1, and a slant helix of type-2 if there exists a non zero fixed direction $\mathbf{U} \in \mathbb{E}_1^3$ such that satisfies, respectively,

$$\langle \mathbf{T}, \mathbf{U} \rangle_L = c_0, \quad \langle \mathbf{N}_q, \mathbf{U} \rangle_L = c_1, \quad \text{and } \langle \mathbf{B}_q, \mathbf{U} \rangle_L = c_2,$$

where c_0, c_1, and c_2 are constants. The fixed direction $\mathbf{U}$ is called axis of the q-helices.

The vector field $\mathbf{U}$ can be written as a combination of q-frame fields as subsequent

$$\mathbf{U} = \lambda_1 \mathbf{T} + \lambda_2 \mathbf{N}_q + \lambda_3 \mathbf{B}_q,$$

where

$$\lambda_1 = -\langle \mathbf{T}, \mathbf{U} \rangle_L \qquad \lambda_2 = \langle \mathbf{N}_q, \mathbf{U} \rangle_L \qquad \lambda_3 = \langle \mathbf{B}_q, \mathbf{U} \rangle_L .$$

Since **U** is a fixed vector filed, its differentiation vanishes, thus the following system is obtained as

$$\begin{aligned} \lambda_1' + \lambda_2 k_1 + \lambda_3 k_2 &= 0, \\ \lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 &= 0, \\ \lambda_3' + \lambda_1 k_2 + \lambda_2 k_3 &= 0. \end{aligned} \tag{3.1}$$

In the following subsections, we study timelike q-helices based on the system (3.1).

3.1. The timelike q-helices of type-0

Theorem 3.1. Let γ be a timelike curve due to the q-frame in $\mathbb{E}_1^3$. Then γ is a timelike q-helix of type-0 if and only if

$$\left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds \right) k_1 + \left(e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds \right) k_2 = 0. \tag{3.2}$$

Proof. A timelike q-helix of type-0 satisfies the condition

$$\lambda_1 = -\langle \mathbf{T}, \mathbf{U} \rangle_L = c_0,$$

where c_0 is constant. Therefore, by substituting $\lambda_1 = c_0$ into the system (3.1), it turns into

$$\begin{aligned} \lambda_2 k_1 + \lambda_3 k_2 &= 0, \\ \lambda_2' - \lambda_3 k_3 - c_0 k_1 &= 0, \\ \lambda_3' + \lambda_2 k_3 - c_0 k_2 &= 0. \end{aligned} \tag{3.3}$$

From $(3.3)_1$,

$$\lambda_3 = -\tfrac{k_1}{k_2} \lambda_2, \qquad \lambda_2 = -\tfrac{k_2}{k_1} \lambda_3. \tag{3.4}$$

By using (3.4) in the equations $(3.3)_2$, and $(3.3)_3$, we get the following linear differential equations of first order:

$$\lambda_2' + \tfrac{k_1 k_3}{k_2} \lambda_2 = c_0 k_1, \tag{3.5}$$

and

$$\lambda_3' - \tfrac{k_2 k_3}{k_1} \lambda_3 = c_0 k_2. \tag{3.6}$$

The solution of (3.5) is

$$\lambda_2 = c_0 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds, \tag{3.7}$$

and the solution of (3.6) is

$$\lambda_3 = c_0 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.8}$$

Substituting (3.7) and (3.8) into $(3.3)_1$ gives the condition to be q-helices of type-0 as follows:

$$\left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds \right) k_1 + \left(e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds \right) k_2 = 0.$$

Conversely, suppose that the relation (3.2) holds, the fixed vector filed $\mathbf{U}$ can also be composed of

$$\begin{aligned} \mathbf{U} = & - c_0 \mathbf{T} + \left(c_0 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds \right) \mathbf{N}_q \\ & + \left(c_0 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds \right) \mathbf{B}_q. \end{aligned} \tag{3.9}$$

We obtain $\mathbf{U}'= \mathbf{0}$ by using (3.2). Hence γ is a timelike q-helix of type-0.

Corollary 3.1. If γ is a timelike q-helix of type-0, an axis of γ is as

$$\begin{aligned} \mathbf{D}_0 = & -c_0 \mathbf{T} + \left(c_0 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds \right) \mathbf{N}_q \\ & + \left(c_0 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds \right) \mathbf{B}_q. \end{aligned}$$

Remark 3.1. If the tangent vector field $\mathbf{T}$ of the curve γ and the fixed axis $\mathbf{D}_0$ are orthogonal to each other, that is, $c_0 = 0$, then the timelike q-helix of type-0 can not occur since the vanishing of the axis $\mathbf{D}_0$.

3.2. The timelike q-helices of type-1

Theorem 3.2. Let γ be a timelike curve due to the q-frame in $\mathbb{E}^3_1$. Then γ is a timelike q-helix of type-1 if and only if

$$\left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds \right) k_1 - \left(e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds \right) k_3 = 0. \tag{3.10}$$

Proof. A timelike q-helix of type-1 satisfies the condition

$$\lambda_2 = \langle \mathbf{N}_q, \mathbf{U} \rangle = c_1, \tag{3.11}$$

where c_1 is constant. Therefore, by substituting $\lambda_2 = c_1$ into the system (3.1), it turns into

$$\begin{aligned} &\lambda_1' + c_1 k_1 + \lambda_3 k_2 = 0, \\ &\lambda_1 k_1 - \lambda_3 k_3 = 0, \\ &\lambda_3' + \lambda_1 k_2 + c_1 k_3 = 0. \end{aligned} \tag{3.12}$$

From $(3.12)_2$,

$$\lambda_3 = \tfrac{k_1}{k_3}\lambda_1, \qquad \lambda_1 = \tfrac{k_3}{k_1}\lambda_3. \tag{3.13}$$

By using (3.13) in the equations $(3.12)_1$, and $(3.12)_3$, we get the following linear differential equations of first order:

$$\lambda_1' + \tfrac{k_1 k_2}{k_3}\lambda_1 = -c_1 k_1, \tag{3.14}$$

and

$$\lambda_3' + \tfrac{k_2 k_3}{k_1}\lambda_3 = -c_1 k_3. \tag{3.15}$$

The solutions of (3.14), and (3.15) are obtained as

$$\lambda_1 = -c_1 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds, \tag{3.16}$$

and

$$\lambda_3 = -c_1 e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds, \tag{3.17}$$

respectively.

Substituting (3.16) and (3.17) into $(3.12)_2$ gives the condition to be timelike q-helices of type-1 as follows:

$$\left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds \right) k_1 - \left(e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds \right) k_3 = 0.$$

Conversely, suppose that the relation (3.10) holds, the fixed vector field $\mathbf{U}$ can also be composed of

$$\begin{aligned} \mathbf{U} = &\left(-c_1 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds \right) \mathbf{T} + c_1 \mathbf{N}_q \\ &- \left(c_1 e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds \right) \mathbf{B}_q. \end{aligned} \tag{3.18}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (3.10) and (3.11). Hence γ is a timelike q-helix of type-1.

Corollary 3.2. If γ is a timelike q-helix of type-1, an axis of γ is as

$$\begin{aligned} \mathbf{D}_1 = &\left(-c_1 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds \right) \mathbf{T} + c_1 \mathbf{N}_q \\ &- \left(c_1 e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds \right) \mathbf{B}_q. \end{aligned}$$

Remark 3.2. If the tangent vector field $\mathbf{N}_q$ of the curve γ and the fixed axis $\mathbf{D}_1$ are orthogonal to each other, that is, $c_1 = 0$, then the timelike q-helix of type-1 can not occur since the vanishing of the axis $\mathbf{D}_1$.

3.3. The timelike q-helices of type-2

Theorem 3.3. Let γ be a timelike curve due to the q-frame in $\mathbb{E}_1^3$. Then γ is a timelike q-helix of type-2 if and only if

$$\left(e^{\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) k_2 - \left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) k_3 = 0. \tag{3.19}$$

Proof. A timelike q-helix of type-2 satisfies the condition

$$\lambda_3 = \langle \mathbf{B}_q, \mathbf{U} \rangle = c_2, \tag{3.20}$$

where c_2 is constant. Therefore, by substituting $\lambda_3 = c_2$ into the system (3.1), it turns into

$$\begin{aligned} &\lambda_1' + \lambda_2 k_1 + c_2 k_2 = 0, \\ &\lambda_2' + \lambda_1 k_1 - c_2 k_3 = 0, \\ &\lambda_1 k_2 + \lambda_2 k_3 = 0. \end{aligned} \tag{3.21}$$

From $(3.21)_3$,

$$\lambda_2 = -\frac{k_2}{k_3}\lambda_1, \qquad \lambda_1 = -\frac{k_3}{k_2}\lambda_2. \tag{3.22}$$

By using (3.22) in the equations $(3.21)_1$, and $(3.21)_2$, we get the following linear differential equations of first order:

$$\lambda_1' - \frac{k_1 k_2}{k_3}\lambda_1 = -c_2 k_2, \tag{3.23}$$

and

$$\lambda_2' - \frac{k_1 k_3}{k_2}\lambda_2 = c_2 k_3. \tag{3.24}$$

The solutions of (3.23) and (3.24) are

$$\lambda_1 = -c_2 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{-\int \frac{k_1 k_2}{k_3} ds} ds, \tag{3.25}$$

and

$$\lambda_2 = c_2 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds, \tag{3.26}$$

respectively.

Substituting (3.25) and (3.26) into $(3.21)_1$ gives the condition to be q-helices of type-2 as follows:

$$\left(e^{\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) k_2 - \left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) k_3 = 0.$$

Conversely, suppose that the relation (3.19) holds, also the fixed vector filed $\mathbf{U}$ can be composed of

$$\begin{aligned}\mathbf{U} = & \left(-c_2 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) \mathbf{T} \\ & + \left(c_2 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) \mathbf{N}_q + c_2 \mathbf{B}_q.\end{aligned} \tag{3.27}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (3.19) and (3.20). Hence γ is a q-helix of type-2.

Corollary 3.3. If γ is a q-helix of type-2, an axis of γ is

$$\begin{aligned}\mathbf{D}_2 = & \left(-c_2 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) \mathbf{T} \\ & + \left(c_2 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) \mathbf{N}_q + c_2 \mathbf{B}_q.\end{aligned}$$

Remark 3.3. If the tangent vector field $\mathbf{B}_q$ of the curve γ and the fixed axis $\mathbf{D}_2$ are orthogonal to each other, that is, $c_2 = 0$, then the timelike q-helix of type-2 can not occur since the vanishing of the axis $\mathbf{D}_2$.

3.4. The relations of timelike q-helices to each other

In this part, we give the relations of timelike q-helices to each other based on the consequences of Theorem 3.1, 3.2, and 3.3.

Corollary 3.4. Let γ be a timelike q-helix of type-0 in $\mathbf{U} \in \mathbb{E}_1^3$. Then γ is a timelike q-helix of type-1 if and only if

$$k_1 = 0 \quad \text{or} \quad k_2 = c_a k_3, \tag{3.28}$$

where c_a is constant.

Proof. Using (3.9) at the condition to be a timelike q-helix of type-1 as follows:

$$\langle \mathbf{N}_q, \mathbf{U} \rangle_L = c_0 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds. \tag{3.29}$$

The expression in (3.29) becomes constant if the cases (3.28) are satisfied.

Corollary 3.5. Let γ be a timelike q-helix of type-0 in $\mathbf{U} \in \mathbb{E}_1^3$. Then γ is a timelike q-helix of type-2 if and only if

$$k_2 = 0 \quad \text{or} \quad k_1 = -c_b k_3, \tag{3.30}$$

where c_b is constant.

Proof. Using (3.9) at the condition to be a timelike q-helix of type-2 as follows:

$$\langle \mathbf{B}_q, \mathbf{U} \rangle_L = -c_0 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.31}$$

The expression in (3.31) becomes constant if the cases (3.30) are satisfied.

Corollary 3.6. Let γ be a timelike q-helix of type-1 in $\mathbf{U} \in \mathbb{E}_1^3$. Then γ is a timelike q-helix of type-0 if and only if

$$k_1 = 0 \quad \text{or} \quad k_3 = c_c k_2, \tag{3.32}$$

where c_c is constant.

Proof. Using (3.18) at the condition to be a timelike q-helix of type-0 as follows:

$$\langle \mathbf{T}, \mathbf{U} \rangle_L = c_1 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds. \tag{3.33}$$

The expression in (3.33) becomes constant if the cases (3.32) are satisfied.

Corollary 3.7. Let γ be a timelike q-helix of type-1 in $\mathbf{U} \in \mathbb{E}_1^3$. Then γ is a timelike q-helix of type-2 if and only if

$$k_3 = 0 \quad \text{or} \quad k_1 = c_d k_2, \tag{3.34}$$

where c_d is constant.

Proof. Using (3.18) at the condition to be a timelike q-helix of type-2 as follows:

$$\langle \mathbf{B}_q, \mathbf{U} \rangle_L = -c_1 e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.35}$$

The expression in (3.35) becomes constant if the cases (3.34) are satisfied.

Corollary 3.8. Let γ be a timelike q-helix of type-2 in $\mathbf{U} \in \mathbb{E}_1^3$. Then γ is a timelike q-helix of type-0 if and only if

$$k_2 = 0 \quad \text{or} \quad k_3 = -c_e k_1, \tag{3.36}$$

where c_e is constant.

Proof. Using (3.27) at the condition to be a timelike q-helix of type-0 as follows:

$$\langle \mathbf{T}, \mathbf{U} \rangle_L = c_2 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds. \tag{3.37}$$

The expression in (3.37) becomes constant if the cases (3.36) are satisfied.

Corollary 3.9. Let γ be a timelike q-helix of type-2 in $\mathbf{U} \in \mathbb{E}_1^3$. Then γ is a timelike q-helix of type-1 if and only if

$$k_3 = 0 \quad \text{or} \quad k_2 = -c_f k_1, \tag{3.38}$$

where c_f is constant.

Proof. Using (3.27) at the condition to be a timelike q-helix of type-1 as follows:

$$\langle \mathbf{N}_q, \mathbf{U} \rangle_L = c_2 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{-\int \frac{k_1 k_3}{k_2} ds} ds. \tag{3.39}$$

The expression in (3.39) becomes constant if the cases (3.38) are satisfied.

The above results can be put together with the following corollary:

Corollary 3.10. Let γ be a curve via q-frame in $\mathbf{U} \in \mathbb{E}_1^3$. Then

(i): The curve γ is both a timelike q-helix of type-0 and a timelike q-helix of type-1 provided that

$$k_1 = 0 \quad \text{or} \quad k_2 = A k_3,$$

where A is an arbitrary constant.

(ii): The curve γ is both a timelike q-helix of type-0 and a timelike q-helix of type-2 provided that

$$k_2 = 0 \quad \text{or} \quad k_1 = B k_3,$$

where B is an arbitrary constant.

(iii): The curve γ is both a timelike q-helix of type-1 and a timelike q-helix of type-2 provided that

$$k_3 = 0 \quad \text{or} \quad k_2 = C k_1,$$

where C is an arbitrary constant.

4. The Darboux q-helices

In this part of the study, we examine the Darboux q-helices of timelike curves. First we research the conditions of q-helices of type-0, type-1, and type-2 to be a Darboux q-helix, respectively. Finally, we obtain the general case for timelike q-helices to be Darboux helices.

Using the relations

$$\mathbf{T}' = \partial \times \mathbf{T}, \qquad \mathbf{N}_q' = \partial \times \mathbf{N}_q, \qquad \mathbf{B}_q' = \partial \times \mathbf{B}_q,$$

The Darboux vector of a timelike curve due to the q-frame is calculated as

$$\partial = -\, k_3 \mathbf{T} + k_2 \mathbf{N}_q - k_1 \mathbf{B}_q. \tag{4.1}$$

We have to give the description of Darboux q-helices as follows:

Definition 4.1. A unit speed timelike curve γ via q-frame whose Darboux vector ∂ is as given in (4.1), is said to be a Darboux helix provided that there exists a non-zero fixed direction $\mathbf{U} \in \mathbb{E}_1^3$ such that satisfies

$$\langle \partial, \mathbf{U} \rangle_L = c, \tag{4.2}$$

where c is constant.

Based upon the system (3.1), we take the timelike q-helices of type-0, type-1, and type-2, and a timelike curve framed by q-frame to be Darboux helices into consideration, respectively, in the subsequent four cases:

Case-1: Let γ be a timelike q-helix of type-0. Hence the equation (3.2) holds. The equation

$$\langle \partial', \mathbf{U} \rangle_L = \lambda_1 k_3' + \lambda_2 k_2' - \lambda_3 k_1' = 0. \tag{4.3}$$

Using (3.5), and (4.3) in the system (3.1) results

$$\begin{array}{l} c_0 k_3' + \lambda_2 k_2' - \lambda_3 k_1' = 0, \\ \lambda_2 k_1 + \lambda_3 k_2 = 0, \\ \lambda_2' + - c_0 k_1 - \lambda_3 k_3 = 0, \\ \lambda_3' + - c_0 k_2 + \lambda_2 k_3 = 0. \end{array} \tag{4.4}$$

Applying $(4.4)_2$ into the equations $(4.4)_3$, and $(4.4)_4$, the functions λ_2, and λ_3 are found as in (3.7), and (3.8). If the values obtained are substituted into the equation $(4.4)_1$, then it follows that

$$k_3' + k_2' e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds - k_1' e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds = 0. \tag{4.5}$$

Also from (3.9), we have

$$\left(e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds \right) = -\frac{k_1}{k_2} \left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds \right). \tag{4.6}$$

Substituting (4.6) into (4.5) gives

$$k_3' + \left(k_2' + \frac{k_1 k_1'}{k_2} \right) \left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds \right) = 0, \tag{4.7}$$

which is the condition for a timelike q-helix of type-0 to be a Darboux helix.

Conversely, suppose that the relation (4.7) holds, it can be seen that the axis given in (3.9) is a fixed one.

Case-2: Let γ be a timelike q-helix of type-1. Hence the equation (3.11) holds. Using (3.11), and (4.3) in the system (3.1), we find the system

$$(4.8)\qquad \begin{array}{l} \lambda_1 k_3' + c_1 k_2' - \lambda_3 k_1' = 0, \\ \lambda_1' + c_1 k_1 + \lambda_3 k_2 = 0, \\ \lambda_1 k_1 - \lambda_3 k_3 = 0, \\ \lambda_3' + \lambda_1 k_2 + c_1 k_3 = 0. \end{array}$$

Applying $(4.8)_2$ into the equations $(4.8)_1$, and $(4.8)_3$, the functions λ_1, and λ_3 are found as in (3.16), and (3.17). If the values obtained are substituted into the equation $(4.8)_1$, then it follows that

(4.9)
$$-k_3' e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds + k_2' + k_1' e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds = 0.$$

Also from (3.23), we obtain

(4.10)
$$\left(e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds \right) = \frac{k_1}{k_3} \left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds \right).$$

Substituting (4.10) into (4.9), we attain the equation

$$(4.11)\qquad k_2' + \left(\frac{k_1 k_1'}{k_3} - k_3' \right) \left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{\int \frac{k_1 k_2}{k_3} ds} ds \right) = 0,$$

which is the condition for a q-helix of type-1 to be a Darboux helix.

Conversely, suppose that the relation (4.11) holds, it can be seen that the axis given in (3.18) is a fixed one.

Case-3: Let γ be a timelike q-helix of type-2. So the equation (3.20) holds. Using (3.20), and (4.3) in the system (3.1), we find the system

$$(4.12)\qquad \begin{array}{l} \lambda_1 k_3' + \lambda_2 k_2' - c_2 k_1' = 0, \\ \lambda_1' + \lambda_2 k_1 + c_2 k_2 = 0, \\ \lambda_2' + \lambda_1 k_1 - c_2 k_3 = 0, \\ \lambda_1 k_2 + \lambda_2 k_3 = 0. \end{array}$$

Applying $(4.12)_3$ into the equations $(4.12)_1$, and $(4.12)_2$, the functions λ_1, and λ_2 are obtained as in (3.25), and (3.26). If the values obtained is put into the equation $(4.12)_1$, then it follows that

(4.13)
$$-k_3' e^{\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{-\int \frac{k_1 k_2}{k_3} ds} ds + k_2' e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds - k_1' = 0.$$

Also from (3.34), we obtain

(4.14)
$$\left(e^{\int\frac{k_1k_2}{k_3}ds}\int k_2 e^{-\int\frac{k_1k_2}{k_3}ds}ds\right)=\frac{k_3}{k_2}\left(e^{\int\frac{k_1k_3}{k_2}ds}\int k_3 e^{-\int\frac{k_1k_3}{k_2}ds}ds\right).$$

Put (4.14) into (4.13), we reach the result

$$k_1'+\left(\frac{k_3k_3'}{k_2}-k_2'\right)\left(e^{\int\frac{k_1k_3}{k_2}ds}\int k_3 e^{-\int\frac{k_1k_3}{k_2}ds}ds\right)=0, \tag{4.15}$$

which is the condition for a timelike q-helix of type-2 to be a Darboux helix.

Conversely, suppose that the relation (4.16) holds, it can be seen that the axis given in (3.27) is a fixed one.

Case 4 (General Case):

Let γ be a timelike curve due to the q-frame in $\mathbb{E}^3_1$. From (4.2), we obtain

$$\lambda_1k_3+\lambda_2k_2-\lambda_3k_1=c. \tag{4.16}$$

Differentiating (4.16) gives

$$\lambda_1k_3'+\lambda_2k_2'-\lambda_3k_1'=0. \tag{4.17}$$

By (4.16) and (4.17), we arrive

$$\lambda_3=\frac{\left(k_2k_3'-k_2'k_3\right)\lambda_2-ck_3'}{k_1k_3'-k_1'k_3}, \tag{4.18}$$

and

$$\lambda_1=\frac{\left(k_2k_1'-k_2'k_1\right)\lambda_2-ck_1'}{k_3'k_1-k_3k_1'}, \tag{4.19}$$

respectively. Substituting (4.18) and (4.19) into $(3.1)_2$ delivers the linear differential equation

$$\lambda_2'+\left(\frac{-k_2'k_1^2+k_2k_1k_1'-k_2'k_3^2-k_2k_3k_3'}{k_3'k_1-k_3k_1'}\right)\lambda_2=c\frac{k_1k_1'-k_3k_3'}{k_3'k_1-k_3k_1'}. \tag{4.20}$$

The solution of (4.20) is

(4.21)
$$\lambda_2=ce^{\int\frac{-k_2k_1k_1'+k_2k_3k_3'+k_2'k_1^2+k_2'k_3^2}{k_3'k_1-k_3k_1'}ds}\int\frac{\left(k_1k_1'-k_3k_3'\right)}{k_3'k_1-k_3k_1'}e^{\int\frac{-k_2'k_1^2+k_2k_1k_1'-k_2'k_3^2-k_2k_3k_3'}{k_3'k_1-k_3k_1'}ds}ds.$$

Using (4.16) and (4.17), we obtain

$$\lambda_1=\frac{\left(k_1'k_2-k_1k_2'\right)\lambda_3-ck_2'}{k_2k_3'-k_2'k_3}, \tag{4.22}$$

and

$$\lambda_2=\frac{\left(k_1'k_3-k_1k_3'\right)\lambda_3-ck_3'}{k_2'k_3-k_2k_3'}, \tag{4.23}$$

respectively. Replacing (4.22) and (4.23) into $(3.1)_3$, we have the following differential equation

$$\lambda_3' + \left(\frac{k_1' k_2^2 - k_1 k_2 k_2' - k_1' k_3^2 + k_1 k_3 k_3'}{k_2 k_3' - k_2' k_3}\right)\lambda_3 = c\frac{k_2 k_2' + k_3 k_3'}{k_2 k_3' - k_2' k_3}. \tag{4.24}$$

The solution of (4.24) is
(4.25)

$$\lambda_3 = c e^{\int \frac{k_1 k_2 k_2' - k_1 k_3 k_3' + k_1' k_3^2 - k_1' k_2^2}{k_2 k_3' - k_2' k_3} ds} \int \frac{k_2 k_2' - k_3 k_3'}{k_2 k_3' - k_2' k_3} e^{\int \frac{k_1' k_2^2 - k_1 k_2 k_2' - k_1' k_3^2 + k_1 k_3 k_3'}{k_2 k_3' - k_2' k_3} ds} ds.$$

From (4.16) and (4.17), we attain

$$\lambda_2 = \frac{\left(k_3 k_1' - k_3' k_1\right)\lambda_1 - c k_1'}{k_2' k_1 - k_2 k_1'}, \tag{4.26}$$

and

$$\lambda_3 = \frac{\left(k_2' k_3 - k_2 k_3'\right)\lambda_1 - c k_2'}{k_1 k_2' - k_1' k_2}, \tag{4.27}$$

respectively. Usage of the equations (4.26), and (4.27) at $(3.1)_1$ allows the equation

$$\lambda_1' + \left(\frac{k_3 k_1' - k_3' k_1 + k_2' k_3 - k_2 k_3'}{k_1 k_2' - k_1' k_2}\right)\lambda_1 = c\frac{k_1 k_1' + k_2 k_2'}{k_1 k_2' - k_1' k_2}. \tag{4.28}$$

The solution of (4.28) is
(4.29)

$$\lambda_1 = c e^{\int \frac{k_3' k_1^2 - k_1 k_1' k_3 - k_2 k_2' k_3 + k_2^2 k_3'}{k_1 k_2' - k_1' k_2} ds} \int \frac{k_1 k_1' + k_2 k_2'}{k_1 k_2' - k_1' k_2} e^{\int \frac{k_1 k_1' k_3 - k_3' k_1^2 + k_2 k_2' k_3 - k_2^2 k_3'}{k_1 k_2' - k_1' k_2} ds} ds.$$

Substituting (4.21), (4.25), and (4.29) into (4.17) gives the condition for a curve to be a Darboux q-helix as follows:
(4.30)

$$\begin{aligned} &\left(e^{\int \frac{k_3' k_1^2 - k_1 k_1' k_3 - k_2 k_2' k_3 + k_2^2 k_3'}{k_1 k_2' - k_1' k_2} ds} \int \frac{k_1 k_1' + k_2 k_2'}{k_1 k_2' - k_1' k_2} e^{\int \frac{k_1 k_1' k_3 - k_3' k_1^2 + k_2 k_2' k_3 - k_2^2 k_3'}{k_1 k_2' - k_1' k_2} ds} ds\right) k_3' \\ &+ \left(e^{\int \frac{-k_2 k_1 k_1' + k_2 k_3 k_3' + k_2' k_1^2 + k_2' k_3^2}{k_3' k_1 - k_3 k_1'} ds} \int \frac{\left(k_1 k_1' - k_3 k_3'\right)}{k_3' k_1 - k_3 k_1'} e^{\int \frac{-k_2' k_1^2 + k_2 k_1 k_1' - k_2' k_3^2 - k_2 k_3 k_3'}{k_3' k_1 - k_3 k_1'} ds} ds\right) k_2' \\ &= \left(e^{\int \frac{k_1 k_2 k_2' - k_1 k_3 k_3' + k_1' k_3^2 - k_1' k_2^2}{k_2 k_3' - k_2' k_3} ds} \int \frac{k_2 k_2' - k_3 k_3'}{k_2 k_3' - k_2' k_3} e^{\int \frac{k_1' k_2^2 - k_1 k_2 k_2' - k_1' k_3^2 + k_1 k_3 k_3'}{k_2 k_3' - k_2' k_3} ds} ds\right) k_1' = 0. \end{aligned}$$

Conversely, suppose that the relation (4.30) holds, also the fixed vector filed **U** can be composed of

(4.31)
$$\mathbf{U}=\left(ce^{\int \frac{k_3'k_1^2-k_1k_1'k_3-k_2k_2'k_3+k_2^2k_3'}{k_1k_2'-k_1'k_2}ds}\int \frac{k_1k_1'+k_2k_2'}{k_1k_2'-k_1'k_2}e^{\int \frac{k_1k_1'k_3-k_3'k_1^2+k_2k_2'k_3-k_2^2k_3'}{k_1k_2'-k_1'k_2}ds}ds\right)\mathbf{T}$$
$$+\left(ce^{\int \frac{-k_2k_1k_1'+k_2k_3k_3'+k_2'k_1^2+k_2'k_3^2}{k_3'k_1-k_3k_1'}ds}\int \frac{\left(k_1k_1'-k_3k_3'\right)}{k_3'k_1-k_3k_1'}e^{\int \frac{-k_2'k_1^2+k_2k_1k_1'-k_2'k_3^2-k_2k_3k_3'}{k_3'k_1-k_3k_1'}ds}ds\right)\mathbf{N}_q$$
$$+\left(ce^{\int \frac{k_1k_2k_2'-k_1k_3k_3'+k_1'k_3^2-k_1'k_2^2}{k_2k_3'-k_2'k_3}ds}\int \frac{k_2k_2'-k_3k_3'}{k_2k_3'-k_2'k_3}e^{\int \frac{k_1'k_2^2-k_1k_2k_2'-k_1'k_3^2+k_1k_3k_3'}{k_2k_3'-k_2'k_3}ds}ds\right)\mathbf{B}_q.$$

We obtain $\mathbf{U}'=\mathbf{0}$ by using (4.16) and (4.30). Hence γ is a Darboux q-helix.

We can give the following theorem containing the cases above as:

Theorem 4.1. Let γ be a timelike curve due to the q-frame in Lorentz-Minkowski 3-space $\mathbb{E}_1^3$. Then

(i) The timelike curve γ is a Darboux q-helix satisfying the condition to be q-helix of type-0 if and only if the equation (4.7) is satisfied,
(ii) The timelike curve γ is a Darboux q-helix satisfying the condition to be q-helix of type-1 if and only if the equation (4.11) is satisfied,
(iii) The timelike curve γ is a Darboux q-helix satisfying the condition to be q-helix of type-2 if and only if the equation (4.15) is satisfied,
(iv) The timelike curve γ is a Darboux q-helix if and only if the equation (4.31) is satisfied, and the fixed axis is given as in (4.31).

5. Conclusion

In the present study, we analyzed timelike q-helices from the point of view of frame fields which describe the behaviour of the curves. The original aspect of our research is to deal quasi-frame (abbv. q-frame) in Lorentz-Minkowski 3-space. For all vector fields of the mentioned frame, timelike slant helices, which are recalled, in the context of the paper, as q-helices, have been worked out in Lorentz-Minkowski 3-space. Additionally, the Darboux q-helices are obtained by Darboux vector which has been formed by q-frame fields.

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Yasin Ünlütürk
Department of Mathematics, Kırklareli University,
Kırklareli 39100, Turkey.
E-mail: yasinunluturk@klu.edu.tr

Cumali Ekici
Department of Mathematics and Computer Sciences,
Eskişehir Osmangazi University,
Eskisehir 26480, Turkey.
E-mail: cekici@ogu.edu.tr

Doğan Ünal
Department of Software Engineering, Kırklareli University,
Kırklareli 39100, Turkey.
E-mail: doganunal@klu.edu.tr