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STUDY OF P-CURVATURE TENSOR IN THE SPACE-TIME OF GENERAL RELATIVITY

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Abstract. The P-curvature tensor has been studied in the space-time of general relativity and it is found that the contracted part of this tensor vanishes in the Einstein space. It is shown that Rainich conditions for the existence of non-null electro variance can be obtained by $P_{\alpha\beta}$. It is established that the divergence of tensor $G_{\alpha\beta}$ defined with the help of $P_{\alpha\beta}$ and scalar P is zero, so that it can be used to represent Einstein field equations. It is shown that for V_4 satisfying Einstein like field equations, the tensor $P_{\alpha\beta}$ is conserved, if the energy momentum tensor is Codazzi type. The space-time satisfying Einstein's field equations with vanishing of P-curvature tensor have been considered and existence of Killing, conformal Killing vector fields and perfect fluid space-time has been established.

1. Introduction

Consider an n-dimensional space V_n in which the curvature tensor W_2 has been defined by

$$W_2(X,Y,Z,T) = Rm(X,Y,Z,T) - \frac{1}{n-1}[g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T)]$$

for any vector fields X, Y, Z and T on V_n [29]. Here $Rm(X,Y,Z,T)$, $Ric(X,Y)$ and $g(X,Y)$ denote the curvature tensor, Ricci tensor and metric tensor of V_n, respectively, for arbitrary vector fields X, Y, Z and T. It is seen that

$$W_2(X,Y,Z,T) = -W_2(Y,X,Z,T)$$

and

$$W_2(X,Y,Z,T) + W_2(Y,Z,X,T) + W_2(Z,X,Y,T) = 0.$$

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Breaking W_2-curvature tensor in skew-symmetric parts in Z and T, the P-curvature tensor has been defined by

$$P(X,Y,Z,T) = Rm(X,Y,Z,T) - \frac{1}{2(n-1)}[g(Y,Z)Ric(X,T)$$

$$-g(X,Z)Ric(Y,T) + g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)], \tag{1}$$

called Pokhariyal and Mishra tensor, which possesses all skew-symmetric and symmetric as well as cyclic properties satisfied by Riemann curvature tensor (see, [28]). The W_2-tensor has been quite widely studied in the space-time of general relativity as well as in differential geometry. Ahsan and Ali [2] have studied space-time satisfying Einstein's field equations with vanishing of W_2-curvature tensor as well as existence of Killing and conformal Killing vector fields. They further examined vanishing and divergence of W_2-tensor in perfect fluid space-time. Matsumoto et al. [24] have studied W_2-curvature tensor in para-Sasakian manifolds. The geometrical and physical properties of W_2-curvature tensor have been studied by several geometers and physicists (for instance, [20], [23], [26], [27], [31], [35], [36], [37]). P-curvature tensor has been defined from W_2-curvature tensor in (1). Various physical and geometrical properties of this curvature tensor are studied in ([3], [11]-[18], [29]) and also by others.

2. P-curvature tensor

We consider the P-curvature tensor in the local coordinates as:

$$P_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2(n-1)}[g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\gamma}R_{\beta\delta} + g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\delta}R_{\alpha\gamma}],$$

where $R_{\alpha\beta\gamma\delta}$ and $R_{\alpha\delta}$ represent the curvature tensor and Ricci tensor, respectively. Here $\alpha, \beta, \gamma, \delta = 1, 2, 3, \ldots, n$. This can be written as:

$$P^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta} - \frac{1}{2(n-1)}[g_{\beta\gamma}R^{\alpha}_{\delta} - g^{\alpha}_{\gamma}R_{\beta\delta} + g^{\alpha}_{\delta}R_{\beta\gamma} - g_{\beta\delta}R^{\alpha}_{\gamma}], \tag{2}$$

where R^{α}_{γ} stands for Ricci operator. Contracting α and δ, we get

$$P_{\beta\gamma} = R_{\beta\gamma} - \frac{1}{2(n-1)}[g_{\beta\gamma}R^{\alpha}_{\alpha} - g^{\alpha}_{\gamma}R_{\beta\alpha} + g^{\alpha}_{\alpha}R_{\beta\gamma} - g_{\beta\alpha}R^{\alpha}_{\gamma}]. \tag{3}$$

On simplification, we get

$$P_{\beta\gamma} = \frac{n}{2(n-1)}[R_{\beta\gamma} - \frac{R}{n}g_{\beta\gamma}],$$

where R denotes the scalar curvature. For $n = 4$, in V_4, we have

$$P_{\beta\gamma} = \frac{2}{3}[R_{\beta\gamma} - \frac{R}{4}g_{\beta\gamma}]. \tag{4}$$

Hence in an Einstein space $P_{\beta\gamma}$ vanishes. Thus, the contracted part of Pokhariyal and Mishra tensor $P_{\alpha\beta\gamma\delta}$ vanishes in Einstein space. This enables us to extend the Pirani formalization of gravitational waves in the Einstein space with these tensors. Further, by multiplying (4) by $g^{\beta\gamma}$, we get

$$g^{\beta\gamma}P_{\beta\gamma} = P = \frac{2}{3}[g^{\beta\gamma}R_{\beta\gamma} - \frac{R}{4}g^{\beta\gamma}g_{\beta\gamma}] = 0.$$

Thus, the scalar invariant P vanishes identically. Misner and Wheeler [25] introduced a vector

$$\Theta_i = \frac{g_{\alpha\beta}\epsilon^{\beta\gamma\mu\nu}R^{\delta}_{\gamma}R_{\delta\mu;\nu}}{\sqrt{-g}R_{\alpha\beta}R^{\alpha\beta}} \tag{5}$$

called complexion vector of a non-null electromagnetic field with no matter and its vanishing implies that field is purely electrical. Here semicolon " ; " is used for covariant derivative and $\epsilon^{\beta\gamma\mu\nu}$ represents the Levi-Civita symbol, which is skew-symmetric in all pairs of indices with $\epsilon^{1234} = 1$ [33]. It was shown by Pokhariyal and Mishra [29] that we can not get purely electric field with the help of $W_2(X, Y, Z, T)$. Rainich [30] has shown that the necessary and sufficient conditions for the existence of non-null electrovariance are

$$R = 0,$$

$$R^{\alpha}_{\beta}R^{\beta}_{\gamma} = \frac{1}{4}\delta^{\alpha}_{\gamma}R_{ab}R^{ab}, \tag{6}$$

$$\Theta_{\alpha;\beta} = \Theta_{\beta;\alpha}. \tag{7}$$

In an electromagnetic field, equation (4) gives

$$P_{\beta\gamma} = \frac{2}{3}R_{\beta\gamma}.$$

By replacing the matter tensor $R_{\alpha\beta}$ by $P_{\alpha\beta}$ in (5), (6) and (7), respectively, we get the Rainich conditions with the help of $P_{\alpha\beta\gamma\delta}$.

2.1. Divergence of $P_{\alpha\beta\gamma\delta}$

We start with the Bianchi differential identity for $P_{\alpha\beta\gamma\delta}$ with the condition that the Ricci tensor is of Codazzi type [19] obtained by Pokhariyal [28]

$$\nabla_X P(Y, Z, T, U) + \nabla_Y P(Z, X, T, U) + \nabla_Z P(X, Y, T, U) = 0.$$

This is expressed in the index notation as

$$\nabla_\sigma P_{\alpha\beta\mu\nu} + \nabla_\nu P_{\alpha\beta\sigma\mu} + \nabla_\mu P_{\alpha\beta\nu\sigma} = 0.$$

Multiply through by $g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}$ (knowing that the metric derivatives are zero, as they act as constants, thus can be taken inside the derivative sign), we get

$$\nabla_\sigma g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}P_{\alpha\beta\mu\nu} + \nabla_\nu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}P_{\alpha\beta\sigma\mu} + \nabla_\mu g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}P_{\alpha\beta\nu\sigma} = 0.$$

Using the property that $P_{\alpha\beta\mu\gamma}$ is symmetric in pair and skew-symmetric in the indices, on simplification, we get

$$\nabla_\sigma(P^{\gamma\sigma} - \frac{1}{2}Pg^{\gamma\sigma}) = 0.$$

We introduce

$$G^{\gamma\sigma} = P^{\gamma\sigma} - \frac{1}{2}Pg^{\gamma\sigma}, \tag{8}$$

and call it Ganesh tensor, whose divergence is zero. Einstein's field equations (that are 10 contained in the tensor equation) with cosmological term are given by

$$E_{\alpha\beta} + \wedge g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{9}$$

with $E_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$ is the Einstein tensor, where $T_{\alpha\beta}$ is the stress-energy tensor and contains all forms of energy and momentum, κ is the coupling constant with the value $\frac{8\pi G}{c^4}$ and $\wedge$ is cosmological constant. These equations describe gravity as a result of spacetimes being curved by means of mass and energy. $E_{\alpha\beta}$ is determined by the curvature of spacetime at a particular point in spacetime which is equated with the energy momentum at that point. The Einstein's field equation (9) can be expressed using the tensor $G_{\alpha\beta}$ defined by (8). Since $G_{\alpha\beta}$ contains extra terms as compared to Einstein tensor $E_{\alpha\beta}$, $R_{\alpha\beta}$ and R is likely to have additional physical and geometrical interpretations derived through the solutions that are the components of metric tensor $g_{\alpha\beta}$ specifying the spacetime geometry.

2.2. Contraction of $P^{\alpha\beta}$

Replacing Einstein tensor $E_{\alpha\beta}$ by $G_{\alpha\beta}$ in (9), the Einstein's field equations without cosmological constant $\wedge$ in the presence of matter can be expressed as

$$P_{\alpha\beta} - \frac{1}{2}Pg_{\alpha\beta} = \kappa T_{\alpha\beta}. \tag{10}$$

Multiplying this equation by $g^{\alpha\beta}$, on simplification we get

$$P = -\kappa T, \tag{11}$$

where $P = P_{\alpha\beta}g^{\alpha\beta}$ and $T = T_{\alpha\beta}g^{\alpha\beta}$. Putting (11) in (10), we get

$$P_{\alpha\beta} = \kappa\{T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\}.$$

It is known that the energy-momentum tensor for the electromagnetic field is given by

$$T_{\alpha\beta} = -F_{\alpha\gamma}F^{\gamma}_{\beta} + \frac{1}{4}g_{\alpha\beta}F_{\delta\gamma}F^{\delta\gamma}, \tag{12}$$

where $F_{\alpha\beta}$ represents skew-symmetric field tensor, satisfying Maxwell's equation [34]. From (12) it is clear that $T^{\alpha}_{\alpha} = T = 0$. Einstein equations written in (10) for purely electromagnetic distribution take the form

$$P_{\alpha\beta} = \kappa T_{\alpha\beta}.$$

From (10) we have

$$\nabla_\gamma P_{\alpha\beta} = \kappa\nabla_\gamma T_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}\nabla_\gamma P.$$

Thus, we can write

$$\nabla_\gamma P_{\alpha\beta} - \nabla_\beta P_{\alpha\gamma} = \kappa\{\nabla_\gamma T_{\alpha\beta} - \nabla_\beta T_{\alpha\gamma}\} + \frac{1}{2}\{g_{\alpha\beta}\nabla_\gamma P - g_{\alpha\gamma}\nabla_\beta P\}. \tag{13}$$

If $T_{\alpha\beta}$ is of Codazzi type, then (13) becomes

$$\nabla_\gamma P_{\alpha\beta} - \nabla_\beta P_{\alpha\gamma} = \frac{1}{2}\{g_{\alpha\beta}\nabla_\gamma P - g_{\alpha\gamma}\nabla_\beta P\}.$$

Multiplying this equation by $g^{\alpha\beta}$, on simplification, we get

$$\nabla_\gamma P^\gamma_\gamma = -\frac{1}{2}\nabla_\gamma P.$$

Multiplying by $g^{\alpha\beta}$, we get on simplification

$$\nabla_\beta P^{\alpha\beta} = 0.$$

Thus, we have the following theorem.

Theorem 2.1. *For V_4 satisfying Einstein (like) field equation, the tensor $P^{\alpha\beta}$ is conserved if the energy-momentum tensor is of Codazzi type.*

2.3. P-flat space-times

Consider the equation (2) for P-curvature tensor.

Definition 2.2. *A space-time is said to be $P-$flat if the tensor $P^\alpha_{\beta\gamma\delta}$, defined by (3), vanishes in it.*

Let us suppose that the space-time is P-flat, then from (2) we have

$$R^\alpha_{\beta\gamma\delta} = \frac{1}{2(n-1)}[g_{\beta\gamma}R^\alpha_\delta - g^\alpha_\gamma R_{\beta\delta} + g^\alpha_\delta R_{\beta\gamma} - g_{\beta\delta}R^\alpha_\gamma].$$

Contracting α and δ yields

$$R_{\beta\gamma} = \frac{1}{2(n-1)}[(n-2)R_{\beta\gamma} + g_{\beta\gamma}R].$$

For V_4, on simplification, we have

$$R_{\beta\gamma} = \frac{R}{4}g_{\beta\gamma}. \tag{14}$$

This shows that P-flat space-time is an Einstein space. Thus, we have

Theorem 2.3. *A P-flat space-time is an Einstein space-time and consequently the scalar curvature R is covariatly constant, that is, $\nabla_\beta R = 0$.*

The gravitational field is adequately described by curvature tensor, as they consist matter part and gravitational part, whose interaction is depicted by Bianchi identities. The main focus of various studies have been the construction of gravitational potential satisfying the Einstein equations for a given distribution of matter. This is accomplished by imposing symmetries on the geometry compatible with the dynamics of the selected distribution of the matter. For the space-times, the geometrical symmetries are given by the following equation

$$\pounds_\xi A - 2\Omega A = 0,$$

where A represents a geometrical/physical quantity, $\pounds_\xi$ denotes the Lie derivative with respect to a vector field ξ and Ω is a scalar [2].

Consider the equation (9) which is written as:

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \wedge g_{\alpha\beta} = \kappa T_{\alpha\beta}. \tag{15}$$

Using (14) in (15), we get on simplification

$$g_{\alpha\beta}(\wedge - \frac{R}{4}) = \kappa T_{\alpha\beta}. \tag{16}$$

Since for a P-flat space-time, R is constant, by taking the Lie derivative of both sides of (16) along ξ gives

$$(\wedge - \frac{R}{4})\pounds_\xi g_{\alpha\beta} = \kappa \pounds_\xi T_{\alpha\beta}, \tag{17}$$

provided $\wedge \neq \frac{R}{4}$. Thus, we have the following theorem.

Theorem 2.4. *For a P-flat space-time satisfying the Einstein's field equations with a cosmological term, there exists a Killing vector field ξ if and only if the Lie derivative of the energy-momentum tensor vanishes with respect to ξ.*

Definition 2.5. *A vector field ξ satisfying the equation*

$$\pounds_\xi g_{\alpha\beta} = 2\Omega g_{\alpha\beta} \tag{18}$$

is called a conformal Killing vector field, where Ω is a scalar. A space-time satisfying (18) is said to admits a conformal motion.

From (17) and (18), we have

$$2\Omega(\wedge - \frac{R}{4})g_{\alpha\beta} = \kappa \pounds_\xi T_{\alpha\beta}.$$

Using (16) as a consequence of P-flat space-time, we get

$$\pounds_\xi T_{\alpha\beta} = 2\Omega T_{\alpha\beta}. \tag{19}$$

The energy momentum tensor $T_{\alpha\beta}$ satisfying equation (19) is said to preserve the symmetry inheritance property [1]. Thus, we have the following theorem.

Theorem 2.6. *A P-flat space-time satisfying the Einstein's field equations with a cosmological term admits a conformal Killing vector field if and only if the energy-momentum tensor has the symmetry inheritance property.*

The energy-momentum tensor for a perfect fluid is given by

$$T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + pg_{\alpha\beta}, \tag{20}$$

where μ is the energy density, p the isotropic pressure, u_α is the velocity of the fluid such that $u_\alpha \cdot u^\alpha = -1$ and $g_{\alpha\beta}u^\alpha = u_\beta$. For more details about the perfect fluid spacetimes, we refer ([4]-[10], [21], [32]) and the references therein.

We now consider a perfect fluid space-time with vanishing P-curvature tensor. From equations (16) and (20), we get

$$g_{\alpha\beta}(\wedge - \frac{R}{4} - \kappa p) = \kappa(\mu + p)u_\alpha u_\beta. \tag{21}$$

Multiplying by $g^{\alpha\beta}$, equation (21) yields on simplification

$$R = \kappa(\mu - 3p) + 4\wedge. \tag{22}$$

Further, contracting equation (21) with $u^\alpha u^\beta$, we get

$$R = 4(\kappa\mu + \wedge). \tag{23}$$

Comparing (22) and (23), yield

$$\mu + p = 0.$$

This means that either $\mu = 0$, $p = 0$ (empty space-time) or the perfect fluid space-time satisfies the vacuum like equation of state [22]. Thus, we have the following theorem.

Theorem 2.7. *In a P-flat perfect fluid space-time satisfying Einstein's field equations with cosmological term, the matter contents of the space-time obey the vacuum like equation of state.*

Discussion

The symmetric nature and other features of P-curvature tensor that are similar to the Riemann curvature become important characteristics for investing its various physical and geometrical as well as applications. The tensor $G_{\alpha\beta}$ can be used to get Einstein (like) field equations and their physical consequences can then be explored. Starting with the various metrics and using the corresponding geodesic equations, the trajectories of the particles are likely to be obtained which may be different from the ones obtained using Einstein tensor $E_{\alpha\beta}$. The comparisons can then be interpreted accordingly.

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