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ON RESULTS OF MIDPOINT-TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL OPERATORS WITH TWICE-DIFFERENTIABLE FUNCTIONS

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Abstract. This article establishes an equality for the case of twice-differentiable convex functions with respect to the conformable fractional integrals. With the help of this identity, we prove sundry midpoint-type inequalities by twice-differentiable convex functions according to conformable fractional integrals. Several important inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Using the specific selection of our results, we obtain several new and well-known results in the literature.

1. Introduction

The convex theory is a suitable and attractive way to analyse a large number of problems from different branches of mathematics. One of the most used inequalities for convex functions is the Hermite–Hadamard-type inequality. Because of this importance, the Hermite–Hadamard-type inequalities have been investigated seriously by many mathematicians in the last century.

Hermite–Hadamard-type inequalities which have been first introduced by C. Hermite and J. Hadamard for the case of convex functions. If $\mathcal{F} : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then the following double inequality holds:

$$\mathcal{F}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \mathcal{F}(x)dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}. \tag{1}$$

Midpoint-type inequality which is the left hand side of (1) and trapezoid-type inequality which is the right hand side of (1). If $\mathcal{F}$ is concave, then both inequalities in (1) hold in the reverse direction.

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Fractional calculus is a field of mathematics that expands the traditional derivative and integral ideas to non-integer orders. Riemann-Liouville fractional integrals, conformable fractional integrals, and many types of fractional integrals have been investigated with Hermite–Hadamard-type inequalities. In recent decades, it has piqued the curiosity of mathematicians, physicists, and engineers [28, 3]. Moreover, fractional derivatives are also used to model a wide range of mathematical biology, as well as chemical processes and engineering problems [4, 9]. With the aid of the derivative's fundamental limit formulation, a newly well-behaved basic fractional derivative known as the conformable derivative is improved in paper [22]. Several major requirements that cannot be implemented by the Riemann-Liouville and Caputo definitions are implemented by the conformable derivative. On the other hand, in paper [2] the author established that the conformable approach in [22] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by several extensions of the conformable approach [29, 14].

Many mathematicians have focused on acquired midpoint-type and trapezoid-type inequalities that give bounds via the left-hand side and right-hand side of the Hermite–Hadamard-type inequalities, respectively. For example, Dragomir and Agarwal [8] first presented trapezoid-type inequalities for the case of convex functions while Kirmacı[21] first established inequalities of midpoint-type inequalities for the case of convex functions. In paper [25], Qaisar and Hussain proved some generalized midpoint-type inequalities. Moreover, Sarikaya et al. and Iqbal et al. established sundry fractional midpoint and trapezoid-type inequalities to the case of convex mappings in papers [16] and [27], respectively. Furthermore, several Hermite–Hadamard-type inequalities for the fractional integrals are presented in paper [15]. For results connected with these type of inequalities one can see Refs. [10, 19] and the references therein.

Large number of mathematicians have investigated the twice-differentiable convex functions to obtain sundry important inequalities. For instance, Barani et al. [5] established inequalities for the case of twice-differentiable convex mappings which are related to Hermite–Hadamard-type inequalities. In paper [23], several generalized fractional integral inequalities of trapezoid-type and midpoint-type for the case of twice-differentiable convex functions are obtained. Moreover, Sarikaya and Aktan [26] established several new inequalities of the Simpson and the Hermite–Hadamard-type for functions whose absolute values of derivatives are convex. The reader is referred to [13, 7, 11] and the references therein for more information and unexplained subjects connected with several properties of Riemann-Liouville fractional integrals and twice-differentiable convex functions.

The aim of this paper is to prove midpoint-type inequalities for the case of twice-differentiable convex functions involving conformable fractional integrals. The whole form of the study takes the form of four sections involving introduction. In Section 2, the fundamental definitions of convex functions, Riemann-Liouville integrals and conformable integrals will be explained for building our principal outcomes. In Section 3, an equality will be proved for the case of twice-differentiable convex functions related to the conformable fractional integrals. With the aid of this identity, we give several midpoint-type inequalities for twice-differentiable convex functions according to conformable fractional integrals. Furthermore, we also present several corollaries and remarks in this section. Finally, summary and concluding remarks are given in Section 4.

2. Preliminaries

This section gives the fundamental definitions of convex functions, Riemann-Liouville integrals and conformable integrals, which are well known in the literature in order to build our main results.

Definition 2.1. [24] *Suppose that I is an interval of real numbers. Then, a function $\mathcal{F}: I \to \mathbb{R}$ is said to be convex, if*

$$\mathcal{F}\left(tx + (1-t)\,y\right) \leq t\mathcal{F}\left(x\right) + (1-t)\,\mathcal{F}\left(y\right)$$

is satisfied $\forall x, y \in I$ and $\forall t \in [0,1]$.

Definition 2.2. *The gamma function, beta function, and incomplete beta function are represented*

$$\Gamma\left(x\right) := \int_0^\infty t^{x-1} e^{-t} dt,$$

$$\mathfrak{B}\left(x,y\right) := \int_0^1 t^{x-1}\left(1-t\right)^{y-1} dt,$$

and

$$\mathscr{B}\left(x,y,r\right) := \int_0^r t^{x-1}\left(1-t\right)^{y-1} dt,$$

respectively for $0 < x, y < \infty$.

Kilbas et al. [20] defined fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 2.3. [20] *The Riemann-Liouville integrals $J_{a+}^{\beta}\mathcal{F}(x)$ and $J_{b-}^{\beta}\mathcal{F}(x)$ of order $\beta > 0$ are given by*

$$J_{a+}^{\beta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(x-t\right)^{\beta-1} \mathcal{F}(t)dt, \quad x > a \tag{2}$$

and

$$J_{b-}^{\beta}\mathcal{F}(x)=\frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1}\mathcal{F}(t)dt, \quad x<b, \tag{3}$$

respectively for $\mathcal{F}\in L_1[a,b]$. Note that the Riemann-Liouville integrals coincides with the classical integrals for the case of $\beta=1$.

Jarad et al. [18] established the fractional conformable integral operators. They also derived certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are described as follows:

Definition 2.4. [18] *The fractional conformable integral operator ${}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(x)$ and ${}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(x)$ of order $\beta\in C$, $Re(\beta)>0$ and $\alpha\in(0,1]$ are presented by*

$${}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(x)=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(t)}{(t-a)^{1-\alpha}}dt, \qquad t>a \tag{4}$$

and

$${}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(x)=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(t)}{(b-t)^{1-\alpha}}dt, \qquad t<b, \tag{5}$$

respectively for $\mathcal{F}\in L_1[a,b]$.

If we choose $\alpha=1$, then the fractional integrals in (4) and (5) equals to the Riemann-Liouville fractional integrals in (2) and (3), respectively. There have been a great number of research papers written on these subjects, [1, 17] and the references therein.

3. Main Results

In this section, midpoint-type inequalities are created for the case of twice-differentiable convex functions related to the conformable fractional integrals. Let us first set up the following identity in order to obtain conformable fractional versions of midpoint-type inequalities.

Lemma 3.1. *Assume that $\mathcal{F}:[a,b]\to\mathbb{R}$ is a twice-differentiable function on (a,b) such that $\mathcal{F}''\in L_1[a,b]$. Then, the following equality holds:*

(6)
$$\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)=\frac{(b-a)^2\alpha^{\beta}}{2}\sum_{i=1}^{4}A_i,$$

where

$$\begin{cases} A_1 = \int\limits_0^{\frac{1}{2}} \left(\int\limits_0^t \left(\frac{1-(1-s)^\alpha}{\alpha} \right)^\beta ds \right) \mathcal{F}''(tb+(1-t)a)\,dt, \\ A_2 = \int\limits_0^{\frac{1}{2}} \left(\int\limits_0^t \left(\frac{1-(1-s)^\alpha}{\alpha} \right)^\beta ds \right) \mathcal{F}''(ta+(1-t)b)\,dt, \\ A_3 = \int\limits_{\frac{1}{2}}^1 \left(\int\limits_t^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-s)^\alpha}{\alpha} \right)^\beta \right] ds \right) \mathcal{F}''(tb+(1-t)a)\,dt, \\ A_4 = \int\limits_{\frac{1}{2}}^1 \left(\int\limits_t^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-s)^\alpha}{\alpha} \right)^\beta \right] ds \right) \mathcal{F}''(ta+(1-t)b)\,dt. \end{cases}$$

Proof. From the fact of the integrating by parts, it yields

$$\begin{aligned} A_1 &= \int\limits_0^{\frac{1}{2}} \left(\int\limits_0^t \left(\frac{1-(1-s)^\alpha}{\alpha} \right)^\beta ds \right) \mathcal{F}''(tb+(1-t)a)\,dt \\ &= \frac{1}{b-a} \left(\int\limits_0^t \left(\frac{1-(1-s)^\alpha}{\alpha} \right)^\beta ds \right) \mathcal{F}'(tb+(1-t)a) \Big|_0^{\frac{1}{2}} \\ &\quad - \frac{1}{b-a} \int\limits_0^{\frac{1}{2}} \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^\beta \mathcal{F}'(tb+(1-t)a)\,dt \\ &= \frac{1}{b-a} \left(\int\limits_0^{\frac{1}{2}} \left(\frac{1-(1-s)^\alpha}{\alpha} \right)^\beta ds \right) \mathcal{F}'\left(\frac{a+b}{2} \right) \\ &\quad - \frac{1}{b-a} \left\{ \frac{1}{b-a} \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^\beta \mathcal{F}(tb+(1-t)a) \Big|_0^{\frac{1}{2}} \right. \\ &\quad \left. - \frac{\beta}{b-a} \int\limits_0^{\frac{1}{2}} \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\beta-1} (1-t)^{\alpha-1} \mathcal{F}(tb+(1-t)a)\,dt \right\} \end{aligned} \tag{7}$$

$$= \frac{1}{b-a}\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta} ds\right)\mathcal{F}'\left(\frac{a+b}{2}\right)$$

$$-\frac{1}{(b-a)^2}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}\mathcal{F}\left(\frac{a+b}{2}\right)$$

$$+\frac{\beta}{(b-a)^2}\int_0^{\frac{1}{2}}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\mathcal{F}\left(tb+(1-t)\,a\right)dt.$$

Similar to foregoing process, we have

$$(8)\qquad A_2 = -\frac{1}{b-a}\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta} ds\right)\mathcal{F}'\left(\frac{a+b}{2}\right)$$

$$-\frac{1}{(b-a)^2}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}\mathcal{F}\left(\frac{a+b}{2}\right)$$

$$+\frac{\beta}{(b-a)^2}\int_0^{\frac{1}{2}}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\mathcal{F}\left(ta+(1-t)\,b\right)dt,$$

$$(9)\qquad A_3 = -\frac{1}{b-a}\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right] ds\right)\mathcal{F}'\left(\frac{a+b}{2}\right)$$

$$-\frac{1}{(b-a)^2}\left(\frac{1}{\alpha^{\beta}}-\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}\mathcal{F}\left(\frac{a+b}{2}\right)$$

$$+\frac{\beta}{(b-a)^2}\int_{\frac{1}{2}}^{1}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\mathcal{F}\left(tb+(1-t)\,a\right)dt,$$

and

$$A_4 = \frac{1}{b-a}\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)\mathcal{F}'\left(\frac{a+b}{2}\right) \tag{10}$$

$$- \frac{1}{(b-a)^2}\left(\frac{1}{\alpha^{\beta}} - \frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}\mathcal{F}\left(\frac{a+b}{2}\right)$$

$$+ \frac{\beta}{(b-a)^2}\int_{\frac{1}{2}}^{1}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\mathcal{F}(ta+(1-t)b)\,dt.$$

If we collect from (7) to (10), then we can obtain

$$\sum_{i=1}^{4}A_i = \frac{\beta}{(b-a)^2}\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\mathcal{F}(tb+(1-t)a)\,dt \tag{11}$$

$$+ \frac{\beta}{(b-a)^2}\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\mathcal{F}(ta+(1-t)b)\,dt$$

$$- \frac{2}{(b-a)^2\alpha^{\beta}}\mathcal{F}\left(\frac{a+b}{2}\right).$$

If we use change of variables in (11), then (11) is converted as follows:

(12)

$$\sum_{i=1}^{4}A_i = -\frac{2}{(b-a)^2\alpha^{\beta}}\mathcal{F}\left(\frac{a+b}{2}\right) + \left(\frac{1}{b-a}\right)^{\alpha\beta+2}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}$$

$$\times\int_a^b\left(\frac{(b-a)^{\alpha}-(b-x)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(b-x)^{1-\alpha}}\mathcal{F}(x)\,dx + \left(\frac{1}{b-a}\right)^{\alpha\beta+2}$$

$$\times\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_a^b\left(\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(x-a)^{1-\alpha}}\mathcal{F}(x)\,dx$$

$$= \frac{\Gamma(\beta+1)}{(b-a)^{\alpha\beta+2}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(b)\right]-\frac{2}{(b-a)^{2}\alpha^{\beta}}\mathcal{F}\left(\frac{a+b}{2}\right).$$

If (12) is multiplied by $\frac{(b-a)^2\alpha^\beta}{2}$, then the proof of Lemma 3.1 is finished. □

Theorem 3.2. *If $\mathcal{F}:[a,b]\to\mathbb{R}$ is a twice-differentiable function on (a,b) such that $\mathcal{F}''\in L_1[a,b]$ and $|\mathcal{F}''|$ is convex on $[a,b]$, then*

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right| \tag{13}$$

$$\leq \frac{(b-a)^2\alpha^\beta}{2}\left\{\varphi_1(\alpha,\beta)+\varphi_2(\alpha,\beta)\right\}\left[|\mathcal{F}''(a)|+|\mathcal{F}''(b)|\right]$$

is valid. Here,

$$\begin{cases} \varphi_1(\alpha,\beta)=\int\limits_0^{\frac12}\left(\int\limits_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)dt \\ =\frac{1}{\alpha^{\beta+1}}\int\limits_0^{\frac12}\left(\mathfrak{B}\left(\frac1\alpha,\beta+1\right)-\mathscr{B}\left(\frac1\alpha,\beta+1,(1-t)^\alpha\right)\right)dt, \\ \varphi_2(\alpha,\beta)=\int\limits_{\frac12}^1\left(\int\limits_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)dt \\ =\frac{1}{\alpha^\beta}\int\limits_{\frac12}^1\left(1-t-\frac1\alpha\left(\mathfrak{B}\left(\beta+1,\frac1\alpha\right)-\mathscr{B}\left(\beta+1,\frac1\alpha,1-(1-t)^\alpha\right)\right)\right)dt, \end{cases} \tag{14}$$

where $\mathfrak{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof. If we take the absolute value of both sides of (6), then we have the following inequality

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right| \tag{15}$$

$$\leq \frac{(b-a)^2\alpha^\beta}{2}\left\{\int\limits_0^{\frac12}\left|\int\limits_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right|\left|\mathcal{F}''(tb+(1-t)a)\right|dt\right.$$

$$+\int_0^{\frac{1}{2}}\left|\int_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right|\left|\mathcal{F}''(ta+(1-t)b)\right|dt$$

$$+\int_{\frac{1}{2}}^1\left|\int_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\left|\mathcal{F}''(tb+(1-t)a)\right|dt$$

$$\left.+\int_{\frac{1}{2}}^1\left|\int_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\left|\mathcal{F}''(ta+(1-t)b)\right|dt\right\}.$$

Because $|\mathcal{F}''|$ is convex on $[a,b]$, we can easily obtain the following inequality

$$\left|\frac{\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_{b-}^\alpha\mathcal{F}(a)+{}^\beta\mathcal{J}_{a+}^\alpha\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)^2\alpha^\beta}{2}\left\{\int_0^{\frac{1}{2}}\left(\int_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)\right.$$

$$\times\left[t\left|\mathcal{F}''(b)\right|+(1-t)\left|\mathcal{F}''(a)\right|+t\left|\mathcal{F}''(a)\right|+(1-t)\left|\mathcal{F}''(b)\right|\right]dt$$

$$+\int_{\frac{1}{2}}^1\left(\int_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)$$

$$\times\left[t\left|\mathcal{F}''(b)\right|+(1-t)\left|\mathcal{F}''(a)\right|+t\left|\mathcal{F}''(a)\right|+(1-t)\left|\mathcal{F}''(b)\right|\right]dt\}$$

$$=\frac{(b-a)^2\alpha^\beta}{2}\left\{\int_0^{\frac{1}{2}}\left(\int_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left(\int_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)dt\right\}\left[\left|\mathcal{F}''(a)\right|+\left|\mathcal{F}''(b)\right|\right].$$

This ends the proof of Theorem 3.2. $\square$

Remark 3.3. *If we set $\alpha = 1$ in Theorem 3.2, then we reduces to*

$$\left|\frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{b-}^{\beta}\mathcal{F}(a) + J_{a+}^{\beta}\mathcal{F}(b)\right] - \mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{2}\left\{\frac{1}{8} - \frac{\beta}{2(\beta+1)(\beta+2)}\right\}\left[\left|\mathcal{F}''(a)\right| + \left|\mathcal{F}''(b)\right|\right],$$

which is established by Hezenci et al. in [12].

Remark 3.4. *Let us consider $\alpha = 1$ and $\beta = 1$ in Theorem 3.2. Then, Theorem 3.2 becomes to*

$$\left|\frac{1}{b-a}\int_a^b \mathcal{F}(x)\,dx - \mathcal{F}\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{48}\left[\left|\mathcal{F}''(a)\right| + \left|\mathcal{F}''(b)\right|\right],$$

which is given in [26, Proposition 1].

Theorem 3.5. *Note that $\mathcal{F} : [a,b] \to \mathbb{R}$ is a twice-differentiable function on (a,b) such that $\mathcal{F}'' \in L_1[a,b]$. Note also that $\left|\mathcal{F}''\right|^q$ is convex on $[a,b]$ with $q > 1$. Then, the following inequalities hold:*

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(a) + {}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(b)\right] - \mathcal{F}\left(\frac{a+b}{2}\right)\right| \tag{16}$$

$$\leq \frac{(b-a)^2\alpha^{\beta}}{2^{1+\frac{1}{q}}}\left(\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}} + \left(\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right)$$

$$\times\left[\left(\frac{\left|\mathcal{F}''(b)\right|^q + 3\left|\mathcal{F}''(a)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|\mathcal{F}''(a)\right|^q + 3\left|\mathcal{F}''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)\alpha^{\beta}}{2^{1+\frac{1}{q}}}\left(\left(4\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}} + \left(4\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right)\left[\left|\mathcal{F}''(a)\right| + \left|\mathcal{F}''(b)\right|\right].$$

Here, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\begin{cases} \psi_1^{\alpha,\beta}(p) = \int_0^{\frac{1}{2}}\left|\int_0^t \left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta} ds\right|^p dt, \\ \psi_2^{\alpha,\beta}(p) = \int_{\frac{1}{2}}^1\left|\int_t^1 \left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right] ds\right|^p dt. \end{cases}$$

Proof. If we use Hölder inequality in (15), then we have

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2}\left\{\left(\int_{0}^{\frac{1}{2}}\left|\int_{0}^{t}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}ds\right|^{p}dt\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\int_{0}^{\frac{1}{2}}\left|\mathcal{F}''(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{0}^{\frac{1}{2}}\left|\mathcal{F}''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\int_{t}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\left.\times\left[\left(\int_{\frac{1}{2}}^{1}\left|\mathcal{F}''(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left|\mathcal{F}''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right]\right\}.$$

Note that $|\mathcal{F}''|^{q}$ is convex on $[a,b]$. Then, we obtain

$$\left|\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2}\left\{\left(\int_{0}^{\frac{1}{2}}\left(\int_{0}^{t}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}ds\right)^{p}dt\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\int_{0}^{\frac{1}{2}}\left(t\left|\mathcal{F}''(b)\right|^{q}+(1-t)\left|\mathcal{F}''(a)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\left(t\left|\mathcal{F}''(a)\right|^q+(1-t)\left|\mathcal{F}''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg]$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\int_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_{\frac{1}{2}}^{1}\left(t\left|\mathcal{F}''(b)\right|^q+(1-t)\left|\mathcal{F}''(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\left(t\left|\mathcal{F}''(a)\right|^q+(1-t)\left|\mathcal{F}''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\Bigg\}$$

$$=\frac{(b-a)^2\alpha^\beta}{2^{1+\frac{1}{q}}}\left(\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}+\left(\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right)$$

$$\times\left[\left(\frac{\left|\mathcal{F}''(b)\right|^q+3\left|\mathcal{F}''(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}''(a)\right|^q+3\left|\mathcal{F}''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

The second inequality of Theorem 3.5 can be obtained simultaneously by letting $\phi_1=3\left|\mathcal{F}''(a)\right|^q$, $\varrho_1=\left|\mathcal{F}''(b)\right|^q$, $\phi_2=\left|\mathcal{F}''(a)\right|^q$ and $\varrho_2=3\left|\mathcal{F}''(b)\right|^q$ and applying the following inequality:

$$\sum_{k=1}^{n}(\phi_k+\varrho_k)^s\le\sum_{k=1}^{n}\phi_k^s+\sum_{k=1}^{n}\varrho_k^s,\qquad 0\le s<1,$$

which finishes the proof of Theorem 3.5. □

Remark 3.6. *Let us consider $\alpha=1$ in Theorem 3.5. Then, we derive*

$$\left|\frac{\Gamma(\beta+1)}{2(b-a)^\beta}\left[J_{b-}^{\beta}\mathcal{F}(a)+J_{a+}^{\beta}\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{2^{1+\frac{1}{q}}}\left(\frac{1}{\beta+1}\left(\frac{1}{(p(\beta+1)+1)\,2^{p(\beta+1)+1}}\right)^{\frac{1}{p}}+\left(\psi_2^{1,\beta}(p)\right)^{\frac{1}{p}}\right)$$

$$\times\left[\left(\frac{|\mathcal{F}''(b)|^q+3|\mathcal{F}''(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}''(a)|^q+3|\mathcal{F}''(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)\,\alpha^\beta}{2^{1+\frac{1}{q}}}\left(\left(\frac{4}{(\beta+1)^p\,(p(\beta+1)+1)\,2^{p(\beta+1)+1}}\right)^{\frac{1}{p}}+\left(4\psi_2^{1,\beta}(p)\right)^{\frac{1}{p}}\right)$$

$$\times\left[|\mathcal{F}''(a)|+|\mathcal{F}''(b)|\right],$$

which are given in [12].

Remark 3.7. *If we choose $\alpha=1$ and $\beta=1$ in Theorem 3.5, then we have*

$$\left|\frac{1}{b-a}\int_a^b \mathcal{F}(x)\,dx-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{|\mathcal{F}'(b)|^q+3|\mathcal{F}'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(a)|^q+3|\mathcal{F}'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)^2}{16}\left(\frac{4}{2p+1}\right)^{\frac{1}{p}}\left[|\mathcal{F}''(a)|+|\mathcal{F}''(b)|\right],$$

which are given in [6, Corollary 4.8].

Theorem 3.8. *Suppose that $\mathcal{F}:[a,b]\to\mathbb{R}$ is a twice-differentiable function on (a,b) such that $\mathcal{F}''\in L_1[a,b]$ and $|\mathcal{F}''|^q$ is convex on $[a,b]$ with $q\geq 1$. Then, the following inequality holds:*

$$\left|\frac{\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[\,{}^\beta\mathcal{J}_{b-}^\alpha\mathcal{F}(a)+\,{}^\beta\mathcal{J}_{a+}^\alpha\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2\,\alpha^\beta}{2}\left\{(\varphi_1(\alpha,\beta))^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\varphi_3(\alpha,\beta)\left|\mathcal{F}''(b)\right|^q+\left(\varphi_1(\alpha,\beta)-\varphi_3(\alpha,\beta)\right)\left|\mathcal{F}''(a)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\varphi_3(\alpha,\beta)\left|\mathcal{F}''(a)\right|^q+\left(\varphi_1(\alpha,\beta)-\varphi_3(\alpha,\beta)\right)\left|\mathcal{F}''(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$+\left(\varphi_2(\alpha,\beta)\right)^{1-\frac{1}{q}}\left[\left(\varphi_4(\alpha,\beta)\left|\mathcal{F}''(b)\right|^q+\left(\varphi_2(\alpha,\beta)-\varphi_4(\alpha,\beta)\right)\left|\mathcal{F}''(a)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\varphi_4(\alpha,\beta)\left|\mathcal{F}''(a)\right|^q+\left(\varphi_2(\alpha,\beta)-\varphi_4(\alpha,\beta)\right)\left|\mathcal{F}''(b)\right|^q\right)^{\frac{1}{q}}\right]\right\}.$$

Here, $\varphi_1(\alpha,\beta)$, $\varphi_2(\alpha,\beta)$ are defined in (14) and

$$\begin{cases}\varphi_3(\alpha,\beta)=\int\limits_0^{\frac{1}{2}} t\left(\int\limits_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)dt\\[2ex] =\frac{1}{\alpha^{\beta+1}}\int\limits_0^{\frac{1}{2}} t\left(\mathfrak{B}\left(\frac{1}{\alpha},\beta+1\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-t)^\alpha\right)\right)dt,\\[2ex] \varphi_4(\alpha,\beta)=\int\limits_{\frac{1}{2}}^{1} t\left(\int\limits_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)dt\\[2ex] =\frac{1}{\alpha^\beta}\int\limits_{\frac{1}{2}}^{1} t\left[1-t-\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathscr{B}\left(\beta+1,\frac{1}{\alpha},1-(1-t)^\alpha\right)\right)\right]dt,\end{cases}$$

where $\mathfrak{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof. Let us apply power-mean inequality in (15). Then, we have

$$\left|\frac{\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}\mathcal{F}(a)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)^2\alpha^\beta}{2}\left\{\left(\int\limits_0^{\frac{1}{2}}\left|\int\limits_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int\limits_0^{\frac{1}{2}}\left|\int\limits_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right|\left|\mathcal{F}''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\left|\int_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right|\left|\mathcal{F}''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$+\left(\int_{\frac{1}{2}}^1\left|\int_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\left(\int_{\frac{1}{2}}^1\left|\int_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\left|\mathcal{F}''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\left|\int_t^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\left|\mathcal{F}''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\Bigg]\Bigg\}.$$

Since $|\mathcal{F}''|^q$ is convex on $[a,b]$, we obtain

$$\left|\frac{\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_{b-}^\alpha\mathcal{F}(a)+{}^\beta\mathcal{J}_{a+}^\alpha\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)^2\alpha^\beta}{2}\Bigg\{\left(\int_0^{\frac{1}{2}}\left(\int_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)dt\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\left(\int_0^{\frac{1}{2}}\left(\int_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)\left(t\left|\mathcal{F}''(b)\right|^q+(1-t)\left|\mathcal{F}''(a)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{1}{2}}\left(\int_0^t\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)\left(t\left|\mathcal{F}''(a)\right|^q+(1-t)\left|\mathcal{F}''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg]$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\int_{t}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_{\frac{1}{2}}^{1}\left(\int_{t}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)\right.\right.$$

$$\times\left(t\left|\mathcal{F}''(b)\right|^{q}+(1-t)\left|\mathcal{F}''(a)\right|^{q}\right)dt\Big)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\int_{t}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)\right.$$

$$\left.\left.\times\left(t\left|\mathcal{F}''(a)\right|^{q}+(1-t)\left|\mathcal{F}''(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right]\right\},$$

which ends the proof of Theorem 3.8. □

Remark 3.9. *If we assign $\alpha=1$ in Theorem 3.8, then we acquire*

$$\left|\frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{b-}^{\beta}\mathcal{F}(a)+J_{a+}^{\beta}\mathcal{F}(b)\right]-\mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)^{2}}{2}\left\{(\varphi_{1}(1,\beta))^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\varphi_{3}(1,\beta)\left|\mathcal{F}''(b)\right|^{q}+(\varphi_{1}(1,\beta)-\varphi_{3}(1,\beta))\left|\mathcal{F}''(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\varphi_{3}(1,\beta)\left|\mathcal{F}''(a)\right|^{q}+(\varphi_{1}(1,\beta)-\varphi_{3}(1,\beta))\left|\mathcal{F}''(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$

$$+(\varphi_{2}(1,\beta))^{1-\frac{1}{q}}\left[\left(\varphi_{4}(1,\beta)\left|\mathcal{F}''(b)\right|^{q}+(\varphi_{2}(1,\beta)-\varphi_{4}(1,\beta))\left|\mathcal{F}''(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\varphi_{4}(1,\beta)\left|\mathcal{F}''(a)\right|^{q}+(\varphi_{2}(1,\beta)-\varphi_{4}(1,\beta))\left|\mathcal{F}''(b)\right|^{q}\right)^{\frac{1}{q}}\right]\right\},$$

which is presented in [12]. *Here,*

$$\begin{cases} \varphi_1(1,\beta) = \frac{1}{(\beta+1)(\beta+2)2^{\beta+2}}, & \varphi_2(1,\beta) = \frac{1}{\beta+1}\left[\frac{\beta+1}{8} - \left(\frac{1+\beta 2^{\beta+1}}{(\beta+2)2^{\beta+2}}\right)\right], \\ \varphi_3(1,\beta) = \frac{1}{(\beta+1)(\beta+3)2^{\beta+3}}, & \varphi_4(1,\beta) = \frac{1}{\beta+1}\left[\frac{2\beta-7}{24} + \frac{2^{\beta+3}-1}{(\beta+3)2^{\beta+3}}\right]. \end{cases}$$

Remark 3.10. *Consider $\alpha = 1$ and $\beta = 1$ in Theorem 3.8. Then, we obtain*

$$\left|\frac{1}{b-a}\int_a^b \mathcal{F}(x)\,dx - \mathcal{F}\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{48}\left[\left(\frac{5\left|\mathcal{F}''(b)\right|^q + 3\left|\mathcal{F}''(a)\right|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{5\left|\mathcal{F}''(a)\right|^q + 3\left|\mathcal{F}''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right],$$

which is given in [26, Proposition 5].

4. Summary and concluding remarks

In this article, we established an equality for the case of convex differentiable functions. By using this identity, we proved midpoint-type inequalities related to the conformable fractional integrals. Furthermore, our results generalized known results in the literature.

In future studies, the ideas and strategies for our results about midpoint-type inequalities via conformable fractional integrals may open new avenues for further research in this field. In addition, one can obtain likewise inequalities of midpoint-type via conformable fractional integrals for twice-differentiable convex functions with the help of the quantum calculus. Furthermore, one can apply these resulting inequalities to different types of fractional integrals.

Availability of data and material

Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

Competing Interests

The authors declare that they have no competing interests.

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References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66.

[2] A. A. Abdelhakim, *The flaw in the conformable calculus: It is conformable because it is not fractional,* Fract. Calc. Appl. Anal. **22** (2019), 242–254.

[3] G. A. Anastassiou, *Generalized fractional calculus: New advancements and aplications*, Springer, Switzerland, 2021.

[4] N. Attia, A. Akgül, D. Seba, and A. Nour, *An efficient numerical technique for a biological population model of fractional order*, Chaos, Solutions & Fractals, **141** (2020), 110349.

[5] A. Barani, S. Barani, S. S. Dragomir, *Refinements of Hermite–Hadamard inequalities for functions when a power of the absolute value of the second derivative is P-convex*, J. Appl. Math. 2012.

[6] H. Budak, F. Ertugral, and E. Pehlivan, *Hermite-Hadamard type inequalities for twice differantiable functions via generalized fractional integrals*, Filomat **33 (**2019**)**, no. 15, 4967–4979.

[7] H. Budak, F. Hezenci, and H. Kara, *On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co–ordinated convex functions via generalized fractional integrals*, Adv. Differential Equations **2021** (2021), 1–32.

[8] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett. **11 (**1998**),** no. 5, 91–95.

[9] A. Gabr, A. H. Abdel Kader, and M. S. Abdel Latif, *The Effect of the Parameters of the Generalized Fractional Derivatives On the Behavior of Linear Electrical Circuits*, Int. J. Appl. Comput. Math. **7** (2021), 247.

[10] F. Hezenci, *A Note on Fractional Midpoint Type Inequalities for Co-ordinated* $(s1, s2)$*-Convex Functions* Cumhuriyet Science Journal **43** (2022)**,** no. 3, 477–491.

[11] F. Hezenci, H. Budak, and H. Kara, *New version of Fractional Simpson type inequalities for twice differentiable functions*, Adv. Differential Equations, **2021** (2021)**,** no. 460.

[12] F. Hezenci, M. Bohner, and H. Budak, *Fractional midpoint-type inequalities for twice-differentiable functions*, submitted.

[13] H. R. Hwang, K. L. Tseng, and K. C. Hsu, *New inequalities for fractional integrals and their applications*, Turkish J. Math. **40** (2016), no. 3, 471–486.

[14] A. Hyder, A.H. Soliman, *A new generalized θ-conformable calculus and its applications in mathematical physics*, Phys. Scr. **96** (2020), 015208.

[15] M. Iqbal, B. M. Iqbal, and K. Nazeer, *Generalization of Inequalities Analogous to Hermite-Hadamard Inequality via Fractional Integrals*, Bull. Korean Math. Soc. **52 (**2015**),** no. 3, 707–716.

[16] M. Iqbal, S. Qaisar, and M. Muddassar, *A short note on integral inequality of type Hermite–Hadamard through convexity*, J. Comput. Anal. Appl. **21 (**2016**)**, no. 5, 946–953.

[17] F. Jarad, T. Abdeljawad, and D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl. **10** (2017), no. 5, 2607–2619.

[18] F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Baleanu, *On a new class of fractional operators*, Adv. Differential Equations, **2017** (2017), 247.

[19] H. Kara, H. Budak, and F. Hezenci, *New Extensions of the Parameterized Inequalities Based on Riemann-Liouville Fractional Integrals*, Mathematics, **10** (2022), no. 18, 3374.

[20] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, Elsevier Sci. B. V., Amsterdam, 2006.

[21] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput. **147** (2004), no. 5, 137–146.

[22] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014), 65–70.

[23] P. O. Mohammed and M. Z. Sarikaya, *On generalized fractional integral inequalities for twice differentiable convex functions*, J. Comput. Appl. Math. **372** (2020), 112740.

[24] J. E. Pecaric, F. Proschan, and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[25] S. Qaisar and S. Hussain, *On Hermite–Hadamard type inequalities for functions whose first derivative absolute values are convex and concave*, Fasc. Math. **58** (2017), no. 1, 155–166.

[26] M. Z. Sarikaya and N. Aktan, *On the generalization of some integral inequalities and their applications*, Math. Comput. Modelling **54** (2011), no. 9-10, 2175–2182.

[27] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling **57** (2013), no. 9-10, 2403–2407.

[28] V. V. Uchaikin, *Fractional derivatives for physicists and engineers*, Springer, Berlin/Heidelberg, Germany, 2013.

[29] D. Zhao and M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, **54** (2017), 903–917.

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