Communications for Statistical Applications and Methods
2023, Vol. 30, No. 3, 227–244
https://doi.org/10.29220/CSAM.2023.30.3.227
Print ISSN 2287-7843 / Online ISSN 2383-4757

Depth-Based rank test for multivariate two-sample scale problem

Digambar Tukaram Shirke[a], Swapnil Dattatray Khorate[1,ab]

[a]Department of Statistics, Shivaji University, Maharashtra, India;
[b]Shri. Yahswantrao Patil Science College, Solankur, Maharashtra, India

Abstract

In this paper, a depth-based nonparametric test for a multivariate two-sample scale problem is proposed. The proposed test statistic is based on the depth-induced ranks and is thus distribution-free. In this article, the depth values of data points of one sample are calculated with respect to the other sample or distribution and vice versa. A comprehensive simulation study is used to examine the performance of the proposed test for symmetric as well as skewed distributions. Comparison of the proposed test with the existing depth-based nonparametric tests is accomplished through empirical powers over different depth functions. The simulation study admits that the proposed test outperforms existing nonparametric depth-based tests for symmetric and skewed distributions. Finally, an actual life data set is used to demonstrate the applicability of the proposed test.

Keywords: data depth, nonparametric test, multivariate skew-normal, skew-cauchy, copula distribution

1. Introduction

Recent advancements in computer technology and every industrial area have made it simple to collect multivariate data. Consequently, multivariate statistical analysis has become increasingly relevant for extracting desired information from multivariate data. The central part of multivariate statistical analysis is hypothesis testing, and testing the scale parameter of two multivariate distributions is often desirable. Several tests are available in the literature for testing scale parameters of two multivariate distributions. However, the majority of them necessitate distributional assumptions like multivariate normality. The fact of the matter is that such a statement is not always satisfied. Therefore, nonparametric tests, which do not include any distributional assumptions, are more applicable in this case.

Data depth is an increasingly growing concept in a multivariate nonparametric setup that transforms multivariate data into a univariate setup and provides a simple way to manage multivariate data. Data depth assigns a value to each multivariate data point that defines how centered or deep the data point is in the given multivariate data set or concerning the given multivariate distribution. The center-outward ordering of multivariate data is the outcome of these depth values of multivariate data points. The concept was first found by Tukey (1975). This paper provides a nonparametric test based on data depth for scale parameters of two multivariate distributions.

In the literature, many tests are available for testing equality of location parameters based on data depth see details in Dovoedo and Chakraborti (2015), Shirke and Khorate (2018), Pawar and

[1] Corresponding author: Shri. Yahswantrao Patil Science College, Solankur, affiliated to Shivaji University, Kolhapur, Maharashtra, India-416212. E-mail:sdkhorate.stat@gmail.com

Published 31 May 2023 / journal homepage: http://csam.or.kr

Shirke (2019). Liu and Singh (1993) developed nonparametric tests based on ranks induced by data depth function to test scale parameters of two multivariate distributions. Rousson (2002) invented depth-based nonparametric tests to assess if two multivariate distributions are identical regarding location and scale parameters. Liu *et al.* (1999) introduced the depth-depth plot (DD-plot) as a simple graphical method for comparing two multivariate distributions in terms of location, scale, and kurtosis difference. By observing the behavior of DD-plot, Li and Liu (2004) developed nonparametric tests based on data depth for location difference and scale difference of two multivariate distributions. Liu and Singh (2006) proposed nonparametric tests for scale difference in two or more multivariate distributions based on the ranks generated by depth function, which generalizes the scale tests of Liu and Singh (1993). With the help of percentile modification by Gastwirth (1965), Chenouri *et al.* (2011) proposed nonparametric tests for scale difference between two or more multivariate distributions based on data depth which are the percentile modification of the tests by Liu and Singh (1993, 2006). Li and Liu (2016) recently proposed a depth-based one-sided nonparametric test for the testing scale parameter of two multivariate distributions based on differences between the depth values and offered a comparison with the test by Liu and Singh (2006) to evaluate the performance of the test. They also provided a depth-based test for the testing scale parameter of more than two distributions. Chavan and Shirke (2020) suggested a depth-based nonparametric test for the testing scale parameter of two multivariate distributions and assessed the test performance by comparing it with the test of Liu and Singh (2006). Some of the scale tests used depth-generated ranks to define test statistics. However, if we change the reference sample or distribution while computing the depth values of data points, the ranks induced by the depth function change, resulting in a change of rank-based test statistics. Here, we modified the reference sample/distribution to calculate the depth values of data points and offered a new test for testing scale parameters of two multivariate distributions based on ranks generated by the depth values of data points.

The following is the layout of the paper. In Section 2, some data depth functions are provided, which are used in the simulation study. The existing depth-based scale tests followed by a proposed test are provided in Section 3. In Section 4, the performance of the proposed test is studied through simulation study for symmetric and skewed distributions. The proposed test is illustrated with a real-life example in Section 5. In the last, concluding remarks are provided in Section 6.

2. Data depth

Tukey (1975) invented the phrase 'depth' to describe the graphical representation of two-dimensional data. The depth function returns a value for each multivariate data point that indicates how deep or centered the data point is in the given data cloud or relation to the given distribution. In other words, it provides a normal center-outward ordering to the data by supplying the scalar value for each multivariate data point. More information can be found in Li and Liu (2004), and the references therein. Let $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_m)$ be the random sample of size m drawn from a p-variate continuous probability distribution $F(\cdot)$. Each $Y_i, i = 1, 2, \ldots, m$ be a p-dimensional vector. The depth of data point y concerning distribution $F(\cdot)$ is denoted by notation $D(y; F)$. It is a nonnegative function defined on $\mathbf{R}^P$ with a range set $[0, \infty)$. In the literature, many depth functions are available. Some of these are halfspace depth (Tukey, 1975), simplicial depth (Liu, 1990), majority depth (Singh, 1991), Mahalanobis depth (Liu and Singh, 1993), regression depth (Rousseeuw and Hubert, 1999), and spatial depth (Serfling, 2002). Many of these depth functions have the desirable properties of affine invariance, monotonicity, maximum at the center, and vanishing at infinity, as described by Zuo and Serfling (2000). In the simulation study, we used the following four depth functions.

2.1. Halfspace depth

This depth function is also known as Tukey depth and has developed by Tukey (1975). Let HD($y; F$) denote the halfspace depth of a data point $y \in \mathbf{R}^p$ with respect to known probability distribution $F(\cdot)$ and is defined as

$$\mathrm{HD}(y; F) = \inf_H \{Pr_F(H) : \text{H is a closed halfspace containing y}\}.$$

When F is unknown, the sample version of halfspace depth is obtained by replacing F with its empirical version F_m and is given by,

$$\mathrm{HD}(y; F_m) = \frac{\min_{\|a\|=1} \#\left\{t : a^T Y_t \leq a^T y,\ t = 1, 2, \ldots, m\right\}}{m}.$$

2.2. Spatial depth

Serfling (2002) proposed a depth function based on Chaudhuri's (1996) spatial quantile called spatial depth. The spatial depth of $y \in \mathbf{R}^p$ concerning F is defined as,

$$\mathrm{SPD}(y; F) = 1 - \left\| \int S(y - \mathbf{Y})\, dF(\mathbf{Y}) \right\| = 1 - \left\| E[S(y - \mathbf{Y})] \right\|,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^p$ and the multivariate spatial sign function $S : \mathbf{R}^p \to \mathbf{R}^p$ can be calculated by

$$S(y) = \begin{cases} \dfrac{y}{\|y\|} & \text{if } y \neq 0, \\ 0 & \text{if } 0 = 0. \end{cases}$$

It's possible to get a sample version of the spatial depth by

$$\mathrm{SPD}(y; F_m) = 1 - \left\| \frac{1}{m} \sum_{i=1}^{m} S(y - Y_i) \right\|.$$

Spatial depth function is also calculated by using the following formula on the standardized version of $\mathbf{Y}$.

$$\mathrm{SPD}(y : F) = 1 - \left\| E_F \left[S\left(\Sigma^{-\frac{1}{2}} (y - \mathbf{Y})\right)\right] \right\|,$$

where Σ is the covariance matrix of F. It's robust version is obtained by using MCD based estimators. Let C be the estimator of Σ based on MCD. Thus, the robust spatial depth is

$$\mathrm{RSPD}(y : F) = 1 - \left\| \frac{1}{m} \sum_{i=1}^{m} S\left[C^{-\frac{1}{2}} (y - Y_i)\right] \right\|.$$

2.3. Mahalanobis depth

This depth function is obtained from the Mahalanobis distance (Mahalanobis, 1936). The Mahalanobis depth of any point $y \in \mathbf{R}^p$ with regards to F is denoted by MD($y; F$) and is given by,

$$\mathrm{MD}(y; F) = \frac{1}{1 + (y - \mu)' \Sigma^{-1} (y - \mu)},$$

where $\mu \in \mathbf{R}^p$ is the location parameter, and Σ is the covariance matrix of order $p \times p$ of distribution F. It is possible to obtain the sample version of this depth function by replacing μ with the sample mean $\bar{Y} \in \mathbf{R}^p$ and Σ with sample covariance matrix S_1. Thus, the sample Mahalanobis depth, MD$(y; F_m)$ is

$$\text{MD}(y; F_m) = \frac{1}{1 + \left(y - \bar{Y}\right)' S_1^{-1} \left(y - \bar{Y}\right)}.$$

Since it is based on the sample mean and covariance matrix, this version of depth function is not robust. MCD (minimum covariance determinant) or MVE (minimum volume ellipsoid) estimates of the sample mean (say $\bar{Y}_1$) and covariance matrix (say C) can be used to obtain a robust version of Mahalanobis depth. Thus, the Robust Mahalanobis depth, RMD$(y; F_m)$ of point y is

$$\text{RMD}(y; F_m) = \frac{1}{1 + \left(y - \bar{Y}_1\right)' C^{-1} \left(y - \bar{Y}_1\right)}.$$

2.4. Projection depth

Zuo (2003) defined this depth function. The projection depth of a point $y \in \mathbf{R}^p$ is denoted by PD$(y; F)$ and is given by,

$$\text{PD}(y; F) = \frac{1}{1 + O(y, F)},$$

where $O(y, F)$ represents the outlyingness of a point y with regards to the distribution F and is achieved in the following way.

$$O(y, F) = \sup_{\|v\|=1} h(y, v, F),$$

where $h(y, v, F) = |v'y - \mu(F_v)|/\sigma(F_v)$. Here, F_v represents the distribution of $v'\mathbf{Y}$, μ is the univariate location and σ is the univariate scale measure. The function $h(y, v, F)$ takes value 0 if $v'y - \mu(F_v) = \sigma(F_v) = 0$. The sample version of this depth function is achieved by considering F with its empirical distribution function, F_m.

3. Scale tests based on data depth

Let $\mathbf{X} = \{X_1, X_2, \ldots, X_{m_1}\}$ and $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_{m_2}\}$ be two random sample from 'p' variate probability distributions F_1 and F_2 respectively. Here, each $X_i, Y_j \in \mathbf{R}^p$, $i = 1, 2, \ldots, m_1$; $j = 1, 2, \ldots, m_2$. We assume here F_1 and F_2 have common location parameter but possibly different scale parameter. The hypothesis of interest is

H_0 : Both the distribution F_1 and F_2 have the same scale,

H_1 : The distribution F_2 has a larger scale than that of F_1.

3.1. Existing scale tests based on data depth

3.1.1. S-test (Li and Liu, 2016):

Li and Liu (2016) proposed a test for the above hypothesis based on depth differences by observing the structure of the DD-plot. Let us denote the combined sample by $\mathbf{Z} = \mathbf{X} \cup \mathbf{Y}$. The test statistic S is

$$S = \sum_{w \in \mathbf{Z}} \left(D(w; F_2) - D(w; F_1)\right).$$

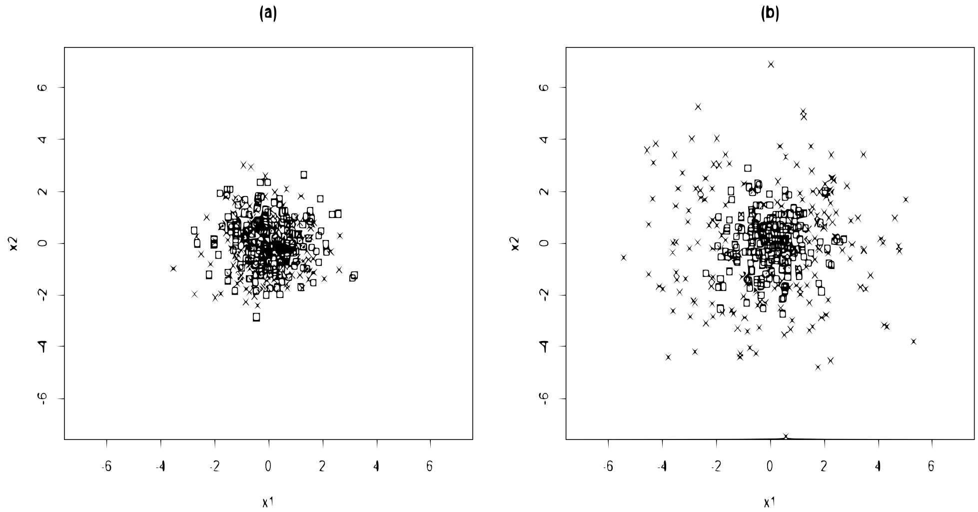

Figure 1: *Two sample bivariate plot.*

Table 1: Bivariate distributions used in the simulation study

Distributions	Distributions with parameters set up
Normal	$F_1 \sim N_2(\mu, \mathbf{I}_2), F_2 \sim N_2(\mu, \Sigma_2)$
Cauchy	$F_1 \sim C_2(\mu, \mathbf{I}_2), F_2 \sim C_2(\mu, \Sigma_2)$
Skew-Normal pattern I	$F_1 \sim SN_2(\mu, \mathbf{I}_2, \theta = (4, 10))$, $F_2 \sim SN_2(\mu, \Sigma_2, \theta = (4, 10))$
Skew-Normal pattern II	$F_1 \sim SN_2(\mu, \mathbf{I}_2, \theta = (10, 4))$, $F_2 \sim SN_2(\mu, \Sigma_2, \theta = (10, 4))$, where $\Sigma_2 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$
Clayton gamma	$F_1 \sim \text{Claygam}_2((1, 1), (1, 1), \delta = 0.5)$, $F_2 \sim \text{Claygam}_2((1, \sigma_1), (1, \sigma_2), \delta = 0.5)$
Frank gamma	$F_1 \sim \text{Frankgam}_2((1, 1), (1, 1), \delta = 0.5)$, $F_2 \sim \text{Frankgam}_2((1, \sigma_1), (1, \sigma_2), \delta = 0.5)$
Gumbel gamma	$F_1 \sim \text{Gumbgam}_2((1, 1), (1, 1), \delta = 1.5)$, $F_2 \sim \text{Gumbgam}_2((1, \sigma_1), (1, \sigma_2), \delta = 1.5)$

If F_2 has a larger scale than F_1, the depth values with respect to F_2 become larger, and consequently, S becomes larger. Thus, hypothesis H_0 is rejected for larger values of S. The p-value for the above defined test is obtained using the fisher permutation test (Efron and Tibshirani, 1994) with the formula below.

$$P_B^S = \frac{\sum_{i=1}^{B} I\left(S_i^* \geq S_{\text{obs}}\right)}{B},$$

where $I(\cdot)$ represents the indicator function, B denotes the permutation number, S_i^* represent the test statistic value for i^{th} permutation, and S_{obs} denotes the observed value of test statistic based on the actual sample.

Table 2: Trivariate distributions used in the simulation study

Distributions	Distributions with parameters set up
Normal	$F_1 \sim N_3(\mu_1, \mathbb{I}_3), F_2 \sim N_3(\mu_1, \Sigma_3)$
Cauchy	$F_1 \sim C_3(\mu_1, \mathbb{I}_3), F_2 \sim C_3(\mu_1, \Sigma_3)$
Skew-Normal	$F_1 \sim SN_3(\mu_1, \mathbb{I}_3, \theta = (10, 4, 4))$, $F_2 \sim SN_3(\mu_1, \Sigma_3, \theta = (10, 4, 4))$, where $\Sigma_3 = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$
Clayton gamma	$F_1 \sim \text{Claygam}_3((1, 1), (1, 1), (1, 1), \delta = 0.5)$, $F_2 \sim \text{Claygam}_3((1, \sigma_1), (1, \sigma_2), (1, \sigma_3), \delta = 0.5)$
Frank gamma	$F_1 \sim \text{Frankgam}_3((1, 1), (1, 1), (1, 1), \delta = 0.5)$, $F_2 \sim \text{Frankgam}_3((1, \sigma_1), (1, \sigma_2), (1, \sigma_3), \delta = 0.5)$
Gumbel gamma	$F_1 \sim \text{Gumbgam}_3((1, 1), (1, 1), (1, 1), \delta = 1.5)$, $F_2 \sim \text{Gumbgam}_3((1, \sigma_1), (1, \sigma_2), (1, \sigma_3), \delta = 1.5)$

Table 3: Distributions (having dimension 10) used in the simulation study

Distributions	Distributions with parameters set up
Normal	$F_1 \sim N_{10}(\mu_2, \mathbb{I}_{10}), F_2 \sim N_{10}(\mu_2, \Sigma_{10})$
Cauchy	$F_1 \sim C_{10}(\mu_2, \mathbb{I}_{10}), F_2 \sim C_{10}(\mu_2, \Sigma_{10})$.
Skew-Normal	$F_1 \sim SN_{10}(\mu_2, \mathbb{I}_{10}, \theta = (10, 10, 4, 4, 4, 4, 4, 4, 4, 4))$, $F_2 \sim SN_{10}(\mu_2, \Sigma_{10}, \theta = (10, 10, 4, 4, 4, 4, 4, 4, 4, 4))$, where $\Sigma_{10} = \begin{bmatrix} \Sigma_{11} & 0_{2,8} \\ 0_{8,2} & \Sigma_{22} \end{bmatrix}$, $\Sigma_{11} = \sigma^1 \mathbb{I}_2, \Sigma_{22} = \sigma^2 \mathbb{I}_8, 0_{2,8}$ = Zero matrix of order 2 X 8, $0_{8,2}$ = Zero matrix of order 8 X 2, $\mathbb{I}_w$ = Identity matrix of order w.
Clayton gamma	$F_1 \sim \text{Claygam}_{10}((1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), \delta = 0.5)$, $F_2 \sim \text{Claygam}_{10}((1, \sigma^1), (1, \sigma^1), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), \delta = 0.5)$
Frank gamma	$F_1 \sim \text{Frankgam}_{10}((1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), \delta = 0.5)$, $F_2 \sim \text{Frankgam}_{10}((1, \sigma^1), (1, \sigma^1), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), \delta = 0.5)$
Gumbel gamma	$F_1 \sim \text{Gumbgam}_{10}((1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), (1, 1), \delta = 1.5)$, $F_2 \sim \text{Gumbgam}_{10}((1, \sigma^1), (1, \sigma^1), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), (1, \sigma^2), \delta = 1.5)$

3.1.2. S_1 and S_2 Test (Chavan and Shirke, 2020):

Chavan and Shirke (2020) proposed depth-based nonparametric tests based on depth differences. The test statistics S_1 and S_2 are

$$S_1 = \sum_{w \in \mathbf{Z}} \sum_{w^* \in \mathbf{Z}} |D(w; F_1) - D(w^*; F_2)|,$$

and

$$S_2 = \sum_{w \in \mathbf{Z}} \sum_{w^* \in \mathbf{Z}} (D(w; F_1) - D(w^*; F_2))^2,$$

larger differences in scale parameters of two distributions lead to higher values of test statistics. Therefore larger values of S_1 and S_2 provide evidence against H_0. The procedure of finding p-value is the same as that of the S-test defined above.

Table 4: Simulated powers for the tests with different depth functions for bivariate normal, cauchy and skew-normal pattern I distribution with $m_1 = m_2 = 50$

Depth function	$\sigma_1 = \sigma_2$	Normal				Cauchy				Skew-Normal pattern I			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
HD	1.0	0.050	0.052	0.048	0.058	0.060	0.051	0.050	0.061	0.056	0.046	0.049	0.049
	1.2	0.193	0.097	0.096	**0.195**	**0.120**	0.055	0.056	0.112	**0.163**	0.088	0.081	0.159
	1.4	**0.415**	0.255	0.227	0.401	0.154	0.085	0.084	**0.159**	**0.278**	0.152	0.141	0.267
	1.6	**0.640**	0.449	0.386	0.635	**0.233**	0.123	0.120	0.223	**0.466**	0.283	0.267	0.452
	1.8	0.808	0.636	0.574	**0.810**	**0.313**	0.184	0.180	0.309	**0.573**	0.350	0.333	0.559
	2.0	**0.912**	0.758	0.701	**0.912**	**0.410**	0.232	0.224	0.403	**0.697**	0.413	0.406	0.673
	2.5	**0.986**	0.937	0.910	0.985	**0.547**	0.352	0.350	0.516	**0.862**	0.599	0.589	0.839
PD	1.0	0.049	0.050	0.050	0.045	0.049	0.044	0.050	0.047	0.049	0.050	0.059	0.053
	1.2	**0.162**	0.082	0.071	0.148	0.098	0.066	0.060	**0.101**	**0.146**	0.070	0.060	0.133
	1.4	**0.384**	0.163	0.140	0.306	**0.164**	0.084	0.078	0.150	**0.330**	0.119	0.100	0.270
	1.6	**0.558**	0.275	0.225	0.453	**0.247**	0.133	0.128	0.232	**0.494**	0.204	0.173	0.380
	1.8	**0.731**	0.386	0.312	0.623	**0.351**	0.179	0.153	0.317	**0.655**	0.283	0.245	0.510
	2.0	**0.817**	0.512	0.435	0.720	**0.418**	0.222	0.205	0.399	**0.785**	0.374	0.323	0.649
	2.5	**0.956**	0.741	0.643	0.893	**0.603**	0.337	0.295	0.568	**0.940**	0.543	0.457	0.842
RMD	1.0	0.047	0.048	0.049	0.045	0.047	0.049	0.049	0.048	0.044	0.046	0.052	0.044
	1.2	0.189	0.126	0.112	**0.197**	**0.071**	0.050	0.049	0.063	**0.187**	0.096	0.101	0.178
	1.4	0.420	0.298	0.292	**0.440**	**0.109**	0.070	0.070	0.107	**0.330**	0.214	0.215	0.325
	1.6	0.697	0.559	0.534	**0.708**	**0.154**	0.082	0.080	0.132	0.512	0.375	0.359	**0.520**
	1.8	0.824	0.713	0.705	**0.839**	**0.178**	0.110	0.109	0.152	**0.670**	0.533	0.525	0.666
	2.0	0.918	0.860	0.843	**0.929**	**0.181**	0.106	0.100	0.156	0.770	0.639	0.627	**0.774**
	2.5	**0.982**	0.970	0.962	0.981	**0.260**	0.135	0.128	0.205	**0.957**	0.849	0.829	0.948
RSPD	1.0	0.045	0.054	0.058	0.046	0.054	0.052	0.055	0.055	0.052	0.042	0.044	0.051
	1.2	0.178	0.096	0.090	**0.184**	**0.100**	0.064	0.064	0.097	0.154	0.091	0.091	**0.155**
	1.4	**0.377**	0.250	0.236	0.368	0.157	0.098	0.095	**0.162**	**0.298**	0.167	0.171	0.289
	1.6	0.656	0.460	0.435	**0.665**	**0.261**	0.158	0.152	**0.261**	**0.447**	0.314	0.310	0.432
	1.8	**0.793**	0.644	0.613	0.786	**0.335**	0.221	0.215	0.323	**0.604**	0.432	0.431	0.583
	2.0	**0.898**	0.792	0.770	0.896	**0.421**	0.261	0.262	0.410	**0.693**	0.542	0.540	0.671
	2.5	0.980	0.942	0.927	**0.982**	**0.580**	0.420	0.418	0.573	**0.907**	0.760	0.746	0.894

The results of these tests are analyzed using the permutation test, and the comparison with the Liu and Singh (2006) test is carried out. These tests showed improved power than the test by Liu and Singh (2006).

3.2. Proposed scale test based on data depth

If we assume F_2 has a larger scale than F_1, then $Y_j^{'s}$ are more scattered than $X_i^{'s}$ around the center. In other words, $X_i^{'s}$ are closer to the center, and $Y_j^{'s}$ are more outlying or dispersed from the center, as shown in Figure 1. Two samples of size 200 are generated from a bivariate normal distribution with mean $(0, 0)$ and covariance matrix $\mathbb{I}_2$. One of them is indicated by $\times$, and another sample is indicated by $\square$, as shown in Figure 1(a). In Figure 1(b), the points indicated by $\square$ are generated from a bivariate normal distribution with mean $(0, 0)$ and covariance matrix $\mathbb{I}$, and the points indicated by $\times$ are generated from a bivariate normal distribution with mean $(0, 0)$ and covariance matrix $5{*}\mathbb{I}_2$. Hence, for measuring the closeness and outlyingness of the multivariate data points from the center, we adopt the concept of data depth, which results in ordering the data from center to outward. So, we obtain the depth values of $X_i, i = 1, 2, \ldots, m_1$ with respect to F_2 and the depth values of $Y_j, j = 1, 2, \ldots, m_2$ with respect to F_1. That is, we obtain the depth values of one sample with respect to another sample and vice-versa. Then assign ranks to the depth values in ascending order, i.e., lower rank assigned to lower depth value. If there is a larger scale in F_2 than that of F_1, then $X_i^{'s}$ are more tightly close to the center, and hence its depth values with respect to F_2 are larger than the depth values of $Y_j's$ calculated with respect to F_1. For example, in Figure 1(b), the depth values of the points marked by $\times$ with respect to the points marked by $\square$ are smaller than the depth values of the points indicated by $\square$ with respect to the points indicated by $\times$.

Table 5: Simulated powers for the tests with different depth functions for bivariate skew-normal pattern II, Clayton gamma, Gumbel gamma distribution with $m_1 = m_2 = 50$

Depth function	$\sigma_1 = \sigma_2$	Skew-Normal pattern II				Clayton gamma				Gumbel gamma			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
HD	1.0	0.061	0.050	0.048	0.051	0.062	0.055	0.049	0.062	0.039	0.052	0.049	0.042
	1.2	**0.147**	0.079	0.079	0.139	**0.100**	0.057	0.064	0.093	0.118	0.078	0.087	**0.123**
	1.4	**0.288**	0.147	0.144	0.277	**0.226**	0.132	0.146	0.220	0.227	0.139	0.142	**0.229**
	1.6	**0.423**	0.248	0.235	0.408	**0.274**	0.151	0.180	0.261	0.297	0.173	0.195	**0.299**
	1.8	**0.595**	0.376	0.352	0.582	**0.394**	0.186	0.214	0.351	**0.393**	0.219	0.266	0.385
	2.0	**0.699**	0.439	0.421	0.694	**0.476**	0.202	0.247	0.396	0.445	0.227	0.303	**0.456**
	2.5	**0.875**	0.630	0.597	0.853	**0.583**	0.148	0.216	0.431	**0.621**	0.209	0.340	0.572
PD	1.0	0.034	0.034	0.039	0.048	0.045	0.049	0.051	0.045	0.045	0.047	0.048	0.054
	1.2	**0.165**	0.078	0.066	0.133	**0.252**	0.092	0.083	0.212	**0.234**	0.081	0.071	0.210
	1.4	**0.332**	0.125	0.122	0.259	**0.510**	0.165	0.150	0.407	**0.482**	0.198	0.179	0.414
	1.6	**0.513**	0.216	0.184	0.383	**0.732**	0.253	0.221	0.618	**0.708**	0.297	0.264	0.623
	1.8	**0.653**	0.290	0.236	0.520	**0.892**	0.337	0.293	0.803	**0.855**	0.444	0.388	0.771
	2.0	**0.763**	0.355	0.320	0.622	**0.954**	0.401	0.330	0.884	**0.945**	0.529	0.447	0.899
	2.5	**0.922**	0.544	0.440	0.808	**0.995**	0.512	0.425	0.983	**0.994**	0.652	0.568	0.980
RMD	1.0	0.046	0.038	0.046	0.047	0.057	0.052	0.055	0.051	0.045	0.039	0.036	0.045
	1.2	**0.145**	0.079	0.076	0.143	**0.151**	0.093	0.088	0.141	**0.138**	0.080	0.083	0.125
	1.4	**0.332**	0.205	0.206	0.327	**0.295**	0.193	0.175	0.241	**0.267**	0.160	0.175	0.222
	1.6	**0.526**	0.380	0.368	**0.526**	**0.415**	0.309	0.307	0.341	**0.438**	0.291	0.310	0.349
	1.8	**0.673**	0.525	0.509	0.669	**0.609**	0.473	0.458	0.494	**0.570**	0.423	0.443	0.459
	2.0	**0.768**	0.648	0.635	0.764	**0.705**	0.554	0.519	0.551	**0.684**	0.497	0.494	0.553
	2.5	**0.935**	0.842	0.829	0.932	**0.902**	0.667	0.621	0.701	**0.867**	0.600	0.615	0.647
RSPD	1.0	0.043	0.041	0.041	0.042	0.050	0.055	0.052	0.045	0.053	0.054	0.051	0.053
	1.2	**0.138**	0.085	0.086	0.135	**0.143**	0.099	0.100	0.138	**0.134**	0.071	0.072	0.127
	1.4	**0.291**	0.171	0.173	0.284	**0.295**	0.170	0.172	0.275	**0.264**	0.174	0.187	0.237
	1.6	**0.462**	0.279	0.277	0.445	**0.434**	0.270	0.285	0.402	**0.425**	0.293	0.299	0.388
	1.8	**0.615**	0.425	0.416	0.592	**0.596**	0.405	0.403	0.548	**0.566**	0.443	0.451	0.524
	2.0	**0.706**	0.552	0.536	0.678	**0.687**	0.492	0.491	0.602	**0.666**	0.559	0.578	0.630
	2.5	**0.888**	0.753	0.749	0.871	**0.890**	0.680	0.671	0.812	**0.856**	0.733	0.737	0.792

As a result, the rank associated with the depth values of $Y_j^{\prime s}$ are smaller than that of $X_i^{\prime s}$, and hence the sum of the rank of depth values of $Y_j^{\prime s}$ with respect to F_1 is also smaller. Thus, if H_0 is true, then the rank sum of depth values of $X_i^{\prime s}$ with respect to F_2 and that of $Y_j^{\prime s}$ with respect to F_1 are approximately equal, and if H_0 is rejected, then the rank sum of depth values of $Y_j^{\prime s}$ with respect to F_1 is smaller than the rank sum of depth values of $X_i^{\prime s}$ with respect to F_2. Hence, we consider the test statistic as the sum of the rank of depth values of $Y_j^{\prime s}$ with respect to F_1. Mathematically, let

$$D^{F_2}(X_i) = D(X_i; F_2); \quad i = 1, 2, \ldots, m_1,$$

and

$$D^{F_1}\left(Y_j\right) = D\left(Y_j; F_1\right); \quad j = 1, 2, \ldots, m_2.$$

It's sample versions are denoted by,

$$D^{\mathbf{Y}}(X_i) = D(X_i; \mathbf{Y}); \quad i = 1, 2, \ldots, m_1,$$

and

$$D^{\mathbf{X}}\left(Y_j\right) = D\left(Y_j; \mathbf{X}\right); \quad j = 1, 2, \ldots, m_2,$$

where $D(X_i; \mathbf{Y})$ denote the sample depth value of $X_i; i = 1, 2, \ldots, m_1$ with respect to the sample $\mathbf{Y}$ and $D(Y_j; \mathbf{X})$ denote the sample depth value of $Y_j; j = 1, 2, \ldots, m_2$ with respect to the sample $\mathbf{X}$. Then we

Table 6: Simulated powers for the tests with different depth functions for bivariate Frank gamma distribution with $m_1 = m_2 = 50$

Depth functions	$\sigma_1 = \sigma_2$	Frank gamma			
		R_O	S_1	S_2	S
HD	1.0	0.047	0.051	0.058	0.052
	1.2	**0.099**	0.069	0.072	0.094
	1.4	**0.156**	0.098	0.121	0.151
	1.6	**0.220**	0.123	0.165	0.202
	1.8	**0.287**	0.138	0.216	0.240
	2.0	**0.305**	0.133	0.223	0.259
	2.5	**0.394**	0.103	0.208	0.251
PD	1.0	0.057	0.059	0.058	0.053
	1.2	**0.233**	0.098	0.090	0.206
	1.4	**0.532**	0.199	0.181	0.458
	1.6	**0.760**	0.292	0.236	0.662
	1.8	**0.895**	0.401	0.345	0.813
	2.0	**0.967**	0.465	0.379	0.906
	2.5	**0.996**	0.527	0.420	0.981
RMD	1.0	0.047	0.055	0.053	0.048
	1.2	**0.129**	0.081	0.083	0.126
	1.4	**0.275**	0.210	0.214	0.243
	1.6	**0.422**	0.343	0.325	0.335
	1.8	**0.553**	0.465	0.447	0.452
	2.0	**0.706**	0.548	0.527	0.506
	2.5	**0.904**	0.622	0.580	0.660
RSPD	1.0	0.056	0.045	0.048	0.055
	1.2	**0.145**	0.084	0.092	0.138
	1.4	**0.261**	0.188	0.200	0.243
	1.6	**0.380**	0.274	0.288	0.347
	1.8	**0.513**	0.385	0.390	0.447
	2.0	**0.637**	0.473	0.489	0.547
	2.5	**0.841**	0.626	0.635	0.695

get

$$\begin{aligned} D^{\text{opp}}(Z) &= \{Z_1, Z_2, \ldots, Z_{m_1}, Z_{m_1+1}, \ldots, Z_{m_1+m_2}\} \\ &= \left\{D^{\mathbf{Y}}(X_1), D^{\mathbf{Y}}(X_2), \ldots, D^{\mathbf{Y}}(X_{m_1}), D^{\mathbf{X}}(Y_1), \ldots, D^{\mathbf{X}}(Y_{m_2})\right\}. \end{aligned}$$

Now, assign ranks to $D^{\text{opp}}(Z)$ according to ascending order and is denoted by

$$\left\{R\left(D^{\mathbf{Y}}(X_1)\right), R\left(D^{\mathbf{Y}}(X_2)\right), \ldots, R\left(D^{\mathbf{Y}}(X_{m_1})\right), R\left(D^{\mathbf{X}}(Y_1)\right), \ldots, R\left(D^{\mathbf{X}}(Y_{m_2})\right)\right\}.$$

Then the proposed test statistic, R_O for testing H_0 is

$$R_O = \sum_{j=1}^{m_2} R\left(D^{\mathbf{X}}\left(Y_j\right)\right).$$

If H_0 is false, then the ranks associated with $D^{\mathbf{X}}(Y_j)$, $j = 1, 2, \ldots, m_2$ are smaller, and hence the sum of rank is also smaller. Hence we reject the null hypothesis for smaller values of R_O. Moreover, computational facilities have improved, we can now use the permutation test to obtain the exact p-value of the test in the presence of ties (see Li and Liu, 2004, 2016). Therefore, the p-value for the above test can be determined using the below equation.

$$p\text{-value} = P_{H_0}\left(R_O < R_O^{\text{obs}}\right),$$

where R_O^{obs} is the observed value of R_O based on the sample $\mathbf{X}$ and $\mathbf{Y}$. To perform Fisher's permutation test, follow the following steps.

Step 1 : Permute $\mathbf{X} \cup \mathbf{Y}$ for a sufficiently large number of times, say B times. Then consider the first m_1 observations as the first sample and the rest m_2 observations as observations of the second sample. When it comes to the l^{th} permutation, the samples are denoted by $\mathbf{X}_l^* = \{X_{l1}^*, X_{l2}^*, \ldots, X_{lm_1}^*\}$, and $\mathbf{Y}_l^* = \{Y_{l1}^*, Y_{l2}^*, \ldots, Y_{lm_2}^*\}, l = 1, 2, \ldots, B$.

Step 2 : For these new samples, evaluate the test statistic value and denote it by $R_{O_l}^*$ for the l^{th} permutation, $l = 1, 2, \ldots, B$.

Step 3 : The p-value can now be calculated using the following equation.

$$p\text{-value} = \sum_{l=1}^{B} \frac{I\left(R_{O_l}^* \leq R_O^{\text{obs}}\right)}{B}.$$

4. Simulation study

The performance of the proposed test is investigated through an extensive simulation study by comparing the proposed test with the existing depth-based nonparametric test by Li and Liu (2016) (denoted by S) and the tests by Chavan and Shirke (2020) (denoted by S_1 and S_2).

We take permutations in large enough number of times to approximate the distribution of test statistics. To examine the performance of the proposed test with regards to empirical power, we have considered two-dimensional, three dimensional, and ten dimensional symmetric and skew distributions as provided in Tables 1–3.

- Normal and Cauchy distribution: As shown in Table 1, for normal distribution, F_1 is taken as a bivariate normal distribution with location parameter μ and scale parameter $\mathbb{I}_2$ and F_2 as a bivariate normal distribution with the same location parameter μ and scale parameter Σ_2, where $\mathbb{I}_2$ is an identity matrix of order 2. For Cauchy distribution, F_1 is taken as bivariate Cauchy distribution with location parameter μ and scale parameter $\mathbb{I}_2$ and F_2 as bivariate Cauchy distribution with the same location parameter μ and scale parameter Σ_2. Similarly, Tables 2 and 3 show the parameter settings for the three- and ten-dimensional distributions employed in the simulation study.

- Skew-Normal distribution: For skew-normal, the first parameter μ indicates the location parameter, the second parameter is the scale parameter, and the third parameter θ be the shape parameter, as shown in Table 1. Two patterns have been taken with two shape parameters. The pattern I is with $\theta = (4, 10)$, and pattern II is with $\theta = (10, 4)$. Similarly, for dimension three and ten, the parameter settings used for skew normal distribution is provided in Tables 2 and 3 respectively. More details about skew-normal have been found in Dovoedo and Chakraborti (2015), Pawar and Shirke (2019).

- Clayton gamma, Frank gamma, and Gumbel gamma distribution: We consider three distributions from Archimedean copula families Clayton, Frank, and Gumbel with gamma marginals. For more details, see Dovoedo and Chakraborti (2015), Pawar and Shirke (2019). As shown in Table 1, we have taken gamma marginals with shape parameter 1 and scale parameter 1 with regards to F_1 for each family. Regarding F_2, the gamma marginals for each family are considered with the shape parameter 1 and scale parameter as σ_1 and σ_2. For Clayton and Frank distribution, the parameter of generator δ is 0.5, and for Gumbel distribution, the parameter of generator δ is 1.5. Similarly, for dimension three and ten, the parameter settings used for these distributions is provided in Tables 2 and 3 respectively. We used the 'copula' package in R software to generate a random sample from these three families of distributions.

Table 7: Simulated powers for the tests with different depth functions for bivariate normal, Cauchy, and skew-normal pattern II distribution with $m_1 = m_2 = 50$ and different values of σ_1 and σ_2

Depth function	(σ_1, σ_2)	Normal				Cauchy				Skew-Normal pattern II			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
HD	(1.0, 1.0)	0.049	0.050	0.054	0.045	0.061	0.051	0.049	0.058	0.052	0.060	0.056	0.053
	(1.0, 1.2)	0.097	0.074	0.076	**0.099**	**0.078**	0.056	0.050	0.075	**0.104**	0.058	0.069	**0.104**
	(1.2, 1.5)	0.353	0.192	0.168	**0.355**	**0.131**	0.073	0.075	**0.131**	0.276	0.154	0.140	**0.280**
	(1.5, 2.0)	**0.751**	0.563	0.509	0.750	**0.315**	0.168	0.161	0.298	0.591	0.345	0.311	**0.594**
	(1.0, 2.5)	**0.595**	0.431	0.409	0.587	**0.224**	0.128	0.134	0.219	0.518	0.390	0.363	**0.528**
	(1.2, 2.0)	**0.585**	0.381	0.341	0.566	**0.217**	0.126	0.124	0.207	**0.467**	0.290	0.259	**0.467**
	(1.4, 1.8)	**0.615**	0.395	0.359	0.599	**0.252**	0.144	0.140	0.248	0.463	0.279	0.252	**0.465**
PD	(1.0, 1.0)	0.053	0.039	0.035	0.053	0.060	0.051	0.050	0.055	0.055	0.067	0.063	0.055
	(1.0, 1.2)	**0.095**	0.053	0.048	0.082	0.073	0.055	0.056	**0.074**	0.100	0.064	0.060	**0.105**
	(1.2, 1.5)	**0.288**	0.111	0.106	0.233	**0.135**	0.071	0.078	0.134	**0.268**	0.093	0.080	0.214
	(1.5, 2.0)	**0.665**	0.351	0.284	0.551	**0.320**	0.158	0.130	0.302	**0.605**	0.252	0.208	0.505
	(1.0, 2.5)	**0.495**	0.272	0.246	0.428	**0.242**	0.111	0.106	0.231	**0.487**	0.230	0.196	0.404
	(1.2, 2.0)	**0.503**	0.238	0.203	0.409	**0.233**	0.110	0.103	0.226	**0.447**	0.197	0.166	0.371
	(1.4, 1.8)	**0.516**	0.242	0.201	0.420	**0.270**	0.139	0.131	0.262	**0.474**	0.201	0.170	0.372
RMD	(1.0, 1.0)	0.046	0.040	0.041	0.044	0.050	0.045	0.045	0.051	0.050	0.053	0.054	0.051
	(1.0, 1.2)	0.101	0.065	0.062	**0.117**	0.056	0.054	0.055	**0.061**	**0.106**	0.073	0.070	0.094
	(1.2, 1.5)	0.355	0.234	0.231	**0.371**	**0.082**	0.053	0.055	0.071	0.289	0.184	0.184	**0.299**
	(1.5, 2.0)	0.758	0.662	0.646	**0.768**	**0.151**	0.098	0.102	0.127	0.648	0.485	0.493	**0.650**
	(1.0, 2.5)	0.613	0.554	0.562	**0.646**	**0.120**	0.065	0.071	0.091	0.543	0.468	0.492	**0.548**
	(1.2, 2.0)	0.591	0.472	0.459	**0.607**	**0.138**	0.069	0.073	0.125	0.505	0.381	0.375	**0.513**
	(1.4, 1.8)	0.639	0.491	0.474	**0.650**	**0.127**	0.080	0.079	0.095	0.510	0.363	0.359	**0.521**
RSPD	(1.0, 1.0)	0.045	0.044	0.040	0.053	0.063	0.055	0.057	0.060	0.047	0.062	0.064	0.052
	(1.0, 1.2)	**0.105**	0.075	0.071	0.102	0.062	0.053	0.051	**0.064**	**0.106**	0.059	0.062	0.105
	(1.2, 1.5)	0.345	0.210	0.196	**0.352**	0.134	0.086	0.091	**0.145**	**0.274**	0.161	0.156	0.267
	(1.5, 2.0)	**0.745**	0.596	0.573	0.743	0.296	0.204	0.198	**0.304**	**0.598**	0.418	0.406	0.580
	(1.0, 2.5)	0.584	0.512	0.505	**0.592**	0.222	0.144	0.145	**0.224**	0.482	0.426	0.417	**0.485**
	(1.2, 2.0)	0.562	0.427	0.410	**0.572**	0.224	0.142	0.138	**0.226**	**0.447**	0.300	0.295	0.427
	(1.4, 1.8)	**0.626**	0.454	0.436	0.624	**0.246**	0.139	0.141	0.236	**0.458**	0.300	0.290	0.440

The random sample of size $m_1 = m_2 = 50$ is generated from each distribution F_1 and F_2, specified in Tables 1–3. The location parameter, $\mu = (0, 0)$, $\mu_1 = (0, 0, 0)$, and $\mu_2 = 0_{10}$, where 0_{10} is the zero vector of dimension 10. We calculate the p-value of the tests by performing 500 permutations of the original sample and then repeating the process 1,000 times to obtain the powers of the tests, which is calculated as the proportion of times the p-value is less than or equal to the nominal significance level 0.05. The powers of the tests for different scale shifts for the distributions with dimension two, three, and ten with halfspace depth (HD), projection depth (PD), Robust Mahalanobis depth (RMD), and robust spatial depth (RSPD) are reported in Tables 4–13.

When the proposed test is compared with existing depth-based nonparametric tests using different depth functions for bivariate and trivariate case, Tables 4–11 proves that the proposed test performs either comparable or better than existing depth-based tests except for normal distribution with RMD. For some distributions with some shifts, the proposed test performs weaker than the S-test. But for the majority of times, the proposed test works better than the S-test. In the simulation study, we have considered seven bivariate distributions, four depth functions, and 12 patterns under H_1(shift in scale). As a result, there are 312 different settings in the bivariate case, and among them, the proposed test provides similar or better powers for 258 settings (see Tables 4–8). Also, there are 192 different settings in the trivariate case, and among them, the proposed test performs equivalent or better for 173 settings (see Tables 9–11).

Tables 4 and 7 reveals that the proposed test performs comparable or better than the existing tests for the maximum number of settings of shift in scale in the case of Cauchy distribution. In other words, we have considered four depth functions and 12 patterns under H_1(shift in scale) for the Cauchy distribution. So, there are total 48 different settings in the bivariate case, and among them, the

Table 8: Simulated powers for the tests with different depth functions for bivariate Clayton gamma, Gumbel gamma, and Frank gamma distribution with $m_1 = m_2 = 50$ and different values of σ_1 and σ_2

Depth function	(σ_1, σ_2)	Clayton gamma				Gumbel gamma				Frank gamma			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
HD	(1.0, 1.0)	0.056	0.068	0.066	0.058	0.043	0.048	0.053	0.042	0.046	0.047	0.042	0.050
	(1.0, 1.2)	0.065	0.050	0.058	**0.066**	0.083	0.056	0.066	**0.085**	0.064	0.057	0.056	**0.065**
	(1.2, 1.5)	**0.168**	0.095	0.098	0.160	0.180	0.098	0.123	**0.189**	**0.142**	0.082	0.099	0.137
	(1.5, 2.0)	**0.358**	0.174	0.221	0.330	**0.379**	0.186	0.250	0.359	**0.267**	0.131	0.205	0.245
	(1.0, 2.5)	**0.243**	0.110	0.202	0.166	**0.239**	0.051	0.169	0.146	**0.164**	0.083	0.197	0.121
	(1.2, 2.0)	**0.244**	0.118	0.180	0.215	**0.269**	0.111	0.187	0.219	**0.211**	0.109	0.184	0.193
	(1.4, 1.8)	**0.268**	0.134	0.160	0.265	**0.327**	0.174	0.216	0.324	**0.216**	0.098	0.154	0.201
PD	(1.0, 1.0)	0.054	0.053	0.051	0.049	0.048	0.046	0.044	0.045	0.036	0.050	0.055	0.038
	(1.0, 1.2)	**0.103**	0.064	0.064	0.095	**0.101**	0.065	0.069	**0.101**	**0.104**	0.058	0.059	0.095
	(1.2, 1.5)	**0.399**	0.142	0.130	0.315	**0.393**	0.164	0.145	0.348	**0.407**	0.155	0.149	0.348
	(1.5, 2.0)	**0.860**	0.367	0.332	0.732	**0.805**	0.395	0.356	0.717	**0.872**	0.380	0.328	0.766
	(1.0, 2.5)	**0.690**	0.381	0.360	0.581	**0.626**	0.414	0.389	0.502	**0.699**	0.386	0.350	0.614
	(1.2, 2.0)	**0.689**	0.271	0.254	0.552	**0.641**	0.333	0.304	0.546	**0.716**	0.300	0.270	0.599
	(1.4, 1.8)	**0.719**	0.260	0.229	0.598	**0.702**	0.323	0.268	0.616	**0.730**	0.284	0.252	0.638
RMD	(1.0, 1.0)	0.052	0.061	0.061	0.051	0.046	0.060	0.060	0.051	0.047	0.047	0.041	0.049
	(1.0, 1.2)	**0.084**	0.057	0.061	0.079	**0.084**	0.064	0.059	0.076	**0.085**	0.067	0.068	0.080
	(1.2, 1.5)	**0.233**	0.157	0.155	0.200	**0.247**	0.156	0.171	0.210	**0.229**	0.165	0.173	0.199
	(1.5, 2.0)	**0.543**	0.429	0.426	0.462	**0.525**	0.361	0.376	0.420	**0.544**	0.425	0.433	0.429
	(1.0, 2.5)	**0.399**	0.408	0.428	0.303	**0.352**	0.367	0.387	0.236	**0.383**	0.404	0.418	0.274
	(1.2, 2.0)	**0.414**	0.352	0.351	0.324	**0.370**	0.285	0.312	0.276	**0.378**	0.355	0.347	0.316
	(1.4, 1.8)	**0.456**	0.329	0.324	0.371	**0.435**	0.305	0.334	0.335	**0.385**	0.319	0.318	0.317
RSPD	(1.0, 1.0)	0.051	0.067	0.070	0.054	0.044	0.049	0.045	0.042	0.040	0.043	0.049	0.037
	(1.0, 1.2)	0.075	0.056	0.061	**0.076**	**0.079**	0.055	0.057	0.075	**0.078**	0.061	0.063	0.073
	(1.2, 1.5)	**0.250**	0.143	0.148	0.232	**0.235**	0.167	0.174	0.220	**0.228**	0.138	0.152	0.218
	(1.5, 2.0)	**0.533**	0.331	0.348	0.483	**0.485**	0.390	0.398	0.448	**0.442**	0.335	0.340	0.392
	(1.0, 2.5)	**0.411**	0.303	0.322	0.325	**0.367**	0.256	0.308	0.268	**0.372**	0.320	0.342	0.300
	(1.2, 2.0)	**0.397**	0.297	0.305	0.363	**0.333**	0.268	0.276	0.303	**0.340**	0.265	0.276	0.312
	(1.4, 1.8)	**0.444**	0.281	0.291	0.412	**0.400**	0.308	0.329	0.372	**0.388**	0.274	0.294	0.355

proposed test gives comparable or better results for 38 settings. Similarly, in the trivariate case, the proposed test gives better results than the other tests in 29 out of 32 settings for Cauchy distribution (see Tables 9 and 11). For Gumbel gamma distribution with the HD, the proposed test performs weaker than the S-test in some settings of the shift in scale. That means, under HD, the proposed test performs better for Gumbel gamma distribution than the other tests in 13 settings out of 22 settings of the shift in scale (see Tables 5, 8, 10, and 11). Furthermore, for the distributions with dimension 10, Tables 12–13 confirms that the performance of the proposed test in terms of power is better than the existing depth-based nonparametric tests considered here. Thus in general, the proposed test is a better alternative to the depth-based tests S, S_1, and S_2.

The proposed test statistic depends on the ranks of the depth values and the depth values varies as the depth function changes. As a result, the ranks based on depth values also changes, and consequently, the value of test statistic also varies. So, the resulting powers of the proposed test varies as a depth function changes. Therefore, in general, we draw the following conclusions. The proposed test performs better than existing S_1 and S_2 tests for all depth functions and distributions considered in this article. In case of the projection depth function, the proposed test gives better results than the existing S-test in all considered distributions. When the robust Mahalanobis depth function is used, the proposed test gives better results than the existing S-test for the distributions from the copula family and provides comparable powers with the existing S-test in all remaining considered distributions. For the remaining two depth functions (halfspace and robust spatial depth function), the proposed test gives comparable powers with the existing S-test in all considered distributions. Thus, the performance of the proposed test changes according to the depth function for the specified distribution due to the different nature of the depth function. The preference of the depth function is given to the projection

Table 9: Simulated powers for the tests with different depth functions for trivariate normal, Cauchy and skew-normal distribution with $m_1 = m_2 = 50$

Depth function	$\sigma_1 = \sigma_2 = \sigma_3$	Normal				Cauchy				Skew-Normal			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
HD	1.0	0.051	0.050	0.055	0.044	0.051	0.057	0.054	0.058	0.047	0.050	0.054	0.052
	1.2	**0.240**	0.119	0.104	0.236	**0.102**	0.064	0.068	0.101	**0.196**	0.098	0.076	0.192
	1.4	**0.564**	0.308	0.248	0.530	**0.173**	0.089	0.081	0.153	**0.442**	0.228	0.196	0.429
	1.6	0.795	0.552	0.457	**0.796**	**0.271**	0.154	0.166	0.244	**0.667**	0.360	0.315	0.631
	1.8	**0.927**	0.738	0.666	0.915	**0.336**	0.182	0.188	0.317	**0.818**	0.509	0.427	0.801
	2.0	**0.980**	0.863	0.797	0.979	**0.419**	0.245	0.254	0.409	**0.925**	0.567	0.512	0.885
	2.5	0.998	0.977	0.966	**0.999**	**0.578**	0.372	0.383	0.543	**0.977**	0.741	0.686	0.969
PD	1.0	0.048	0.042	0.045	0.055	0.046	0.058	0.063	0.046	0.054	0.057	0.057	0.046
	1.2	**0.218**	0.079	0.073	0.191	**0.125**	0.069	0.069	0.121	**0.206**	0.102	0.088	0.188
	1.4	**0.483**	0.232	0.207	0.387	0.200	0.104	0.102	**0.204**	**0.501**	0.233	0.182	0.417
	1.6	**0.728**	0.436	0.385	0.615	**0.316**	0.169	0.156	0.299	**0.709**	0.361	0.315	0.609
	1.8	**0.888**	0.623	0.548	0.799	**0.397**	0.221	0.201	0.393	**0.871**	0.542	0.469	0.786
	2.0	**0.947**	0.725	0.675	0.886	**0.497**	0.296	0.275	0.470	**0.931**	0.659	0.585	0.851
	2.5	**0.997**	0.943	0.914	0.988	0.678	0.434	0.413	**0.679**	**0.993**	0.836	0.755	0.968
RMD	1.0	0.055	0.059	0.055	0.061	0.052	0.060	0.059	0.056	0.048	0.051	0.053	0.055
	1.2	0.268	0.161	0.149	**0.279**	**0.083**	0.054	0.051	0.069	**0.238**	0.135	0.125	0.227
	1.4	0.584	0.414	0.392	**0.588**	**0.115**	0.071	0.079	0.097	**0.503**	0.314	0.301	0.481
	1.6	0.837	0.710	0.701	**0.841**	**0.143**	0.074	0.071	0.117	**0.746**	0.539	0.509	0.721
	1.8	0.949	0.878	0.851	**0.952**	**0.183**	0.071	0.071	0.142	**0.902**	0.734	0.709	0.874
	2.0	**0.980**	0.942	0.935	**0.980**	**0.212**	0.097	0.091	0.153	**0.960**	0.820	0.790	0.949
	2.5	**1.000**	0.998	0.997	**1.000**	**0.289**	0.118	0.117	0.190	**0.999**	0.937	0.917	0.993
RSPD	1.0	0.049	0.055	0.055	0.051	0.053	0.058	0.062	0.049	0.048	0.048	0.054	0.055
	1.2	0.268	0.140	0.128	**0.270**	**0.102**	0.069	0.071	**0.102**	**0.226**	0.101	0.102	0.222
	1.4	**0.536**	0.334	0.331	0.535	0.189	0.119	0.119	**0.195**	**0.480**	0.287	0.282	0.464
	1.6	**0.812**	0.614	0.594	0.806	**0.242**	0.163	0.161	0.236	**0.682**	0.467	0.471	0.671
	1.8	0.930	0.826	0.815	**0.931**	**0.361**	0.224	0.221	0.348	**0.863**	0.638	0.631	0.847
	2.0	**0.983**	0.923	0.916	0.980	**0.443**	0.295	0.297	0.440	**0.931**	0.782	0.770	0.923
	2.5	0.997	0.995	0.994	**0.998**	**0.615**	0.445	0.446	0.594	**0.993**	0.955	0.947	0.992

depth function while implementing the proposed test.

5. Illustration with real data

The performance of the proposed test is demonstrated with the painted turtle data by Jolicoeur and Mosimann (1960). They have considered the painted turtles to study the relationship between the size and shape of groups of living organisms, emphasizing the application of principal component analysis. In this data collection, the carapaces of 24 male and 24 female turtles were measured for carapace length, carapace width, and carapace height. The observation of male and female turtles suggests two samples, and the dilemma is to see whether the female turtle's population has a larger scale than the male turtle's population. That is, the hypothesis of interest is,

H_{A0} : There is no significant difference between the scale parameter of female and male turtle populations.

H_{A1} : Female turtle population has a larger scale than the male turtle population.

To run the tests, we first transform the data by subtracting the most central observation (deepest observation in the data cloud) from each data point, ensuring that the center of both samples remains the same. The DD-plots for the original data (without centering) and the centred data using the projection depth function are shown in Figure 2. Figure 2(b) shows that there is a significant difference in the scale parameter of male and female turtle populations. The proposed test, as well as existing depth-based tests, are then used to test the hypothesis mentioned above. The 'ddalpha' package in 'R' software is used to acquire depth values of the multivariate data points using the depth functions

Table 10: Simulated powers for the tests with different depth functions for trivariate Clayton gamma, Gumbel gamma, and Frank gamma distribution with $m_1 = m_2 = 50$

Depth function	$\sigma_1 = \sigma_2 = \sigma_3$	Clayton gamma				Gumbel gamma				Frank gamma			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
HD	1.0	0.052	0.051	0.048	0.056	0.056	0.070	0.064	0.050	0.048	0.047	0.046	0.044
	1.2	**0.157**	0.069	0.067	0.147	0.153	0.076	0.071	**0.156**	0.109	0.057	0.067	**0.115**
	1.4	**0.267**	0.115	0.125	0.266	**0.294**	0.131	0.148	0.279	**0.164**	0.072	0.091	0.157
	1.6	**0.397**	0.126	0.156	0.350	0.418	0.155	0.208	**0.425**	**0.204**	0.063	0.146	0.196
	1.8	**0.488**	0.132	0.178	0.411	**0.537**	0.167	0.236	0.535	**0.284**	0.056	0.144	0.252
	2.0	**0.622**	0.152	0.200	0.490	0.644	0.223	0.301	**0.648**	**0.297**	0.039	0.130	0.267
	2.5	**0.718**	0.093	0.162	0.505	**0.761**	0.183	0.297	0.730	**0.323**	0.016	0.123	0.211
PD	1.0	0.043	0.048	0.050	0.037	0.047	0.045	0.050	0.042	0.041	0.051	0.048	0.051
	1.2	**0.289**	0.101	0.088	0.237	**0.303**	0.124	0.118	0.274	**0.323**	0.107	0.097	0.278
	1.4	**0.643**	0.245	0.212	0.548	**0.646**	0.290	0.249	0.591	**0.702**	0.289	0.248	0.617
	1.6	**0.886**	0.389	0.309	0.797	**0.879**	0.529	0.451	0.819	**0.934**	0.461	0.384	0.870
	1.8	**0.963**	0.491	0.403	0.939	**0.979**	0.670	0.583	0.953	**0.988**	0.570	0.487	0.960
	2.0	**0.992**	0.581	0.479	0.967	**0.995**	0.782	0.679	0.987	**0.998**	0.627	0.520	0.992
	2.5	**1.000**	0.662	0.501	1.000	**1.000**	0.854	0.753	1.000	**1.000**	0.656	0.501	0.998
RMD	1.0	0.056	0.057	0.058	0.054	0.048	0.039	0.040	0.042	0.044	0.047	0.042	0.046
	1.2	**0.231**	0.111	0.101	0.179	**0.228**	0.089	0.084	0.172	**0.211**	0.097	0.092	0.185
	1.4	**0.483**	0.247	0.242	0.377	**0.464**	0.201	0.191	0.361	**0.474**	0.237	0.226	0.359
	1.6	**0.737**	0.402	0.371	0.607	**0.676**	0.313	0.308	0.518	**0.700**	0.383	0.341	0.518
	1.8	**0.871**	0.489	0.443	0.721	**0.866**	0.428	0.413	0.649	**0.843**	0.438	0.408	0.665
	2.0	**0.947**	0.574	0.517	0.821	**0.920**	0.542	0.505	0.764	**0.942**	0.508	0.446	0.759
	2.5	**0.994**	0.647	0.549	0.891	**0.990**	0.630	0.601	0.862	**0.994**	0.511	0.426	0.847
RSPD	1.0	0.050	0.058	0.061	0.046	0.054	0.044	0.045	0.053	0.062	0.061	0.064	0.068
	1.2	**0.226**	0.105	0.108	0.203	**0.230**	0.120	0.126	0.225	**0.211**	0.116	0.123	**0.211**
	1.4	**0.466**	0.229	0.232	0.442	**0.476**	0.275	0.275	0.450	**0.469**	0.267	0.258	0.416
	1.6	**0.711**	0.457	0.456	0.662	**0.689**	0.459	0.453	0.655	**0.641**	0.400	0.388	0.582
	1.8	**0.862**	0.635	0.610	0.824	**0.847**	0.632	0.626	0.817	**0.816**	0.572	0.565	0.741
	2.0	**0.932**	0.742	0.712	0.906	**0.910**	0.748	0.751	0.882	**0.890**	0.681	0.674	0.842
	2.5	**0.996**	0.911	0.882	0.980	**0.992**	0.914	0.900	0.981	**0.987**	0.864	0.814	0.944

Table 11: Simulated powers for the tests with different depth functions for trivariate normal, Cauchy, skew-normal, Clayton gamma, Gumbel gamma, and Frank gamma distribution with $m_1 = m_2 = 50$ and different values of σ_1, σ_2 and σ_3

Depth function	$(\sigma_1, \sigma_2, \sigma_3)$	Normal				Cauchy				Skew-Normal			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
HD	(1.0, 1.0, 1.0)	0.044	0.050	0.057	0.042	0.050	0.064	0.063	0.047	0.044	0.039	0.039	0.043
	(1.0, 1.2, 2.0)	**0.452**	0.215	0.185	0.439	**0.143**	0.077	0.071	0.130	**0.431**	0.203	0.179	0.417
	(1.2, 1.5, 2.0)	**0.740**	0.445	0.366	0.730	**0.226**	0.133	0.136	0.213	**0.615**	0.337	0.299	0.621
	(1.4, 1.6, 1.8)	**0.811**	0.538	0.430	0.807	**0.232**	0.114	0.118	0.217	**0.672**	0.365	0.312	0.660
	(1.8, 2.0, 2.5)	**0.985**	0.891	0.834	0.984	**0.452**	0.273	0.281	0.436	**0.932**	0.655	0.610	0.916
RMD	(1.0, 1.0, 1.0)	0.039	0.060	0.053	0.043	0.040	0.040	0.044	0.052	0.042	0.041	0.044	0.038
	(1.0, 1.2, 2.0)	0.456	0.319	0.309	**0.465**	**0.099**	0.051	0.051	0.086	**0.435**	0.275	0.274	0.418
	(1.2, 1.5, 2.0)	**0.735**	0.597	0.570	0.732	**0.131**	0.066	0.065	0.104	**0.684**	0.471	0.457	0.665
	(1.4, 1.6, 1.8)	0.812	0.663	0.637	**0.820**	**0.151**	0.082	0.078	0.112	**0.743**	0.546	0.519	0.735
	(1.8, 2.0, 2.5)	0.988	0.960	0.955	**0.989**	**0.221**	0.098	0.094	0.158	**0.974**	0.876	0.842	0.962
Depth function	$(\sigma_1, \sigma_2, \sigma_3)$	Clayton gamma				Gumbel gamma				Frank gamma			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
HD	(1.0, 1.0, 1.0)	0.054	0.050	0.054	0.062	0.051	0.057	0.050	0.048	0.044	0.043	0.047	0.036
	(1.0, 1.2, 2.0)	**0.172**	0.046	0.090	0.148	**0.204**	0.012	0.075	0.180	**0.127**	0.032	0.098	0.120
	(1.2, 1.5, 2.0)	**0.316**	0.092	0.149	0.270	**0.369**	0.081	0.136	0.332	**0.198**	0.054	0.111	0.183
	(1.4, 1.6, 1.8)	**0.354**	0.112	0.150	0.324	**0.409**	0.150	0.194	0.406	**0.206**	0.051	0.123	0.196
	(1.8, 2.0, 2.5)	**0.576**	0.101	0.147	0.460	**0.636**	0.123	0.247	0.603	**0.295**	0.020	0.109	0.238
RMD	(1.0, 1.0, 1.0)	0.051	0.037	0.039	0.048	0.037	0.062	0.062	0.045	0.043	0.041	0.039	0.045
	(1.0, 1.2, 2.0)	**0.369**	0.253	0.265	0.294	**0.340**	0.196	0.210	0.248	**0.349**	0.215	0.223	0.258
	(1.2, 1.5, 2.0)	**0.661**	0.358	0.341	0.527	**0.622**	0.277	0.270	0.441	**0.593**	0.320	0.320	0.431
	(1.4, 1.6, 1.8)	**0.692**	0.404	0.371	0.559	**0.698**	0.355	0.346	0.522	**0.697**	0.357	0.340	0.525
	(1.8, 2.0, 2.5)	**0.955**	0.596	0.515	0.834	**0.937**	0.500	0.472	0.741	**0.945**	0.534	0.479	0.758

provided in Section 2. Finally, to determine the p-value of the tests, the sample is permuted 100,000 times. Table 14 shows the test statistic values for the proposed test and existing depth-based nonpara-

Table 12: Simulated powers for the tests with RMD for normal, Cauchy, and skew-normal distribution having dimension 10 with $m_1 = m_2 = 50$ and different values of σ^1 and σ^2

σ^1	σ^2	Normal				Cauchy				Skew-Normal			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
1.0	1.0	0.053	0.055	0.051	0.056	0.051	0.054	0.050	0.054	0.051	0.049	0.051	0.047
1.0	1.2	0.388	0.171	0.132	**0.389**	**0.069**	0.044	0.037	0.060	0.388	0.166	0.138	**0.389**
1.0	1.5	**0.919**	0.658	0.539	0.913	**0.135**	0.054	0.048	0.079	**0.908**	0.642	0.523	0.901
1.0	2.0	**1.000**	0.980	0.931	1.000	**0.197**	0.059	0.056	0.108	**0.999**	0.979	0.912	0.999
1.0	2.5	**1.000**	1.000	0.986	1.000	**0.234**	0.066	0.056	0.102	**1.000**	0.999	0.978	1.000
1.2	1.2	**0.437**	0.180	0.138	0.431	**0.086**	0.054	0.047	0.059	**0.414**	0.163	0.126	0.391
1.3	1.0	0.112	0.056	0.060	**0.122**	0.053	0.054	**0.055**	0.052	**0.129**	0.055	0.057	0.121
1.3	1.2	**0.613**	0.304	0.226	0.599	0.070	0.050	0.048	**0.071**	**0.565**	0.290	0.216	0.548
1.3	1.5	**0.984**	0.804	0.693	0.975	**0.137**	0.070	0.063	0.094	**0.978**	0.780	0.654	0.972
1.3	2.0	**1.000**	0.992	0.961	1.000	**0.210**	0.077	0.071	0.114	**1.000**	0.992	0.949	1.000
1.3	2.5	**1.000**	1.000	0.997	1.000	**0.279**	0.089	0.076	0.134	**1.000**	1.000	0.998	1.000
1.5	1.0	**0.189**	0.088	0.086	0.189	**0.064**	0.045	0.046	0.060	**0.166**	0.068	0.061	0.159
1.5	1.2	**0.690**	0.384	0.300	0.683	**0.070**	0.049	0.044	0.060	**0.668**	0.335	0.271	0.664
1.5	1.5	**0.960**	0.697	0.563	0.949	**0.128**	0.043	0.035	0.070	**0.943**	0.698	0.554	0.940
1.8	1.5	0.995	0.903	0.819	**0.996**	**0.150**	0.068	0.056	0.094	**0.993**	0.894	0.788	0.991
1.8	2.0	**1.000**	0.999	0.990	1.000	**0.233**	0.062	0.061	0.114	**1.000**	0.995	0.977	1.000
1.8	2.5	**1.000**	1.000	0.999	1.000	**0.309**	0.099	0.093	0.167	**1.000**	1.000	0.997	1.000
2.0	1.2	**0.832**	0.551	0.423	0.829	**0.102**	0.054	0.058	0.087	**0.838**	0.505	0.384	0.821
2.0	1.5	**0.996**	0.945	0.852	0.997	**0.146**	0.071	0.063	0.100	**0.998**	0.919	0.821	0.998
2.0	2.0	**1.000**	0.999	0.990	1.000	**0.260**	0.093	0.082	0.147	**1.000**	0.996	0.975	1.000
2.0	2.5	**1.000**	1.000	0.999	1.000	**0.316**	0.110	0.088	0.167	**1.000**	1.000	0.997	1.000
2.5	2.5	**1.000**	1.000	0.994	**1.000**	**0.287**	0.098	0.089	0.141	**1.000**	0.998	0.975	**1.000**

Table 13: Simulated powers for the tests with RMD for Clayton gamma, Gumbel gamma, and Frank gamma distribution having dimension 10 with $m_1 = m_2 = 50$ and different values of σ^1 and σ^2

σ^1	σ^2	Clayton gamma				Gumbel gamma				Frank gamma			
		R_O	S_1	S_2	S	R_O	S_1	S_2	S	R_O	S_1	S_2	S
1.0	1.0	0.039	0.056	0.062	0.052	0.047	0.051	0.051	0.049	0.051	0.052	0.050	0.055
1.0	1.2	**0.415**	0.151	0.131	0.357	**0.387**	0.131	0.099	0.310	**0.467**	0.151	0.124	0.378
1.0	1.5	**0.908**	0.424	0.316	0.837	**0.924**	0.421	0.299	0.808	**0.957**	0.362	0.250	0.867
1.0	2.0	**0.998**	0.575	0.376	0.993	**1.000**	0.564	0.417	0.974	**1.000**	0.363	0.211	0.993
1.0	2.5	**1.000**	0.461	0.281	0.997	**1.000**	0.480	0.322	0.984	**1.000**	0.199	0.093	0.999
1.2	1.2	**0.452**	0.155	0.120	0.420	**0.466**	0.145	0.107	0.392	**0.527**	0.167	0.126	0.421
1.3	1.0	0.101	0.056	0.058	**0.104**	**0.128**	0.055	0.058	0.113	**0.140**	0.059	0.051	0.105
1.3	1.2	**0.579**	0.246	0.199	0.498	**0.618**	0.218	0.168	0.496	**0.651**	0.209	0.173	0.539
1.3	1.5	**0.981**	0.594	0.446	0.935	**0.975**	0.570	0.436	0.908	**0.987**	0.469	0.306	0.952
1.3	2.0	**0.999**	0.740	0.555	0.999	**1.000**	0.753	0.599	0.990	**1.000**	0.444	0.242	0.997
1.3	2.5	**1.000**	0.656	0.457	1.000	**1.000**	0.707	0.535	0.994	**1.000**	0.272	0.123	0.999
1.5	1.0	**0.182**	0.060	0.055	0.162	**0.194**	0.081	0.069	0.154	**0.177**	0.076	0.076	0.155
1.5	1.2	**0.713**	0.269	0.205	0.630	**0.745**	0.274	0.209	0.617	**0.754**	0.246	0.174	0.606
1.5	1.5	**0.962**	0.495	0.345	0.936	**0.975**	0.508	0.361	0.918	**0.981**	0.428	0.271	0.920
1.8	1.5	**0.991**	0.657	0.501	0.967	**0.996**	0.668	0.510	0.968	**0.997**	0.530	0.354	0.988
1.8	2.0	**1.000**	0.817	0.631	1.000	**1.000**	0.862	0.718	0.999	**1.000**	0.468	0.264	1.000
1.8	2.5	**1.000**	0.745	0.553	1.000	**1.000**	0.841	0.706	0.999	**1.000**	0.235	0.099	0.999
2.0	1.2	**0.847**	0.314	0.226	0.764	**0.848**	0.307	0.237	0.682	**0.920**	0.318	0.220	0.789
2.0	1.5	**0.998**	0.665	0.498	0.984	**0.996**	0.670	0.510	0.958	**1.000**	0.541	0.355	0.995
2.0	2.0	**1.000**	0.810	0.618	1.000	**1.000**	0.862	0.747	0.998	**1.000**	0.445	0.242	1.000
2.0	2.5	**1.000**	0.737	0.546	1.000	**1.000**	0.882	0.763	0.999	**1.000**	0.230	0.094	1.000
2.5	2.5	**1.000**	0.531	0.309	0.999	**1.000**	0.793	0.626	0.997	**1.000**	0.135	0.041	0.996

metric tests with their p-values. Thus, the proposed test is recommended for actual real-life data sets. Table 14 shows that the proposed test yields approximately equal p-values for both the depth functions, but test S_1 yields a slightly different p-value. In other words, at a 5% level of significance, the proposed test reported the same conclusion (i.e., reject the null hypothesis, H_{A0}) for both the depth functions, but the S_1 test reject H_{A0} in case of RSPD and fails to reject H_{A0} in case of RMD at a 5% level of significance. This admits that the proposed test performs uniformly over the different depth functions. Thus, the proposed test concludes that there is a large scale in the measurements of the

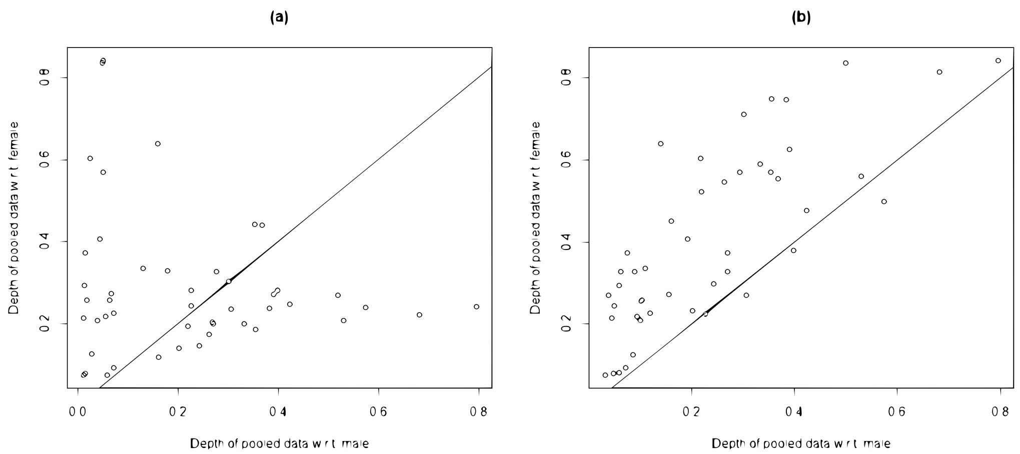

Figure 2: *DD-plot for turtles data set using RMD for (a) original data (b) centered data.*

Table 14: The values of test statistic and p-values of the tests

	RSPD			
Test	R_O	S_1	S_2	S
Value of test statistic	371	526.6698	183.2495	6.1380
p-value	0.0034	0.0234	0.0315	0.0032
	RMD			
Test	R_O	S_1	S_2	S
Value of test statistic	381	511.9843	195.4974	5.9770
p-value	0.0061	0.1043	0.0556	0.0053

female turtle population than that of the male turtle population at a 5% level of significance.

6. Concluding remarks

The majority of depth-based nonparametric scale tests are based on the ranks of depth values of multivariate data. However, the depth value of the data points is calculated with respect to the given distribution or given data cloud, which we refer to as reference distribution or reference sample. If the reference distribution or sample is changed while calculating depth values, then it changes the depth values of the data points and results in different ranks of data points. Here, we provided a new way of calculating depth values of data points regarding reference distribution or reference data and suggested a new scale test based on data depth. The test that is being suggested is nonparametric. The performance of the proposed test is studied by comparing the test with the existing depth-based nonparametric scale tests through a simulation study, and it confesses that the proposed test gives comparable results with the existing depth-based S-test and provides better results than S_1 and S_2 tests for symmetric as well as skewed distributions. The proposed test is also used for higher dimension data. For higher dimension data, the proposed test usually gives better powers than the existing tests considered in this article. The proposed test statistic depends on the ranks of the depth values and the depth values varies as the depth function changes. As a result, the ranks based on depth values also changes, and consequently, the value of test statistic also varies. So, the resulting powers of the proposed test varies as a depth function changes. Thus, the performance of the proposed test changes according to the depth function for the specified distribution due to the different nature of the depth function. Hence the performance of the proposed test depends on the depth function. In general, for the projection depth function, the proposed test gives better results than the existing S_1, S_2 and S tests in all considered distributions. So, the preference of the depth function is given to the projection depth function while implementing the proposed test. Lastly, an illustrative example is provided for the applicability of the proposed test and reported that the test works well on actual data sets.

References

Chaudhuri P (1996). On a geometric notion of quantiles for multivariate data, *Journal of the American Statistical Association*, **91**, 862–872.

Chavan AR and Shirke DT (2020). Nonparametric two sample tests for scale parameters of multivariate distributions, *Communications for Statistical Applications and Methods*, **27**, 397–412.

Chenouri S, Small CG, and Farrar TJ (2011). Data depth-based nonparametric scale tests, *Canadian Journal of Statistics*, **39**, 356–369.

Dovoedo YH and Chakraborti S (2015). Power of depth-based nonparametric tests for multivariate locations, *Journal of Statistical Computation and Simulation*, **85**, 1987–2006.

Efron B and Tibshirani RJ (1994). *An Introduction to the Bootstrap*, CRC press, London.

Gastwirth JL (1965). Percentile modifications of two sample rank tests, *Journal of the American Statistical Association*, **60**, 1127–1141.

Jolicoeur P and Mosimann JE (1960). Size and shape variation in the painted turtle. A principal component analysis, *Growth*, **24**, 339–354.

Li J and Liu RY (2004). New nonparametric tests of multivariate locations and scales using data depth, *Statistical Science*, **19**, 686–696.

Li J and Liu RY (2016). New nonparametric tests for comparing multivariate scales using data depth, In *Robust Rank-Based and Nonparametric Methods* (pp. 209–226), Springer, Michigan, USA.

Liu RY (1990). On a notion of data depth based on random simplices, *The Annals of Statistics*, **18**, 405–414.

Liu RY and Singh K (1993). A quality index based on data depth and multivariate rank tests, *Journal of the American Statistical Association*, **88**, 252–260.

Liu RY, Parelius JM, and Singh K (1999). Multivariate analysis by data depth: Descriptive statistics, graphics and inference, (with discussion and a rejoinder by liu and singh), *The Annals of Statistics*, **27**, 783–858.

Liu RY and Singh K (2006). Rank tests for multivariate scale difference based on data depth, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, **72**, 17–36.

Mahalanobis PC (1936). On the generalized distance in statistics, National Institute of Science of India, 1936.

Pawar SD and Shirke DT (2019). Nonparametric tests for multivariate locations based on data depth, *Communications in Statistics-Simulation and Computation*, **48**, 753–776.

Rousseeuw PJ and Hubert M (1999). Regression depth, *Journal of the American Statistical Association*, **94**, 388–402.

Rousson V (2002). On distribution-free tests for the multivariate two-sample location-scale model, *Journal of Multivariate Analysis*, **80**, 43–57.

Serfling R (2002). A depth function and a scale curve based on spatial quantiles, In *Statistical Data Analysis Based on the L1-Norm and Related Methods* (pp. 25–38), Birkhäuser, Basel.

Serfling R (2002). A depth function and a scale curve based on spatial quantiles, In *Statistical Data Analysis Based on the L1-Norm and Related Methods* (pp. 25–38), Birkhäuser, Basel.

Shirke DT and Khorate SD (2018). Two-Sample nonparametric test for testing equality of locations based on data depth, *Journal of the Indian Society for Probability and Statistics*, **19**, 9–23.

Singh K (1991). A notion of majority depth, Unpublished document, 1991.

Tukey JW (1975). Mathematics and the picturing of data, In *Proceedings of the International Congress of Mathematicians*, Vancouver, 2, 523–531.

Zuo Y and Serfling R (2000). General notions of statistical depth function, *Annals of Statistics*, **28**, 461–482.

Zuo Y (2003). Projection-Based depth functions and associated medians, *The Annals of Statistics*, **31**, 1460–1490.

Received January 06, 2022; Revised January 21, 2023; Accepted February 23, 2023