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# A REMARK ON STATISTICAL MANIFOLDS WITH TORSION

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Abstract. Consider a Riemannian manifold $M$ equipped with a metric $g$. In this article, we study a notion for statistical manifolds $(M, g, \nabla)$, which can have a non-zero torsion, abbreviated to SMT. Then it turns out that the tensor fields $\nabla g$ and $\tilde{\nabla} g$ obtained from two different SMT-connections are different.

## 1. Introduction

Let $M$ be a manifold with a metric $g$. Given a linear connection $\nabla$ there exists a unique connection $\nabla^*$ such that

$$d(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla^* Y)$$

and we then say that $\nabla$, $\nabla^*$ are dual connections with respect to the metric $g$.

A statistical manifold can be defined using the notion of dual connections, that is, a manifold $(M, g, \nabla, \nabla^*)$ satisfying

$$T^{\nabla} = T^{\nabla^*} = 0$$

where the torsion of $\nabla$ is given by

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

There are a few equivalent ways in which statistical manifolds have been introduced; for details we refer to [1, 3, 7, 11].

In this article, we consider statistical manifolds whose torsions are not necessarily zero. We will use a notion of "statistical manifolds admitting torsion" as introduced in [6] and abreviate it as "SMT".

The difference between a linear connection $\nabla$ and the Levi-Civita connection $\nabla^g$ is a $(2,1)$-tensor field denoted by $A$, that is

$$\nabla_X Y = \nabla^g_X Y + A(X,Y). \tag{1}$$

The notation $A$ is also used for the $(3,0)$-tensor defined by

$$A(X,Y,Z) = g(A(X,Y), Z).$$

In [5], given a SMT $(M, g, \nabla)$ with $\nabla = \nabla^g + A$, an equivalent condition for the difference tensor $A$ is computed, see (8). In this article, we will consider the space

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of $A$ satisfying this condition and denote the space by $\mathcal{SMT}$. We also consider the symmetric space of $\mathcal{A}^S$ consisting of $(3,0)$-tensor fields $A$ which are symmetric with respect to the second and third variables.

In the main Theorem, we will then construct a bijection between $\mathcal{SMT}$ and $\mathcal{A}^S$. Here we observe that $\mathcal{A}^S$ is actually the space of $\nabla g$'s where $(M, g, \nabla)$ is a SMT, so we conclude that $\nabla g \neq \tilde{\nabla} g$ for two different SMT-connections $\nabla$ and $\tilde{\nabla}$.

## 2. Preliminaries

Let $(M, g)$ be a Riemannian manifold and $\Gamma(M)$, $\Gamma^*(M)$ the set of sections of the tangent bundle $TM$, $T^*M$, respectively.

A linear connection $\nabla$ is then a map

$$\nabla : \Gamma(M) \otimes \Gamma(M) \to \Gamma(M)$$

with some properties and gives a way how to transport a vector field along a direction.

A metric connection $\nabla$ is a linear connection, which gives isometries between tangent spaces by parallel transport, that is

$$V(g(X,Y)) = g(\nabla_V X, Y) + g(X, \nabla_V Y). \tag{2}$$

The condition (2) is equivalent to $\nabla g = 0$, since for $(2,0)$- tensor field $g$

$$(\nabla_V g)(X,Y) = V(g(X,Y)) - g(\nabla_V X, Y) - g(X, \nabla_V Y).$$

The Levi-Civita connection, denoted by $\nabla^g$, is the unique metric connection with torsion $T = 0$.

The difference between a linear connection $\nabla$ and the Levi-Civita connection $\nabla^g$ is a $(2,1)$-tensor (field) $A$, that is, for any tangent vector fields $X, Y \in \Gamma(M)$,

$$\nabla_X Y = \nabla^g_X Y + A(X,Y).$$

Using the same notation, a $(3,0)$-tensor $A$ is defined by

$$A(X,Y,Z) = \langle A(X,Y), Z\rangle.$$

We now consider the case where isometries between tangent spaces are obtained by parallel transports with respect to two connections $\nabla$, $\nabla^*$ as follows.

Definition 2.1 (Dual Connections). For a linear connection $\nabla$, the dual connection $\nabla^*$ of $\nabla$ with respect to $g$ is defined by

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y)).$$

By the expression (1) let

$$\begin{aligned} \nabla_X Y &= \nabla^g + A(X,Y) && (3)\\ \nabla^*_X Y &= \nabla^g + A^*(X,Y). && (4) \end{aligned}$$

We can then easily check the following.

Lemma 2.2. *Given a linear connection $\nabla$ and its dual connection $\nabla^*$ defined as above, the following equality holds:*

$$\langle A(Z,X), Y\rangle + \langle X, A^*(Z,Y)\rangle = A(Z,X,Y) + A^*(Z,Y,X) = 0. \tag{5}$$

*So, a linear connection $\nabla$ has a unique dual connection $\nabla^*$*

## 3. Statistical manifolds admitting torsion

A statistical manifold in a classical sense is a torsion-free manifold with some properties.

In [6] a notion for generalized statistical manifolds is introduced. There are some well-known equivalent properties of these statistical manifolds. In this article, we take the following properties as definitions.

Definition 3.1. [2, 3, 6, 8]

(i) A Riemannian manifold $(M, g, \nabla)$ is a statistical manifold if

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = 0, \tag{6}$$

for $X, Y, Z \in \Gamma(TM)$.

(ii) A Riemannian manifold $(M, g, \nabla)$ is a statistical manifold admitting torsion, (SMT) for short, if

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = -g(T^\nabla(X, Y), Z), \tag{7}$$

for $X, Y, Z \in \Gamma(TM)$, where $T^\nabla$ is the torsion tensor of $\nabla$.

Considering the difference tensor field $A$ as in (3), we obtain the following result.

Proposition 3.2. [5, 8] *Given a Riemannian manifold $(M, g, \nabla)$ the following conditions are equivalent.*

*(i)* $(M, g, \nabla, \nabla^*)$ *is a SMT.*

*(ii) Let* $\nabla = \nabla^g + A$. *Then it holds*

$$A(X, Y, Z) = A(Z, Y, X) \quad \text{for } X, Y, Z \in \Gamma(TM). \tag{8}$$

*(iii)* $T^{\nabla^*} = 0$.

Here we note that a statistical manifold $(M, g, \nabla, \nabla^*)$ in a classical sense is the manifold with $T^\nabla = T^{\nabla^*} = 0$.

We consider the $(3, 0)$- tensor field $A$ as an element of $\otimes^3 TM$, identifying $TM$ with $TM^*$. Then by Proposition 3.2 $(ii)$, for the set of SMT's we can consider a space as follows:

$$\mathcal{SMT} = \{A \in \otimes^3 TM | A(X, Y, Z) = A(Z, Y, X)\}.$$

We also take a symmetric space:

$$\mathcal{A}^S = \{A \in \otimes^3 TM | A(X, Y, Z) = A(X, Z, Y)\} = TM \otimes S^2 TM.$$

We will then find a bijection between the above two spaces in the following theorem.

Theorem 3.3. *A bijection between $\mathcal{SMT}$ and $\mathcal{A}^S$ arises from the following:*

*For $S \in \mathcal{SMT}$, we associate $G \in \mathcal{A}^S$ by*

$$G(X, Y, Z) = S(X, Y, Z) + S(X, Z, Y). \tag{9}$$

*And for $G \in \mathcal{A}^S$, we associate $S \in \mathcal{SMT}$ by*

$$2S(X, Y, Z) = G(X, Y, Z) - G(Y, Z, X) + G(Z, X, Y).$$

*Proof.* Given $S \in \mathcal{SMT}$, we get $G \in \mathcal{A}^S$ by

$$G(X,Y,Z) = S(X,Y,Z) + S(X,Z,Y) \in \mathcal{A}^S.$$

Now since $S \in \mathcal{SMT}$,

$$\begin{aligned} & G(X,Y,Z) - G(Y,Z,X) + G(Z,X,Y) \\ = \; & S(X,Y,Z) + S(X,Z,Y) - S(Y,Z,X) - S(Y,X,Z) \\ & + S(Z,X,Y) + S(Z,Y,X) \\ = \; & 2S(X,Y,Z). \end{aligned}$$

We note that the above (9) gives a linear map for each $T_xM$, $x \in M$.

Finally, the elements of $\mathcal{SMT}$ and $\mathcal{A}^S$ are symmetric with respect to two variables, namely, first and third ones for $\mathcal{SMT}$, second and third ones for $\mathcal{A}^S$. So, $\mathcal{SMT}$ and $\mathcal{A}^S$ have the same dimension.

We now conclude that the mapping (9) is a bijection from $\mathcal{SMT}$ to $\mathcal{A}^S$. □

Corollary 3.4. *Two different SMT-connections $\nabla$ and $\tilde{\nabla}$ give two different and tensor fields $\nabla g$ and $\tilde{\nabla} g$.*

*Proof.* For $\nabla = \nabla^g + A$, recall that

$$\nabla g = A(X,Y,Z) + A(X,Z,Y). \tag{10}$$

So, by the bijection in Theorem 3.3, we have two different tensor fields $\nabla g$ and $\tilde{\nabla} g$ for two different SMT- connection $\nabla$, $\tilde{\nabla}$.

Here (10) follows from

$$\begin{aligned} \nabla g(X,Y,Z) &= (\nabla_X g)(Y,Z) \\ &= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &= \nabla^g g(X,Y,Z) + A(X,Y,Z) + A(X,Z,Y) \\ &= A(X,Y,Z) + A(X,Z,Y). \end{aligned}$$

□

Remark 3.5. Since $\otimes^2 TM = \Lambda^2 TM \oplus S^2 TM$ where the tensor product $\Lambda^2$ and $S^2$ is skew-symmetric and symmetric tensor products, respectively, we have

$$\otimes^3 TM = \mathcal{A}^M \oplus \mathcal{A}^S$$

with

$$\mathcal{A}^M = \{A \in \otimes^3 TM | A(X,Y,Z) = -A(X,Z,Y)\} = TM \otimes \Lambda^2 TM.$$

So, from the bijection in Theorem 3.3 we also have a bijection between $\mathcal{SMT}$ and $\otimes^3 TM/\mathcal{A}^M$. Note that $\mathcal{A}^M$ is the space of $A$'s of metric connections $\nabla$, that is, linear connections with $\nabla g = 0$.

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