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CERTAIN FORM OF HILBERT-TYPE INEQUALITY USING NON-HOMOGENEOUS KERNEL OF HYPERBOLIC FUNCTIONS

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Abstract. In this article, we establish Hilbert-type integral inequalities with the help of a non-homogeneous kernel of hyperbolic function with best constant factor. We also study the obtained inequalities's equivalent form. Additionaly, several specific Hilbert's type inequalities with constant factors in the term of the rational fraction expansion of higher order derivatives of cotangent and cosine functions are presented.

1. Introduction

For $f(u), g(u) \geq 0$, define

$L^p[0,\infty) = \Big\{ f : \int_0^\infty f^p(u)du < \infty \Big\}$ and $L^p_\mu[0,\infty) = \Big\{ f : \int_0^\infty \mu(u) f^p(u)du < \infty \Big\}$.

Also

$||f||_p = \Big(\int_0^\infty f^p(u)du \Big)^{1/p}$ and $||f||_{p,\mu} = \Big(\int_0^\infty \mu(u) f^p(u)du \Big)^{1/p}$.

Through out in this paper, we consider all the functions are measurable and non-negative; and $p, p' \in \mathbb{R}^+$ with $1/p + 1/p' = 1$.

For $1 \leq p, p' < \infty$ with $1/p + 1/p' = 1$ and $\tau \in L^p[0,\infty), \sigma \in L^{p'}[0,\infty)$, we have the well known Hilbert's inequality [3]

$$\int_0^\infty \int_0^\infty \frac{\tau(u)\sigma(v)}{u+v} dudv < \pi ||\tau||_p ||\sigma||_{p'}, \tag{1}$$

where π is the best possible constant.

For $h(t) > 0, \phi(s) = \int_0^\infty h(t)t^{s-1}dt \in \mathbb{R}^+, \tau(u), \sigma(v) \geq 0$, we have the followings [3]:

$$\int_0^\infty \int_0^\infty h(uv)\tau(u)\sigma(v)dudv < \phi(\frac{1}{p}) \Big(\int_0^\infty u^{p-2}\tau^p(u)du \Big)^{\frac{1}{p}} \Big(\int_0^\infty \sigma^{p'}(v)dv \Big)^{\frac{1}{p'}}, \tag{2}$$

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$$\int_0^\infty \Big(\int_0^\infty h(uv)\tau(u)du\Big)^p dv < \phi^p(\frac{1}{p})\int_0^\infty u^{p-2}\tau^p(u)du, \tag{3}$$

$$\int_0^\infty v^{p-2}\Big(\int_0^\infty h(uv)\tau(u)du\Big)^p dv < \phi^p(\frac{1}{p'})\int_0^\infty \tau^p(u)du, \tag{4}$$

where the right-hand sides of above inequalities are positive.

In [11], Yang gave an extension of (1), by introducing the parameter $\lambda \in (0,1]$ and optimized weight coefficients, which is as follows.

If $0 < \int_0^\infty u^{1-\lambda}\tau^2(u)du < \infty$ and $0 < \int_0^\infty v^{1-\lambda}\sigma^2(v)dv < \infty$, then

$$0 < \int_0^\infty\int_0^\infty \frac{\tau(u)\sigma(v)}{(u+v)^\lambda}dudv < B\big(\frac{\lambda}{2},\frac{\lambda}{2}\big)\Big(\int_0^\infty u^{1-\lambda}\tau^2(u)du\int_0^\infty v^{1-\lambda}\sigma^2(v)dv\Big)^{\frac{1}{2}}, \tag{5}$$

where the constant factor $B\big(\frac{\lambda}{2},\frac{\lambda}{2}\big)$ is the best possible.

In [12], Yang introduce an independent parameter $\lambda > 0$ and pairs (p,p'), (r,r') with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}+\frac{1}{r'}=1$, such that if $p,r>1$, and the right-hand side are positive, then

$$\int_0^\infty\int_0^\infty \frac{\tau(u)\sigma(v)}{u^\lambda+v^\lambda}dudv < \frac{\pi}{\lambda\sin(\frac{\pi}{r})}\Big[\int_0^\infty u^{p(1-\frac{\lambda}{r})-1}\tau^p(u)du\Big]^{\frac{1}{p}}\Big[\int_0^\infty v^{p'(1-\frac{\lambda}{r'})-1}\sigma^{p'}(v)dv\Big]^{\frac{1}{p'}},$$

where the contant factor $\frac{\pi}{\lambda\sin(\frac{\pi}{r})}$ is the best possible.

In [13], Author gave an another form of Hilbert-type inequality using exponential kernel. That is

$$\int_0^\infty\int_0^\infty e^{-uv}\tau(u)\sigma(v)dudv < \sqrt{\pi}\Big[\int_0^\infty \tau^2(u)du\Big]^{\frac{1}{2}}\Big[\int_0^\infty \sigma^2(v)dv\Big]^{\frac{1}{2}},$$

where $\sqrt{\pi}$ is the best constant.

In the literature [1], [11], [12], [13], [4], [5], [6], [7], [8]; there are several extensions, generalizations and variants of the inequality (1).

In this study, we will discuss the Hilbert-type inequality by introducing multiple parameters and the following non-homogeneous kernel

$$\kappa(u,v) := \frac{e^{\beta_2(uv)^m}+\rho e^{\beta_3(uv)^m}}{e^{\beta_1(uv)^m}+\delta e^{-\beta_1(uv)^m}}, \tag{6}$$

where $\rho,\delta=\pm1, \beta_1>0, \beta_3\le\beta_2<\beta_1(\beta_2\ne\beta_3,\ \rho=-1)$, for $\rho\delta=-1, \beta>0$; and for $\rho\delta=1, \beta\ge 0$.

For $m=1$, the above kernel becomes

$$\kappa(u,v) := \frac{e^{\beta_2(uv)}+\rho e^{\beta_3(uv)}}{e^{\beta_1(uv)}+\delta e^{-\beta_1(uv)}},$$

which is studied by Minghui You and Yue Guan [14].

We also establish some particular inequalities having the constant factor containing the rational fraction expansion of higher order derivatives of cotangent function and cosine function.

2. Preliminaries and Lemmas

DEFINITION 2.1. [10] We define

$$\mathbb{B}(r,s)=\int_0^1 u^{r-1}(1-u)^{s-1}du,\ r>0,s>0,$$

and

$$\Gamma(m)=\int_0^\infty u^{m-1}e^{-u}du,\ m>0,$$

where $\mathbb{B}(m,n)$ and $\Gamma(z)$ are Beta function and Gamma function respectively for $m\in\mathbb{N},\Gamma(m)=(m-1)!$.

LEMMA 2.2. [10,14] *Let $x,y>0,x+y=1,\phi(u)=\cot u,n\in\mathbb{N}\cup\{0\}$, then*

$$\phi^{2n}(x\pi)=\frac{(2n)!}{\pi^{2n+1}}\sum_{j=0}^{\infty}\Big(\frac{1}{(j+x)^{2n+1}}-\frac{1}{(j+y)^{2n+1}}\Big), \tag{7}$$

$$\phi^{2n+1}(x\pi)=-\frac{(2n+1)!}{\pi^{2n+2}}\sum_{j=0}^{\infty}\Big(\frac{1}{(j+x)^{2n+2}}+\frac{1}{(j+y)^{2n+2}}\Big). \tag{8}$$

LEMMA 2.3. [10,14] *Let $x,y>0,x+y=1,\chi(u)=\csc u,n\in\mathbb{N}\cup\{0\}$, then*

$$\chi^{2n}(x\pi)=\frac{(2n)!}{\pi^{2n+1}}\sum_{j=0}^{\infty}(-1)^j\Big(\frac{1}{(j+x)^{2n+1}}+\frac{1}{(j+y)^{2n+1}}\Big), \tag{9}$$

$$\chi^{2n+1}(x\pi)=-\frac{(2n+1)!}{\pi^{2n+2}}\sum_{j=0}^{\infty}(-1)^j\Big(\frac{1}{(j+x)^{2n+2}}-\frac{1}{(j+y)^{2n+2}}\Big). \tag{10}$$

LEMMA 2.4. *Let $\rho,\delta=\pm1,\beta_1>0,\beta_3\le\beta_2<\beta_1(\beta_2\ne\beta_3,\rho=-1),\beta>m\ge1$, for $\rho\delta=-1,\beta\ge m\ge1;\kappa(u,v)$ as in (6) and*

$$C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)=\sum_{j=0}^{\infty}\frac{(-\delta)^j}{(2\beta_1 j+\beta_1-\beta_2)^{\frac{\beta+1}{m}}}+\rho\sum_{j=0}^{\infty}\frac{(-\delta)^j}{(2\beta_1 j+\beta_1-\beta_3)^{\frac{\beta+1}{m}}}, \tag{11}$$

then

$$W(u)=\int_0^\infty\kappa(u,v)v^\beta dv=u^{-\beta-1}\frac{\Gamma(\frac{\beta+1}{m})}{m}C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta), \tag{12}$$

$$\breve{W}(v)=\int_0^\infty\kappa(u,v)u^\beta du=v^{-\beta-1}\frac{\Gamma(\frac{\beta+1}{m})}{m}C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta). \tag{13}$$

Proof. Using the transformation $uv=t$, we get

$$W(u)=u^{-\beta-1}\int_0^\infty\kappa(1,t)t^\beta dt, \tag{14}$$

since $t\in(0,\infty),\delta=\pm1$, we find

$$\frac{1}{1+\delta e^{-2\beta_1 t^m}}=\sum_{j=0}^{\infty}(-\delta)^j e^{-2\beta_1 j t^m},$$

therefore

$$\int_0^\infty \kappa(1,t)t^\beta dt = \sum_{j=0}^{\infty}(-\delta)^j \int_0^\infty e^{-(2\beta_1 j+\beta_1-\beta_2)t^m} t^\beta dt + \rho \sum_{j=0}^{\infty}(-\delta)^j \int_0^\infty e^{-(2\beta_1 j+\beta_1-\beta_3)t^m} t^\beta dt$$

$$= I_1 + \rho I_2, \tag{15}$$

Now on putting $u = (2\beta_1 j + \beta_1 - \beta_2)t^m$, we get

$$I_1 = \sum_{j=0}^{\infty}(-\delta)^j \frac{1}{m} \frac{1}{(2\beta_1 j + \beta_1 - \beta_2)^{\frac{\beta+1}{m}}} \int_0^\infty e^{-u} u^{\frac{\beta+1}{m}-1} du$$

$$= \frac{\Gamma(\frac{\beta+1}{m})}{m} \sum_{j=0}^{\infty} \frac{(-\delta)^j}{(2\beta_1 j + \beta_1 - \beta_2)^{\frac{\beta+1}{m}}}, \tag{16}$$

Similarly , by substituting $u = (2\beta_1 j + \beta_1 - \beta_3)t^m$, we obtained

$$I_2 = \frac{\Gamma(\frac{\beta+1}{m})}{m} \sum_{j=0}^{\infty} \frac{(-\delta)^j}{(2\beta_1 j + \beta_1 - \beta_3)^{\frac{\beta+1}{m}}}, \tag{17}$$

Combining (15), (16) and (17), we have

$$\int_0^\infty \kappa(1,t)t^\beta dt = \frac{\Gamma(\frac{\beta+1}{m})}{m} \Big[\sum_{j=0}^{\infty} \frac{(-\delta)^j}{(2\beta_1 j + \beta_1 - \beta_2)^{\frac{\beta+1}{m}}} + \rho \sum_{j=0}^{\infty} \frac{(-\delta)^j}{(2\beta_1 j + \beta_1 - \beta_3)^{\frac{\beta+1}{m}}} \Big]$$

$$= \frac{\Gamma(\frac{\beta+1}{m})}{m} C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta), \tag{18}$$

In the same way , we can easily show that (13) holds. Hence

$$W(u) = \int_0^\infty \kappa(u,v) v^\beta dv = u^{-\beta-1} \frac{\Gamma(\frac{\beta+1}{m})}{m} C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta),$$

$$\breve{W}(v) = \int_0^\infty \kappa(u,v) u^\beta du = v^{-\beta-1} \frac{\Gamma(\frac{\beta+1}{m})}{m} C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta).$$

□

Remark 2.5. If any of the following conditions:
(i) $\rho = 1, \delta = -1, \beta > m \geq 1$; *(ii)* $\rho = -1, \delta = 1, \beta > m \geq 1$; *(iii)* $\rho = 1, \delta = 1, \beta \geq m \geq 1$; *(iv)* $\rho = -1, \delta = -1, \beta > m \geq 1$; is hold then the series in (11) is converges. Therefore in the condition of Lemma 2.4 (11) is convergent.

Lemma 2.6. *Let $\rho, \delta = \pm 1, \beta > 0, \beta_3 \leq \beta_2 < \beta_1 (\beta_2 \neq \beta_3, \rho = -1), \beta > m \geq 1$ for $\rho\delta = -1, \beta \geq m \geq 1$ for $\rho\delta = 1; \kappa(u,v)$ and $C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)$ are as in (6) and (11) respectively, for sufficiently small $\epsilon > 0$, $f_\epsilon(u), g_\epsilon(u)$ are defined as follows:*

$$f_\epsilon(u) = \begin{cases} u^{\beta+\frac{\epsilon}{p}} & u \in (0,1] \\ 0 & u \in (1,\infty) \end{cases}, \tag{19}$$

$$g_\epsilon(u) = \begin{cases} 0 & u \in (0,1] \\ u^{\beta-\frac{\epsilon}{p'}} & u \in (1,\infty) \end{cases}, \tag{20}$$

then

$$lim_{\epsilon\to 0^+}\epsilon J = \epsilon\int_0^\infty\int_0^\infty \kappa(u,v)f_\epsilon(u)g_\epsilon(v)dudv$$

$$= \frac{\Gamma(\frac{\beta+1}{m})}{m}C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta). \tag{21}$$

Proof. Substituting $uv = t$, we have

$$\epsilon J = \epsilon\int_1^\infty v^{\beta-\frac{\epsilon}{p'}}\Big(\int_0^1 \kappa(u,v)u^{\beta+\frac{\epsilon}{p}}du\Big)dv$$

$$= \epsilon\int_1^\infty v^{\beta-\frac{\epsilon}{p'}-\beta-\frac{\epsilon}{p}-1}\Big(\int_0^v \kappa(1,t)t^{\beta+\frac{\epsilon}{p}}dt\Big)dv$$

$$= \epsilon\int_1^\infty v^{-\epsilon-1}\Big(\int_0^1 \kappa(1,t)t^{\beta+\frac{\epsilon}{p}}dt\Big)dv + \epsilon\int_1^\infty v^{-1-\epsilon}\Big(\int_1^v \kappa(1,t)t^{\beta+\frac{\epsilon}{p}}dt\Big)dv$$

by Fubini's theorem, we have

$$= \int_0^1 \kappa(1,t)t^{\beta+\frac{\epsilon}{p}}dt + \epsilon\int_1^\infty \kappa(1,t)t^{\beta+\frac{\epsilon}{p}}\Big(\int_t^\infty v^{-\epsilon-1}dv\Big)dt$$

$$= \int_0^1 \kappa(1,t)t^{\beta+\frac{\epsilon}{p}}dt + \int_1^\infty \kappa(1,t)t^{\beta-\frac{\epsilon}{p'}}dt. \tag{22}$$

let $\epsilon \to 0^+$, we get

$$= \int_0^1 \kappa(1,t)t^{\beta}dt + \int_1^\infty \kappa(1,t)t^{\beta}dt = \int_0^\infty \kappa(1,t)t^{\beta}dt.$$

using (18) of 2.4 , we have

$$lim_{\epsilon\to 0^+}\epsilon J = \frac{\Gamma(\frac{\beta+1}{m})}{m}C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta).$$

□

3. Main Results

THEOREM 3.1. *Let* $p > 1, \frac{1}{p} + \frac{1}{p'} = 1, \rho,\delta = \pm 1, \beta_1 > 0, m \geq 1, \beta_3 \leq \beta_2 < \beta_1(\beta_2 \neq \beta_3, \rho = -1)$. *Let* $\beta > m$ *for* $\rho\delta = -1; \beta \geq m$ *for* $\rho\delta = 1$. *Let* $\mu(u) = u^{-(p\beta+1)}, \nu(v) = v^{-(p'\beta+1)}$ *and define* $\tau(u), \sigma(u) \geq 0$ *such that* $\tau(u) \in L^p_\mu[0,\infty), \sigma(u) \in L^{p'}_\nu[0,\infty)$. *Furthermore, define* $\kappa(u,v)$ *and* $C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)$ *are defined by* (6) *and* (11) *respectively, then*

$$\int_0^\infty\int_0^\infty \kappa(u,v)\tau(u)\sigma(v)dudv < \frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)||\tau||_{p,\mu}||\sigma||_{p',\nu}, \tag{23}$$

where $\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)$ *is the best possible constant.*

Proof. We use the idea as in [9]

$$\int_{\omega*\omega}\kappa(u,v)\tau(u)\sigma(v)d\mu_1(u)d\mu_2(v)\le\bigg(\int_\omega\phi^p(u)\tau(u)\tau^p(u)d\mu_1(u)\bigg)^{\frac{1}{p}}\times$$

$$\bigg(\int_\omega\chi^{p'}(v)G(v)\sigma^{p'}(v)d\mu_2(v)\bigg)^{\frac{1}{p'}},\tag{24}$$

where $p>1, \frac{1}{p}+\frac{1}{p'}=1;\mu_1,\mu_2$ are positive and $\sigma-$ finite measures,
$\kappa:\omega*\omega\to\mathbb{R}\ ;\ \tau,\sigma,\phi,\chi:[0,\infty)\to\mathbb{R}$ are non-negative, measurable functions and

$$F(u)=\int_\omega\frac{\kappa(u,v)}{\chi^p(v)}d\mu_2(v),\ G(v)=\int_\omega\frac{\kappa(u,v)}{\phi^{p'}(u)}d\mu_1(u).$$

In (24) equality holds if and only if $\tau^p(u)=k_1\phi^{-(p+p')}(u)$ and $\sigma^{p'}(v)=k_2\chi^{-(p+p')}(v)$ for arbitrary constants k_1,k_2.
Let $\omega=[0,\infty)\,,\phi(u)=u^{-\frac{\beta}{p'}},\chi(v)=v^{-\frac{\beta}{p}}$ and let $\kappa(u,v)$ be defined in (6), then $F(u)=\int_\omega\kappa(u,v)v^\beta dv=W(u)$ and $G(v)=\breve{W}(v)$ as defined in Lemma 2.4.

Using (12), (13) in (24), we have

$$\int_0^\infty\int_0^\infty\kappa(u,v)\tau(u)\sigma(v)dudv\le\bigg(\int_0^\infty u^{-\frac{p\beta}{p'}}W(u)\tau^p(u)du\bigg)^{\frac{1}{p}}\bigg(\int_0^\infty v^{-\frac{p'\beta}{p}}\breve{W}(v)\sigma^{p'}(v)dv\bigg)^{\frac{1}{p'}}$$

$$\le\bigg(\int_0^\infty u^{-\frac{p\beta}{p'}-\beta-1}\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\tau^p(u)du\bigg)^{\frac{1}{p}}\times$$

$$\bigg(\int_0^\infty v^{-\frac{p'\beta}{p}-\beta-1}\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\sigma^{p'}(v)dv\bigg)^{\frac{1}{p'}}$$

$$=\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\bigg(\int_0^\infty u^{-(p\beta+1)}\tau^p(u)du\bigg)^{\frac{1}{p}}\times\bigg(\int_0^\infty v^{-(p'\beta+1)}\sigma^{p'}(v)dv\bigg)^{\frac{1}{p'}}$$

$$=\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)||\tau||_{p,\mu}||\sigma||_{p',\nu}.\tag{25}$$

If we consider equality in (25), we will have $u^{-(p\beta+1)}\tau^p(u)=\frac{k_1}{u}$ and $v^{-(p'\beta+1)}\sigma^{p'}(v)=\frac{k_2}{v}$, which is contradiction to the case $\tau(u)\in L^p_\mu[0,\infty)\,,\sigma(u)\in L^{p'}_\nu[0,\infty)$.
So, in (25) strict inequality holds, which yields (23).

Now, we will show that the factor $\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)$ in (23) is the best constant. For this, we assume that there be a constant $\theta\bigg(0<\theta<\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\bigg)$, such that (23) is holds if we take θ on the place of $\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)$. That is

$$\int_0^\infty\int_0^\infty\kappa(u,v)\tau(u)\sigma(v)dudv<\theta||\tau||_{p,\mu}||\sigma||_{p',\nu},\tag{26}$$

In particular, if we take f_ϵ and g_ϵ defined in Lemma 2.6, on the place of τ and σ in (26) respectively, we have

$$\epsilon\int_0^\infty\int_0^\infty \kappa(u,v)f_\epsilon(u)g_\epsilon(v)dudv < \epsilon\theta\Big(\int_0^1 u^{-\epsilon-1}du\Big)^{\frac{1}{p}}\Big(\int_1^\infty v^{-\epsilon-1}dv\Big)^{\frac{1}{p'}} = \theta.$$

By Lemma 2.6, we have

$$\frac{1}{m}\Gamma\big(\frac{\beta+1}{m}\big)C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta) < \theta.$$

Take $\epsilon\to 0^+$, we have $\frac{1}{m}\Gamma\big(\frac{\beta+1}{m}\big)C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\le\theta$, which contradicts the existance of θ, which gives the constant factor $\frac{1}{m}\Gamma\big(\frac{\beta+1}{m}\big)C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)$ is the best possile in (23).

□

THEOREM 3.2. *Let* $p>1, \frac{1}{p}+\frac{1}{p'}=1, \rho,\delta=\pm1, \beta_1>0, m\ge 1, \beta_3\le\beta_2<\beta_1(\beta_2\ne\beta_3,\rho=-1)$, *for* $\rho\delta=-1, \beta>m$; *for* $\rho\delta=1,\beta\ge m; \mu(u)=u^{-(p\beta+1)}, \nu(v)=v^{-(p'\beta+1)}, \tau(u)\ge 0, \tau(u)\in L^p_\mu[0,\infty), \kappa(u,v)$ *and* $C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)$ *are same as in Theorem 3.1, then*

$$\int_0^\infty v^{p(\beta+1)-1}\Big(\int_0^\infty \kappa(u,v)\tau(u)du\Big)^p dv < \Big(\frac{1}{m}\Gamma\big(\frac{\beta+1}{m}\big)C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\Big)^p\|\tau\|^p_{p,\mu}, \tag{27}$$

where the constant factor $\Big(\frac{1}{m}\Gamma\big(\frac{\beta+1}{m}\big)C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\Big)^p$ *is the best possible, and* (27) *is equivalent to* (23).

Proof. Substituting $\sigma(v)=v^{p(\beta+1)-1}\Big(\int_0^\infty\kappa(u,v)\tau(u)du\Big)^{p-1}$, by (23), we have

$$0<\Big(\|\sigma\|_{p',\nu}\Big)^{pq}=\Big(\int_0^\infty v^{-(p'\beta+1)}\sigma^{p'}(v)dv\Big)^p$$

$$=\Big(\int_0^\infty v^{p(\beta+1)-1}\Big(\int_0^\infty\kappa(u,v)\tau(u)du\Big)^p dv\Big)^p=\Big(\int_0^\infty\int_0^\infty\kappa(u,v)\tau(u)\sigma(v)dudv\Big)^p$$

$$\le\Big(\frac{1}{m}\Gamma\big(\frac{\beta+1}{m}\big)C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\Big)^p\|\tau\|^p_{p,\mu}\|\sigma\|^p_{p',\nu}. \tag{28}$$

Therefore

$$0<\Big(\|\sigma\|_{p',\nu}\Big)^{p'}=\int_0^\infty v^{p(\beta+1)-1}\Big(\int_0^\infty\kappa(u,v)\tau(u)du\Big)^p dv$$

$$\le\Big(\frac{1}{m}\Gamma\big(\frac{\beta+1}{m}\big)C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\Big)^p\|\tau\|^p_{p,\mu}. \tag{29}$$

Since $\tau(u)\in L^p_\mu[0,\infty)$, by (29), it follows that $\sigma(u)\in L^{p'}_\nu[0,\infty)$; by using (23) again, both (28) and (29) have strict inequalities, thus (27) proved.
Secondly, assume that (27) holds, by Hölder's inequality, we get

$$\int_0^\infty \int_0^\infty \kappa(u,v)\tau(u)\sigma(v)dudv = \int_0^\infty \Big(v^{\beta+\frac{1}{p'}} \int_0^\infty \kappa(u,v)\tau(u)du\Big)\Big(v^{-(\beta+\frac{1}{p'})}\sigma(v)\Big)dv$$

$$\leq \Bigg(\int_0^\infty v^{p\beta+p-1}\Big(\int_0^\infty \kappa(u,v)\tau(u)du\Big)^p dv\Bigg)^{\frac{1}{p}} ||\sigma||_{p',\nu}. \tag{30}$$

Applying (27) to (30), we obtain (23). Hence (23) and (27) are equivalent. On the contrary if we assume that the constant factor $\Big(\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\Big)^p$ in (27) is not the best possible, then by the equivalence of (23) and (27), we have, the constant appeared in (23) is not the best constant, which is contradiction. Hence, the constant $\Big(\frac{1}{m}\Gamma(\frac{\beta+1}{m})C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta)\Big)^p$ is the best.

□

4. Conclusion

For $\rho=-1,\delta=-1,\beta_3=-\beta_2,\beta=2nm+m-1(n\in\mathbb{N}\cup\{0\}),m\geq 1$; by (7) and (11); Theorem 3.1 gives the following:

COROLLARY 4.1. *Let* $p>1,\frac{1}{p}+\frac{1}{p'}=1;\beta_2>0,\beta_2<\beta_1;n\in\mathbb{N}\cup\{0\},\phi(u)=\cot u,\mu(u)=u^{p-2nmp-mp-1},\nu(v)=v^{p'-2nmq-mq-1},\tau(u),\sigma(u)\geq 0,\tau(u)\in L^p_\mu[0,\infty)$, $\sigma(u)\in L^{p'}_\nu[0,\infty)$; *then*

$$\int_0^\infty \int_0^\infty \sinh\big(\beta_2(uv)^m\big)\operatorname{csch}\big(\beta_1(uv)^m\big)\tau(u)\sigma(v)dudv$$

$$< -\frac{1}{m}\Big(\frac{\pi}{2\beta_1}\Big)^{2n+1}\phi^{2n}\Big(\frac{\beta_1+\beta_2}{2\beta_1}\pi\Big)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{31}$$

Proof. Since

$$\kappa(u,v) := \frac{e^{\beta_2(uv)^m}+\rho e^{\beta_3(uv)^m}}{e^{\beta_1(uv)^m}+\delta e^{-\beta_1(uv)^m}}.$$

After putting $\rho=-1,\delta=-1,\beta_3=-\beta_2$, we have

$$\kappa(u,v) := \frac{e^{\beta_2(uv)^m}-e^{-\beta_2(uv)^m}}{e^{\beta_1(uv)^m}-e^{-\beta_1(uv)^m}} = \sinh\big(\beta_2(uv)^m\big)\operatorname{csch}\big(\beta_1(uv)^m\big),$$

and

$$C_{\rho,\delta}(\beta_1,\beta_2,\beta_3,\beta) = \sum_{j=0}^\infty \frac{1}{(2\beta_1 j+\beta_1-\beta_2)^{\frac{\beta+1}{m}}} - \sum_{j=0}^\infty \frac{1}{(2\beta_1 j+\beta_1+\beta_2)^{\frac{\beta+1}{m}}},$$

using these two, we have

$$\int_0^\infty \int_0^\infty \sinh\big(\beta_2(uv)^m\big)\operatorname{csch}\big(\beta_1(uv)^m\big)\tau(u)\sigma(v)dudv$$

$$< \frac{1}{m}\Gamma(\frac{\beta+1}{m})\Bigg(\sum_{j=0}^\infty \frac{1}{(2\beta_1 j+\beta_1-\beta_2)^{\frac{\beta+1}{m}}} - \sum_{j=0}^\infty \frac{1}{(2\beta_1 j+\beta_1+\beta_2)^{\frac{\beta+1}{m}}}\Bigg)||\tau||_{p,\mu}||\sigma||_{p',\nu}.$$

Now putting $\frac{\beta+1}{m} = 2n+1$, in right hand side of above inequality, it becomes

$$\frac{1}{m}\Gamma\big(2n+1\big)\bigg(\sum_{j=0}^{\infty}\frac{1}{(2\beta_1 j+\beta_1-\beta_2)^{2n+1}}-\sum_{j=0}^{\infty}\frac{1}{(2\beta_1 j+\beta_1+\beta_2)^{2n+1}}\bigg)||\tau||_{p,\mu}||\sigma||_{p',\nu}.$$

Using (7) by setting $x = \frac{\beta_1+\beta_2}{2\beta_1}$ and $y = \frac{\beta_1-\beta_2}{2\beta_1}$, above inequality takes form

$$\int_0^\infty\int_0^\infty \sinh\big(\beta_2(uv)^m\big)\operatorname{csch}\big(\beta_1(uv)^m\big)\tau(u)\sigma(v)dudv$$
$$< -\frac{1}{m}\frac{(2n)!}{(2\beta)^{2n+1}}\bigg(\sum_{j=0}^{\infty}\Big(\frac{1}{(j+x)^{2n+1}}-\frac{1}{(j+y)^{2n+1}}\Big)\bigg)||\tau||_{p,\mu}||\sigma||_{p',\nu}$$
$$= -\frac{1}{m}\bigg(\frac{\pi}{2\beta_1}\bigg)^{2n+1}\phi^{2n}\bigg(\frac{\beta_1+\beta_2}{2\beta_1}\pi\bigg)||\tau||_{p,\mu}||\sigma||_{p',\nu},$$

which proves (31). □

For $\beta_1 = 2\lambda, \beta_2 = \lambda$ in (31); and since

$$\frac{e^{\lambda(uv)^m}-e^{-\lambda(uv)^m}}{e^{2\lambda(uv)^m}-e^{-2\lambda(uv)^m}} = \frac{1}{2}\operatorname{sech}\big(\lambda(uv)^m\big),$$

we have

$$\int_0^\infty\int_0^\infty \operatorname{sech}\big(\lambda(uv)^m\big)\tau(u)\sigma(v)dudv < -\frac{2}{m}\bigg(\frac{\pi}{4\lambda}\bigg)^{2n+1}\phi^{2n}\bigg(\frac{3\pi}{4}\bigg)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{32}$$

Now, letting $\lambda = 1, n = 0$ in (32), we get $\mu(u) = u^{p-pm-1}, \nu(v) = v^{p'-qm-1}$ and

$$\int_0^\infty\int_0^\infty \operatorname{sech}\big((uv)^m\big)\tau(u)\sigma(v)dudv < \frac{\pi}{2m}||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{33}$$

For $\beta_1 = 3\lambda, \beta_2 = \lambda, \lambda > 0$, (31) becomes

$$\int_0^\infty\int_0^\infty \sinh\big(\lambda(uv)^m\big)\operatorname{csch}\big(3\lambda(uv)^m\big)\tau(u)\sigma(v)dudv$$
$$< -\frac{1}{m}\bigg(\frac{\pi}{6\lambda}\bigg)^{2n+1}\phi^{2n}\bigg(\frac{2\pi}{3}\bigg)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{34}$$

Particularly, setting $\lambda = 1, n = 0$ in (34), we have

$$\int_0^\infty\int_0^\infty \sinh\big((uv)^m\big)\operatorname{csch}\big(3(uv)^m\big)\tau(u)\sigma(v)dudv < \frac{1}{m}\frac{\sqrt{3}\pi}{18}||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{35}$$

Let $\beta_1 = 3\lambda, \beta_2 = 2\lambda, \lambda > 0$ in (31); then

$$\int_0^\infty\int_0^\infty \sinh\big(2\lambda(uv)^m\big)\operatorname{csch}\big(3\lambda(uv)^m\big)\tau(u)\sigma(v)dudv$$
$$< -\frac{1}{m}\bigg(\frac{\pi}{6\lambda}\bigg)^{2n+1}\phi^{2n}\bigg(\frac{5\pi}{6}\bigg)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{36}$$

Particularly, putting $\lambda = 1, n = 0$ in (36); we have

$$\int_0^\infty \int_0^\infty \sinh\left(2(uv)^m\right)\operatorname{csch}\left(3(uv)^m\right)\tau(u)\sigma(v)dudv < \frac{1}{m}\frac{\sqrt{3}\pi}{6}||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{37}$$

Let $\beta_1 = 4\lambda, \beta_2 = \lambda$ in (31) and note that

$$\frac{e^{\lambda(uv)^m} - e^{-\lambda(uv)^m}}{e^{4\lambda(uv)^m} - e^{-4\lambda(uv)^m}} = \frac{1}{4}\operatorname{sech}\left(\lambda(uv)^m\right)\operatorname{sech}\left(2\lambda(uv)^m\right),$$

so, we have

$$\int_0^\infty \int_0^\infty \operatorname{sech}\left(\lambda(uv)^m\right)\operatorname{sech}\left(2\lambda(uv)^m\right)\tau(u)\sigma(v)dudv$$

$$< -\frac{4}{m}\left(\frac{\pi}{8\lambda}\right)^{2n+1}\phi^{2n}\left(\frac{5\pi}{8}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{38}$$

Particularly, using $\lambda = 1, n = 0$ in (38), we have

$$\int_0^\infty \int_0^\infty \operatorname{sech}\left((uv)^m\right)\operatorname{sech}\left(2(uv)^m\right)\tau(u)\sigma(v)dudv < \frac{(\sqrt{2}-1)\pi}{2m}||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{39}$$

As like above, on using $\rho = 1, \delta = -1, \beta_3 = -\beta_2, \beta = 2nm + 2m - 1 (n \in \mathbb{N} \cup \{0\})$ in Theorem 3.1 and by (8), (11); we have the following:

COROLLARY 4.2. *Let* $p > 1, \frac{1}{p} + \frac{1}{p'}; \beta_2 \geq 0, \beta_2 < \beta_1; n \in \mathbb{N} \cup \{0\}, \phi(u) = \cot u, \mu(u) = u^{p-2nmp-2mp-1}, \nu(v) = v^{p'-2nmq-2mq-1}, \tau(u), \sigma(u) \geq 0, \tau(u) \in L^p_\mu[0,\infty), \sigma(u) \in L^{p'}_\nu[0,\infty)$; *then*

$$\int_0^\infty \int_0^\infty \cosh\left(\beta_2(uv)^m\right)\operatorname{csch}\left(\beta_1(uv)^m\right)\tau(u)\sigma(v)dudv$$

$$< -\frac{1}{m}\left(\frac{\pi}{2\beta}\right)^{2n+2}\phi^{2n+1}\left(\frac{\beta_1 + \beta_2}{2\beta_1}\pi\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{40}$$

For $\beta_1 = \lambda, \beta_2 = 0, \lambda > 0$ in (40), we obtain

(41)
$$\int_0^\infty \int_0^\infty \operatorname{csch}\left(\lambda(uv)^m\right)\tau(u)\sigma(v)dudv < -\frac{1}{m}\left(\frac{\pi}{2\lambda}\right)^{2n+2}\phi^{2n+1}\left(\frac{\pi}{2}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}.$$

For $\beta_1 = 2\lambda, \beta_2 = \lambda$ in (40) and since

$$\frac{e^{\lambda(uv)^m} + e^{-\lambda(uv)^m}}{e^{2\lambda(uv)^m} - e^{-2\lambda(uv)^m}} = \frac{1}{2}\operatorname{csch}\left(\lambda(uv)^m\right),$$

we have

(42)
$$\int_0^\infty \int_0^\infty \operatorname{csch}\left(\lambda(uv)^m\right)\tau(u)\sigma(v)dudv < -\frac{2}{m}\left(\frac{\pi}{4\lambda}\right)^{2n+2}\phi^{2n+1}\left(\frac{3\pi}{4}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}.$$

REMARK 4.3. Note that (41) and (42) are equivalent . Since

$$\sum_{j=0}^{\infty}\left(\frac{1}{(4j+1)^{2n+2}}+\frac{1}{(4j+3)^{2n+2}}\right)=\sum_{j=0}^{\infty}\frac{1}{(2j+1)^{2n+2}}.$$

using this in (8) it can be easy to prove that

$$\phi^{2n+1}\left(\frac{3\pi}{4}\right)=2^{2n+1}\phi^{2n+1}\left(\frac{\pi}{2}\right).$$

Thus (41) and (42) are equivalent.

Assuming $\beta_1=3\lambda,\beta_2=\lambda,\lambda>0$ or $\beta_1=3\lambda,\beta_2=2\lambda,\lambda>0$ in (40); then we have the following inequalities respectively:

$$\int_0^{\infty}\int_0^{\infty}\cosh\big(\lambda(uv)^m\big)\operatorname{csch}\big(3\lambda(uv)^m\big)\tau(u)\sigma(v)dudv$$

$$< -\frac{1}{m}\left(\frac{\pi}{6\lambda}\right)^{2n+2}\phi^{2n+1}\left(\frac{2\pi}{3}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}, \tag{43}$$

$$\int_0^{\infty}\int_0^{\infty}\cosh\big(2\lambda(uv)^m\big)\operatorname{csch}\big(3\lambda(uv)^m\big)\tau(u)\sigma(v)dudv$$

$$< -\frac{1}{m}\left(\frac{\pi}{6\lambda}\right)^{2n+2}\phi^{2n+1}\left(\frac{5\pi}{6}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{44}$$

Assuming $\beta_1=4\lambda,\beta_2=\lambda,\lambda>0$ in (40); then

$$\int_0^{\infty}\int_0^{\infty}\operatorname{csch}\big(\lambda(uv)^m\big)\operatorname{sech}\big(2\lambda(uv)^m\big)\tau(u)\sigma(v)dudv$$

$$< -\frac{1}{m}\frac{1}{4^{3n+2}}\big(\frac{\pi}{\lambda}\big)^{2n+2}\phi^{2n+1}\big(\frac{5\pi}{8}\big)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{45}$$

Now, for $\rho=1,\delta=1,\beta=2nm+m-1(n\in\mathbb{N}\cup\{0\})$ and applying (9), (11) in Theorem 3.1, we have

COROLLARY 4.4. *Let* $p>1,\frac{1}{p}+\frac{1}{p'}=1;\beta_2\geq 0,\beta_2<\beta_1;n\in\mathbb{N}\cup\{0\},\chi(u)=\csc u,\mu(u)=u^{p-2nmp-mp-1},\nu(v)=v^{p'-2nmq-mq-1},\tau(u),\sigma(u)\geq 0,\tau(u)\in L^p_\mu[0,\infty),$ $\sigma(u)\in L^{p'}_\nu[0,\infty)$, *then*

$$\int_0^{\infty}\int_0^{\infty}\cosh\big(\beta_2(uv)^m\big)\operatorname{sech}\big(\beta_1(uv)^m\big)\tau(u)\sigma(v)dudv$$

$$< \frac{1}{m}\left(\frac{\pi}{2\beta_1}\right)^{2n+1}\chi^{2n}\left(\frac{\beta_1+\beta_2}{2\beta_1}\pi\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{46}$$

Let $\beta_1=\lambda,\lambda>0,\beta_2=0$ in (46), we have

$$\int_0^{\infty}\int_0^{\infty}\operatorname{sech}\big(\lambda(uv)^m\big)\tau(u)\sigma(v)dudv<\frac{1}{m}\left(\frac{\pi}{2\lambda}\right)^{2n+1}\chi^{2n}\left(\frac{\pi}{2}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{47}$$

REMARK 4.5. The inequalities in (32) and (47) are equivalent. With the help of (7), (9) and the below result

$$\sum_{j=0}^{\infty}\left(\frac{1}{(4j+1)^{2n+1}}-\frac{1}{(4j+3)^{2n+1}}\right)=\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2n+1}},$$

we have $\phi^{2n+1}\left(\frac{3\pi}{4}\right)=-2^{2n}\chi^{2n}\left(\frac{\pi}{2}\right)$. Therefore (47) is equivalent to (32).

Let $\beta_1=2\lambda,\beta_2=\lambda,\lambda>0$ in (46) and in view of

$$\frac{e^{\lambda(uv)^m}+e^{-\lambda(uv)^m}}{e^{2\lambda(uv)^m}+e^{-2\lambda(uv)^m}}=\frac{1}{2}\operatorname{csch}\left(\lambda(uv)^m\right)\tanh\left(2\lambda(uv)^m\right),$$

we have

$$\int_0^\infty\int_0^\infty \operatorname{csch}\left(\lambda(uv)^m\right)\tanh\left(2\lambda(uv)^m\right)\tau(u)\sigma(v)dudv$$

$$<\frac{2}{m}\left(\frac{\pi}{4\lambda}\right)^{2n+1}\chi^{2n}\left(\frac{3\pi}{4}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{48}$$

Now, for $\lambda=1,n=0$ in (48), we have $\mu(u)=u^{p-pm-1},\nu(v)=v^{p'-mq-1}$ and

$$\int_0^\infty\int_0^\infty \operatorname{csch}\left((uv)^m\right)\tanh\left(2(uv)^m\right)\tau(u)\sigma(v)dudv<\frac{\pi}{\sqrt{2}m}||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{49}$$

Let $\beta_1=\alpha,\beta_2=2-\alpha$ in (46), $1<\alpha\le 2$, then

$$\int_0^\infty\int_0^\infty \cosh\left((2-\alpha)(uv)^m\right)\operatorname{sech}\left(\alpha(uv)^m\right)\tau(u)\sigma(v)dudv$$

$$<\frac{1}{m}\left(\frac{\pi}{2\alpha}\right)^{2n+1}\chi^{2n}\left(\frac{\pi}{\alpha}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{50}$$

Using $n=0$ in (50), we have

$$\int_0^\infty\int_0^\infty \cosh\left((2-\alpha)(uv)^m\right)\operatorname{sech}\left(\alpha(uv)^m\right)\tau(u)\sigma(v)dudv$$

$$<\frac{1}{m}\left(\frac{\pi}{2\alpha}\right)csc\left(\frac{\pi}{\alpha}\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{51}$$

Similarly, by setting $\rho=-1,\delta=1,\beta_3=-\beta_2,\beta=2nm+2m-1(n\in\mathbb{N}\cup\{0\})$ in Theorem 3.1, we have

COROLLARY 4.6. *Let* $p>1,\frac{1}{p}+\frac{1}{p'}=1;\beta_2>0,\beta_2<\beta_1;n\in\mathbb{N}\cup\{0\};\chi(u)=\csc u,\mu(u)=u^{p-2nmp-2mp-1},\nu(v)=v^{p'-2nmq-2mq-1},\tau(u),\sigma(u)\ge 0,\tau(u)\in L^p_\mu[0,\infty),\sigma(u)\in L^{p'}_\nu[0,\infty)$; *then*

$$\int_0^\infty\int_0^\infty \sinh\left(\beta_2(uv)^m\right)\operatorname{sech}\left(\beta_1(uv)^m\right)\tau(u)\sigma(v)dudv$$

$$< \frac{1}{m}\left(\frac{\pi}{2\beta_1}\right)^{2n+2}\chi^{2n+1}\left(\frac{\beta_1+\beta_2}{2\beta_1}\pi\right)||\tau||_{p,\mu}||\sigma||_{p',\nu}. \tag{52}$$

As above, for different values of β_1, β_2 and n in (52), we will find some other certain form of Hilbert's-type inequality.

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