Korean J. Math. **31** (2023), No. 2, pp. 203–216
http://dx.doi.org/10.11568/kjm.2023.31.2.203

A HOPF BIFURCATION IN AN ATTRACTION-ATTRACTION CHEMOTAXIS SYSTEM WITH GLOBAL COUPLING

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Abstract. We consider a bistable attraction-attraction chemotaxis system with global coupling term. The study in this paper asserts that conditions for chemotactic coefficients for attraction and attraction and the global coupling constant to show existence of stationary solutions and Hopf bifurcation in the interfacial problem as the bifurcation parameters vary are obtained analytically.

1. Introduction

We are concerned with the chemotaxis system (see [3, 6, 9, 14, 15, 20]):

$$\begin{cases} \varepsilon\sigma U_t = \varepsilon^2\nabla^2 U - \varepsilon\nabla\cdot(\kappa_1 U\,\nabla\chi(V)) + \varepsilon\nabla\cdot(\kappa_2 U\,\nabla\xi(W)) + F(U,V), \\ V_t = \nabla^2 V + \mu U - V, \\ W_t = \nabla^2 W + U + V - W, \quad 0 < x < 1,\ t > 0, \end{cases} \tag{1}$$

where $U(x,t)$ is a cell density, $V(x,t)$ and $W(x,t)$ are the chemical concentrations of attractant and repellent and $F(U,V) = -U - V + H(U - a_0)$. The parameters $\varepsilon, \sigma, \kappa_1, \kappa_2, \mu$ and a_0 are positive constants, ∇ is the gradient operator, $\chi(V)$ and $\xi(W)$ are the chemical sensitivity functions.

The system with $\varepsilon = 1$, $F = 0$ and a domain $x \in (0,\infty)$ is globally well-posed in the sense that $\kappa_2 - \mu\kappa_1 > 0$ in [7]. The solution behavior of (1) with $F = 0$ in the multi-dimensional case was essentially determined by the competition of attraction and repulsion which is characterized by the sign of $\kappa_2 - \mu\kappa_1$ in [16, 17].

In this paper, we consider the attraction-attraction system satisfying that $\chi'(V) > 0$ for $V > 0$ and $\xi'(W) < 0$ for $W > 0$([16]) and the threshold a_0 is replaced by the total activator and inhibitor concentration in the medium ([8, 13, 19]):

$$a = a_0 + \nu\Big(\int_0^1 (U+V)dx - s_0\Big) \tag{2}$$

where a_0 and s_0 are positive constants, and ν characterizes the intensity of global coupling.

Received March 8, 2023. Revised June 21, 2023. Accepted June 21, 2023.
2010 Mathematics Subject Classification: 35K57, 35R35, 35B32, 35B25, 35K55, 35K57, 58J55.
Key words and phrases: attraction-attraction, chemotaxis, free boundary problem, Hopf bifurcation.
This work was supported by Kyonggi University Research Grant 2021.

We shall deal with the following a free boundary problem which is reduced from (1) with the global coupling term (2) and investigate the existence of time periodic solutions. A free boundary problem of (1) with the global coupling term (2) is given by :

$$\text{(3)} \qquad \begin{cases} V_t = V_{xx} - (\mu+1)V + \mu H(x-\eta(t)),\ 0<x<1,\ t>0 \\ W_t = W_{xx} - W + H(x-\eta(t)),\ 0<x<1,\ t>0 \\ V_x(0,t) = 0 = V_x(1,t),\ W_x(0,t) = 0 = W_x(1,t),\ t>0. \end{cases}$$

Let A be an operator defined by $A := -\frac{d^2}{dx^2} + \mu + 1$ with domain $D(A) = \{V \in H^{2,2}((0,1)) : V_x(0,t) = 0,\ V_x(1,t) = 0\}$. Let $A_0 := -\frac{d^2}{dx^2} + 1$ with domain $D(A_0) = \{W \in H^{2,2}((0,1)) : W_x(0,t) = 0,\ W_x(1,t) = 0\}$. The velocity of the interface $\eta(t)$ is given by (see [12, 18]);

$$\text{(4)} \qquad \frac{d\,\eta(t)}{dt} = \frac{1}{\sigma}\Big(C(V(\eta(t)); a) + \kappa_1 V_x(\eta(t),t) - \kappa_2 W_x(\eta(t),t)\Big),\ \eta(0) = \eta_0,$$

where C is a continuously differentiable function defined on an interval $I := (-a, 1-a)$, which is given by ([2, 8, 12])

$$\text{(5)} \qquad C(r\,;\,a) = -\frac{1-2a-2r}{\sqrt{(r+a)(1-a-r)}}$$

with $a = a_0 + \nu(1-\eta-s_0)$.

The organization of the paper is as follows: In section 2, a change of variables is given which regularizes problem (3) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we obtain a regularity of the solution which is sufficient for an analysis of the bifurcation. We show the existence of equilibrium solutions for (3) and obtain the linearization of problem (3) under the condition of global coupling constant ν. In section 3, we investigate the conditions to obtain the periodic solutions and the bifurcation of the interface problem as the parameter σ varies and examine the global coupling effect.

2. Equilibrium solutions and Linearization of the interface equation

In order to apply semigroup theory to (3), we choose the space $X := L_2(0,1)$ with norm $\|\cdot\|_2$. To get differential dependence on initial conditions, we decompose V in (3) into two parts: u, which is a solution to a more regular problem and g, which is less regular but explicitly known in terms of the Green's function G of the operator A. Namely, we define $g : [0,1]\times[0,1] \to \mathbb{R}$, by

$$g(x,\eta) := A^{-1}(\mu H(\cdot-\eta)(x)) = \mu \int_0^1 G(x,y)\, H(y-\eta)\, dy,$$

where $G : [0,1]\times[0,1] \to \mathbb{R}$ is a Green's function of A satisfying the Neumann boundary conditions, and $\gamma : [0,1] \to \mathbb{R}$, $\gamma(\eta) := g(\eta,\eta)$. If we take a transformation $u(t)(x) = V(x,t) - g(x,\eta(t))$, we have $(u_x)(t)(x) = V_x(x,t) - g_x(x,\eta(t))$. Since $G_x(x,\eta)$ is discontinuous, we cannot obtain one step more regular than that of (3). Let $p(x,t) = V_x(x,t)$ and define $\hat{g} : [0,1]\times[0,1] \to \mathbb{R}$,

$$\hat{g}(x,\eta) := A^{-1}(\mu\delta(\cdot-\eta)(x)) = \mu \int_0^1 \hat{G}(x,y)\, \delta(y-\eta)\, dy,$$

where $\hat{G} : [0,1]\times[0,1] \to \mathbb{R}$ is a Green's function of A satisfying the Dirichlet boundary conditions, and $\hat{\gamma} : [0,1] \to \mathbb{R},\ \hat{\gamma}(\eta) := \hat{g}(\eta,\eta)$. We define $j : [0,1] \times [0,1] \to \mathbb{R}$,

$$j(x,\eta) := A_0^{-1}(H(\cdot - \eta)(r)) = \int_0^1 J(x,y)H(y-\eta)\,dy$$

and $\alpha : [0,1] \to \mathbb{R},\ \ \alpha(\eta) := j(\eta,\eta)$. Here $J : [0,1]^2 \to \mathbb{R}$ is a Green's function of A_0 satisfying the boundary conditions. Define $w(t)(x) = W(x,t) - j(x,\eta(t))$, $q(x,t) = W_x(x,t)$ and define $\hat{j} : [0,1]) \times [0,1] \to \mathbb{R}$,

$$\hat{j}(x,\eta) := A_0^{-1}(\delta(\cdot - \eta)(x)) = \int_0^1 \hat{J}(x,y)\,\delta(y-\eta)\,dy,$$

where $\hat{J} : [0,1] \times [0,1] \to \mathbb{R}$ is a Green's function of A_0 satisfying the Dirichlet boundary conditions and $\hat{\alpha} : [0,1] \to \mathbb{R},\ \hat{\alpha}(\eta) := \hat{j}(\eta,\eta)$.

Applying the transformations $u(t)(x) = V(x,t) - g(x,\eta(t))$, $v(t)(x) = p(x,t) - \hat{g}(x,\eta(t))$ and $w(t)(x) = W(x,t) - j(x,\eta(t))$, $z(t)(x) = q(x,t) - \hat{j}(x,\eta(t))$, then (3) becomes

(6)
$$\begin{cases} u_t + Au = \frac{1}{\sigma}\mu G(x,\eta)\big(C(u(\eta)+\gamma(\eta);a) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(z(\eta)+\hat{\alpha}(\eta))\big) \\ v_t + Av = -\frac{1}{\sigma}\frac{\mu}{\eta}\hat{G}(x,\eta)\big(C(u(\eta)+\gamma(\eta);a) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(z(\eta)+\hat{\alpha}(\eta))\big) \\ w_t + A_0 w = \frac{1}{\sigma}J(x,\eta)\big(C(u(\eta)+\gamma(\eta);a) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(z(\eta)+\hat{\alpha}(\eta))\big) \\ z_t + A_0 z = -\frac{1}{\sigma}\frac{1}{\eta}\hat{J}(x,\eta)\big(C(u(\eta)+\gamma(\eta);a) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(z(\eta)+\hat{\alpha}(\eta))\big) \\ \eta'(t) = \frac{1}{\sigma}\big(C(u(\eta)+\gamma(\eta);a) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(z(\eta)+\hat{\alpha}(\eta))\big),\ t > 0. \end{cases}$$

Thus, we obtain an abstract evolution equation equivalent to (3) :

$$\text{(7)} \qquad \begin{cases} \frac{d}{dt}(u,v,w,z,\eta) + \widetilde{A}(u,v,w,z,\eta) = \frac{1}{\sigma}\,f(u,v,w,z,\eta), \\ (u,v,w,z,\eta)(0) = (u_0(x), v_0(x), w_0(x), z_0(x), \eta_0), \end{cases}$$

where $\widetilde{A}$ is a 5×5 matrix where (1,1) and (2,2)-entries are an operator A, (3,3) and (4,4)-entries are an operator A_0 and all the others are zero. The nonlinear forcing term f is

$$f(u,v,w,z,\eta) = \begin{pmatrix} f_1(\eta)\cdot(f_{21}(u,v,w,z,\eta) + f_{22}(u,v,w,z,\eta)) \\ f_2(\eta)\cdot(f_{21}(u,v,w,z,\eta) + f_{22}(u,v,w,z,\eta)) \\ f_3(\eta)\cdot(f_{21}(u,v,w,z,\eta) + f_{22}(u,v,w,z,\eta)) \\ f_4(\eta)\cdot(f_{21}(u,v,w,z,\eta) + f_{22}(u,v,w,z,\eta))) \\ f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,z,\eta) \end{pmatrix},$$

where $f_1 : (0,1) \to X,\ f_1(\eta)(x) := \mu G(x,\eta)$, $f_2 : (0,1) \to X,\ f_2(\eta)(x) := -\frac{\mu}{\eta}\hat{G}(x,\eta)$, $f_3 : (0,1) \to X,\ f_3(\eta)(x) := J(x,\eta)$, $f_4 : (0,1) \to X,\ f_4(\eta)(x) := -\frac{1}{\eta}\hat{J}(x,\eta)$, $f_{21} : Y \to \mathbb{C},\ f_{21}(u,v,w,z,\eta) := C(u(\eta)+\gamma(\eta);a)$, $f_{22} : Y \to \mathbb{C},\ f_{22}(u,v,w,z,\eta) := \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(z(\eta)+\hat{\alpha}(\eta))$ and $Y := \{(u,v,w,z,\eta) \in C^1(0,1)\times C^1(0,1)\times C^1(0,1)\times C^1(0,1)\times(0,1)\ :\ u(\eta)+\gamma(\eta) \in I,\ v(\eta)+\hat{\gamma}(\eta) \in I,\ w(\eta)+\alpha(\eta) \in I,\ z(\eta)+\hat{\alpha}(\eta) \in I\} \subset_{\text{open}} C^1(\mathbb{R})\times C^1(\mathbb{R})\times C^1(\mathbb{R})\times C^1(\mathbb{R})\times\mathbb{R}$.

The well-posedness of solutions of (7) is shown in [5, 9, 18] with the help of the semigroup theory using domains of fractional powers $\theta \in (3/4, 1]$ of A, A_0 and $\widetilde{A}$. Moreover, the nonlinear term f is a continuously differentiable function from $W \cap \widetilde{X}^\theta$ to $\widetilde{X}$, where $\widetilde{X} := D(\widetilde{A}) = D(A) \times D(A) \times D(A_0) \times D(A_0) \times \mathbb{R}$, $X^\theta := D(A^\theta)$; $X_0^\theta := D(A_0^\theta)$ and $\widetilde{X}^\theta := D(\widetilde{A}^\theta) = X^\theta \times X^\theta \times X_0^\theta \times X_0^\theta \times \mathbb{R}$.

The derivative of f can be obtained from the following in [4]:

Lemma 2.1. *The functions $G(\cdot, \eta) : (0,1) \to X$, $\hat{G}(\cdot, \eta) : (0,1) \to X$, $J(\cdot, \eta) : (0,1) \to X$, $\hat{J}(\cdot, \eta) : (0,1) \to X$, $C(\cdot) : Y \to \mathbb{C}$ and $f : Y \to X \times \mathbb{R}$ are continuously differentiable with derivatives given by*

$$Df_{21}(u,v,w,z,\eta)(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) = C'(u(\eta)+\gamma(\eta);a)\cdot(u'(\eta)\tilde{\eta}+\tilde{u}(\eta)+\gamma'(\eta)\tilde{\eta})$$

$$Df_{22}(u,v,w,z,\eta)(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) = \kappa_1\big(v'(\eta)\hat{\eta}+\hat{v}(\eta)+\hat{\gamma}'(\eta)\hat{\eta}\big)$$

$$\begin{aligned} &Df(u,v,w,z,\eta)(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) \\ &= (f_{21}(u,v,w,z,\eta)+f_{22}(u,v,w,z,\eta))\cdot(f_1'(\eta),f_2'(\eta),f_3'(\eta),f_4'(\eta),0)\,\hat{\eta} \\ &\quad +(Df_{21}(u,v,w,z,\eta)+Df_{22}(u,v,w,z,\eta))(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta})\cdot(f_1(\eta),f_2(\eta),f_3(\eta),f_4(\eta),1). \end{aligned}$$

We shall examine the existence of equilibrium solutions of (7) and thus, we look for $(u^*, v^*, w^*, z^*, \eta^*) \in D(\widetilde{A}) \cap Y$ satisfying the following equations:

$$(8)\qquad \begin{cases} Au = \frac{1}{\sigma}\,\mu\, G(\cdot,\eta)P(u,v,w,z,\eta) \\ Av = -\frac{1}{\sigma}\frac{\mu}{\eta}\,\hat{G}(\cdot,\eta)P(u,v,w,z,\eta) \\ A_0 w = \frac{1}{\sigma}\,J(\cdot,\eta)P(u,v,w,z,\eta) \\ A_0 z = -\frac{1}{\sigma}\frac{1}{\eta}\,\hat{J}(\cdot,\eta)P(u,v,w,z,\eta) \\ 0 = P(u,v,w,z,\eta) \\ u'(0)=0=u'(1),\ v(0)=0=v(1),\ w'(0)=0=w'(1),\ z(0)=0=z(1), \end{cases}$$

where $P(u,v,w,z,\eta) := C(u(\eta)+\gamma(\eta);a)+\kappa_1\chi'(u(\eta)+\gamma(\eta))(v(\eta)+\hat{\gamma}(\eta))-\kappa_2\xi'(w(\eta)+\alpha(\eta))(z(\eta)+\hat{\alpha}(\eta))$.

Theorem 2.2. *Suppose that $0 < \frac{1}{2} - a_0 + \nu s_0 < \frac{2}{3}\nu + \gamma(\frac{1}{3})$ for $\eta \in (\frac{1}{3}, 1)$. Then equation (7) has at least one equilibrium solution $(0,0,0,0,\eta^*)$, $\eta^* \in (\frac{1}{3}, 1)$. The linearization of f at the stationary solution $(0,0,0,0,\eta^*)$ is*

$$Df(0,0,0,0,\eta^*)(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) = \begin{pmatrix} \mu G(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) \\ -\frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) \\ J(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) \\ -\frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) \\ Q(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) \end{pmatrix},$$

where $Q(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\eta}) = 4(\hat{u}(\eta^*)+\gamma'(\eta^*)\hat{\eta}-\nu\hat{\eta}) + \kappa_1\chi''(\gamma(\eta^*)\hat{\gamma}(\eta^*)(\hat{u}(\eta^*)+\gamma'(\eta^*)\hat{\eta}) + \kappa_1\chi'(\gamma(\eta^*))(\hat{v}(\eta^*)+\hat{\gamma}'(\eta^*)\hat{\eta}) - \kappa_2\xi''(\alpha(\eta^*))\hat{\alpha}'(\eta^*)(\hat{w}(\eta^*)+\alpha'(\eta^*)\hat{\eta}) - \kappa_2\xi'(\alpha(\eta^*))(\hat{z}(\eta^*)+\hat{\alpha}'(\eta^*)\hat{\eta})$. The pair $(0,0,0,0,\eta^*)$ corresponds to a unique steady state $(V^*, p^*, W^*, q^*, \eta^*)$ of (3) for $\sigma \neq 0$ with $V^*(x) = g(x,\eta^*)$, $p^*(x) = \hat{g}(x,\eta^*)$, $W^*(x) = j(x,\eta^*)$ and $q^*(x) = \hat{j}(x,\eta^*)$.

Proof. From the system of equations (8), we have $u^* = 0, v^* = 0, w^* = 0$ and $z^* = 0$. In order to show existence of η^*, we define

$$\Gamma(\eta) := C(\gamma(\eta); a) + \kappa_1 \chi'(\gamma(\eta))\,\hat{\gamma}(\eta) - \kappa_2\xi'(\alpha(\eta))\hat{\alpha}(\eta).$$

Then

$$\begin{aligned}\Gamma'(\eta) = \;& C'(\gamma(\eta); a) + \kappa_1\chi''(\gamma(\eta))\hat{\gamma}(\eta)\gamma'(\eta) + \kappa_1\chi'(\gamma(\eta))\hat{\gamma}'(\eta) - \kappa_2\xi'(\alpha(\eta))\hat{\alpha}'(\eta)\\ & -\kappa_2\xi''(\alpha(\eta))\alpha'(\eta)\hat{\alpha}(\eta).\end{aligned}$$

Since $\gamma'(\eta) < 0$ for $\eta \in (0,1)$, $C'(\gamma(\eta); a) < 0$ for $\eta \in (0,1)$, and since $\chi'(\gamma(\eta)) > 0$, $\xi'(\alpha(\eta)) < 0$ and $\hat{\gamma}'(\eta) < 0, \hat{\alpha}'(\eta) < 0$ for $\eta \in (1/3, 1)$, we have $\Gamma'(\eta) < 0$ for $(\frac{1}{3}, 1)$. Thus, we must have $\Gamma(\frac{1}{3}) > 0$ and $\Gamma(1) < 0$ in order for η^* to exist in $(\frac{1}{3}, 1)$.

The formula for $Df(0,0,0,0,\eta^*)$ follows from the relation $C'(1/2 - a_0 - \nu(1 - s_0)) = 4$, and the corresponding steady state $(V^*, p^*, W^*, q^*, \eta^*)$ for (3) is obtained by using the transformation and Theorem 2.1 in [4].

□

3. A Hopf bifurcation with a global coupling effect

In this section, we shall show that there exists a Hopf bifurcation from the curve $\sigma \mapsto (0,0,0,0,\eta^*)$ of the equilibrium solution. First, let us introduce the following relevant definition.

Definition 3.1. Under the assumptions of Theorem 2.2, define (for $1 \geq \theta > 3/4$) the linear operator B from $\widetilde{X}^\theta$ to $\widetilde{X}$ by

$$B := Df(0,0,0,0,\eta^*)\,.$$

We then define $(0,0,0,0,\eta^*)$ to be a Hopf point for (7) if and only if there exists an $\epsilon_0 > 0$ and a C^1-curve

$$(-\epsilon_0 + \tau^*, \tau^* + \epsilon_0) \mapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times \widetilde{X}_{\mathbf{C}}$$

($X_{\mathbf{C}}$ denotes the complexification of the real space X) of eigendata for $-\widetilde{A} + \tau B$ with

(i) $(-\widetilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau), \quad (-\widetilde{A} + \tau B)(\overline{\phi(\tau)}) = \overline{\lambda(\tau)}\,\overline{\phi(\tau)}$;
(ii) $\lambda(\tau^*) = i\beta$ with $\beta > 0$;
(iii) $\mathrm{Re}\,(\lambda) \neq 0$ for all λ in the spectrum of $(-\widetilde{A} + \tau^* B) \setminus \{\pm i\beta\}$;
(iv) $\mathrm{Re}\,\lambda'(\tau^*) \neq 0$ (transversality);

where $\tau = 1/\sigma$.

Next, we check (7) for Hopf points. For this we have to solve the eigenvalue problem:

$$-\widetilde{A}(u,v,w,z,\eta) + \tau B(u,v,w,z,\eta) = \lambda I_5(u,v,w,z,\eta),$$

where I_5 is an 5×5 identity matrix. This is equivalent to

$$(9)\qquad \begin{cases} (A+\lambda)u = \tau\mu\, G(\cdot,\eta^*)R(u,v,w,z,\eta), \\ (A+\lambda)v = -\tau\,\frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)\,R(u,v,w,z,\eta), \\ (A_0+\lambda)w = \tau\, J(\cdot,\eta^*)R(u,v,w,z,\eta), \\ (A_0+\lambda)z = -\tau\frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)R(u,v,w,z,\eta), \\ \lambda\,\eta = \tau R(u,v,w,z,\eta), \end{cases}$$

where $R(u,v,w,z,\eta) = 4(u(\eta^*)+\gamma'(\eta^*)\eta - \nu\,\eta) + \kappa_1 d_2(u(\eta^*)+\gamma'(\eta^*)\eta) + \kappa_1 d_1(v(\eta^*) + \hat{\gamma}'(\eta^*)\eta) - \kappa_2 d_3(w(\eta^*)+\alpha'(\eta^*)\eta) - \kappa_2 d_4(z(\eta^*)+\hat{\alpha}'(\eta^*)\eta))$, $d_1 = \chi'(\gamma(\eta^*))$, $d_2 = \chi''(\gamma(\eta^*))\hat{\gamma}(\eta^*)$, $d_3 = \xi''(\alpha(\eta))\hat{\alpha}(\eta^*)$ and $d_4 = \xi'(\alpha(\eta^*))$.

We shall show that an equilibrium solution becomes a Hopf point.

THEOREM 3.2. *Suppose that* $0 < \frac{1}{2} - a_0 + \nu s_0 < \frac{2}{3}\nu + \gamma(\frac{1}{3})$, $d_3 < \frac{d_4}{\eta^*} < 0$ *and* $0 < \frac{d_1}{\eta^*} < d_2$. *Suppose the operator* $-\widetilde{A} + \tau^* B$ *has a unique pair* $\{\pm i\beta\}$, $\beta > 0$ *of purely imaginary eigenvalues for some* $\tau^* > 0$. *Then* $(0,0,0,0,\eta^*,\tau^*)$ *is a Hopf point for (7).*

Proof. We assume without loss of generality that $\beta > 0$, and Φ^* is the (normalized) eigenfunction of $-\widetilde{A}+\tau^* B$ with eigenvalue $i\beta$. We have to show that $(\Phi^*, i\beta)$ can be extended to a C^1-curve $\tau \mapsto (\Phi(\tau), \lambda(\tau))$ of eigendata for $-\widetilde{A}+\tau B$ with $\mathrm{Re}(\lambda'(\tau^*)) \neq 0$.

For this, let $\Phi^* = (\psi_0, v_0, w_0, z_0, \eta_0) \in D(A)\times D(A)\times D(A_0)\times D(A_0)\times \mathbb{R}$. First, we note that $\eta_0 \neq 0$. Otherwise, by (9), $(A+i\beta)\psi_0 = i\beta\,\mu\,\eta_0\, G(\cdot,\eta^*) = 0$ and $(A+i\beta)v_0 = -\frac{\mu}{\eta^*}\, i\beta\,\eta_0\,\hat{G}(\cdot,\eta^*) = 0$, which is not possible given A is symmetric. So, without loss of generality, let $\eta_0 = 1$. Then $E(\psi_0, v_0, w_0, z_0, i\beta, \tau^*) = 0$ by (9), where

$$E : D(A)_{\mathbf{C}} \times D(A)_{\mathbf{C}} \times D(A_0)_{\mathbf{C}} \times D(A_0)_{\mathbf{C}} \times \mathbb{C}\times\mathbb{R} \longrightarrow X_{\mathbf{C}}\times X_{\mathbf{C}}\times X_{\mathbf{C}}\times X_{\mathbf{C}}\times\mathbb{C},$$

$$E(u,v,w,z,\lambda,\tau) := \begin{pmatrix} (A+\lambda)u - \tau\mu\, G(\cdot,\eta^*)R^*(\eta^*) \\ (A+\lambda)v + \tau\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)R^*(\eta^*) \\ (A_0+\lambda)w - \tau J(\cdot,\eta^*)R^*(\eta^*) \\ (A_0+\lambda)z + \tau\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)R^*(\eta^*) \\ \lambda - \tau R^*(\eta^*) \end{pmatrix},$$

where $R^*(\eta^*) = (4+\kappa_1 d_2)(u(\eta^*)+\gamma'(\eta^*)) - 4\nu + \kappa_1 d_1(v(\eta^*)+\hat{\gamma}'(\eta^*)) - \kappa_2 d_3(w(\eta^*) + \alpha'(\eta^*)) - \kappa_2 d_4(z(\eta^*)+\hat{\alpha}'(\eta^*))$. The equation $E(u,v,w,z,\lambda,\tau) = 0$ is equivalent to λ being an eigenvalue of $-\widetilde{A}+\tau B$ with eigenfunction $(u,v,w,z,1)$. We shall apply the implicit function theorem to E. For this we check that E is of C^1- class and that

(10)

$$D_{(u,v,w,z,\lambda)}E(\psi_0,v_0,w_0,z_0,i\beta,\tau^*) \in L(D(A)_{\mathbf{C}}\times D(A)_{\mathbf{C}}\times D(A_0)_{\mathbf{C}}\times D(A_0)_{\mathbf{C}}\times\mathbb{C}\times\mathbb{R}, X_{\mathbf{C}}\times\mathbb{C})$$

is an isomorphism. In addition, the mapping

$$D_{(u,v,w,z,\lambda)}E(\psi_0,v_0,w_0,z_0,i\beta,\tau^*)(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\lambda})$$

$$=\begin{pmatrix} (A+i\beta)\hat{u}+\hat{\lambda}\psi_0-\tau^*\mu G(\cdot,\eta^*)\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\\ (A+i\beta)\hat{v}+\hat{\lambda}v_0+\tau^*\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\\ (A_0+i\beta)\hat{w}+\hat{\lambda}w_0-\tau^*\,J(\cdot,\eta^*)\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\\ (A_0+i\beta)\hat{z}+\hat{\lambda}z_0+\tau^*\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\\ \hat{\lambda}-\tau^*\,\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\end{pmatrix}$$

is a compact perturbation of the mapping

$$(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\lambda})\longmapsto\big((A+i\beta)\hat{u},(A+i\beta)\hat{v}\,,(A_0+i\beta)\hat{w},(A_0+i\beta)\hat{z},\hat{\lambda}\big)$$

which is invertible. Thus, $D_{(u,v,w,z,\lambda)}E(\psi_0,v_0,w_0,z_0,i\beta,\tau^*)$ is a Fredholm operator of index 0. Therefore, in order to verify (10), it suffices to show that the system of equations

$$D_{(u,v,w,z,\lambda)}E(\psi_0,v_0,w_0,z_0,i\beta,\tau^*)(\hat{u},\hat{v},\hat{w},\hat{z},\hat{\lambda})=0$$

which is equivalent to

(11)
$$\begin{cases}(A+i\beta)\hat{u}+\hat{\lambda}\psi_0=\tau^*\mu\,G(\cdot,\eta^*)\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\\ (A+i\beta)\hat{v}+\hat{\lambda}v_0=-\tau^*\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\\ (A_0+i\beta)\hat{w}+\hat{\lambda}w_0=\tau^*\,J(\cdot,\eta^*)\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\\ (A_0+i\beta)\hat{z}+\hat{\lambda}z_0=-\tau^*\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\\ \hat{\lambda}=\tau^*\,\big((4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*)\big)\end{cases}$$

necessarily implies that $\hat{u}=0$, $\hat{v}=0$, $\hat{w}=0$, $\hat{z}=0$ and $\hat{\lambda}=0$. If we define $\phi:=\psi_0-\mu G(\cdot,\eta^*)$, $\xi:=v_0+\frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)$, $\rho:=w_0-J(\cdot,\eta^*)$ and $\zeta:=z_0+\frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)$, then (11) becomes

(12) $$(A+i\beta)\hat{u}+\hat{\lambda}\phi=0,$$
(13) $$(A+i\beta)\hat{v}+\hat{\lambda}\xi=0,$$
(14) $$(A_0+i\beta)\hat{w}+\hat{\lambda}\rho=0,$$
(15) $$(A_0+i\beta)\hat{z}+\hat{\lambda}\zeta=0,$$
(16) $$\frac{\hat{\lambda}}{\tau^*}=(4+\kappa_1 d_2)\hat{u}(\eta^*)+\kappa_1 d_1\hat{v}(\eta^*)-\kappa_2 d_3\hat{w}(\eta^*)-\kappa_2 d_4\hat{z}(\eta^*).$$

On the other hand, since $E(\psi_0,v_0,w_0,z_0,i\beta,\tau^*)=0$, ϕ, ξ,ρ and ζ are solutions to the equations, we have:

(17) $$(A+i\,\beta)\phi=-\mu\,\delta_{\eta^*},$$
(18) $$(A+i\beta)\xi=\frac{\mu}{\eta^*}\delta_{\eta^*},$$
(19) $$(A_0+i\,\beta)\rho=-\delta_{\eta^*},$$
(20) $$(A_0+i\,\beta)\zeta=\frac{1}{\eta^*}\delta_{\eta^*},$$

(21)
$$\frac{i\beta}{\tau^*} = (4+\kappa_1 d_2)(\phi(\eta^*)+\mu G(\eta^*,\eta^*)+\gamma'(\eta^*)) - 4\nu + \kappa_1 d_1(\xi(\eta^*) - \frac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*)+\hat{\gamma}'(\eta^*))$$
$$-\kappa_2 d_3(\rho(\eta^*)+J(\eta^*,\eta^*)+\alpha'(\eta^*)) - \kappa_2 d_4(\zeta(\eta^*) - \frac{1}{\eta^*}\hat{J}(\eta^*,\eta^*)+\hat{\alpha}'(\eta^*)).$$

Multiplying (13) and (18) by ϕ, and (12) and (17) by ξ and subtracting one from the other, we now obtain

$$\hat{u}(\eta^*) = -\eta^*\,\hat{v}(\eta^*),\ \hat{w}(\eta^*) = -\eta^*\,\hat{z}(\eta^*), \tag{22}$$
$$\phi(\eta^*) = -\eta^*\,\xi(\eta^*),\ \rho(\eta^*) = -\eta^*\,\zeta(\eta^*). \tag{23}$$

Multiplying (12) by $(4+\kappa_1 d_2)\overline{\phi}$, (13) by $-\eta^*\kappa_1 d_1\overline{\xi}$, (14) by $-\mu\kappa_2 d_3\overline{\rho}$ and (15) by $\mu\eta^*\kappa_2 d_4\overline{\zeta}$ and adding the resultants to each, and multiplying (17) by $(4+\kappa_1 d_2)\overline{\phi}$, (18) by $-\eta^*\kappa_1 d_1\overline{\xi}$, (19) by $-\mu\kappa_2 d_3\overline{\rho}$ and (15) by $\mu\eta^*\kappa_2 d_4\overline{\zeta}$ and adding the resultants to each, and from (21), we obtain

$$\frac{\mu}{\tau^*} = (4+\kappa_1 d_2)||\phi||^2 - \eta^*\kappa_1 d_1||\xi||^2 - \mu\kappa_2 d_3||\rho||^2 + \mu\eta^*\kappa_2 d_4||\zeta||^2. \tag{24}$$

and thus

$$\int \big((4+\kappa_1 d_2)\hat{u}\overline{\phi} - \eta^*\kappa_1 d_1\hat{v}\overline{\xi} - \mu\kappa_2 d_3\hat{w}\overline{\rho} + \mu\eta^*\kappa_2 d_4\hat{z}\overline{\zeta}\big) = 0. \tag{25}$$

Now, multiplying (12) by $(4+\kappa_1 d_2)\overline{\hat{u}}$, (17) by $-\eta^*\kappa_1 d_1\overline{\hat{v}}$, (14) by $-\mu\kappa_2 d_3\overline{\hat{w}}$ and (15) by $\mu\eta^*\kappa_2 d_4\overline{\hat{z}}$ and adding the resultants to each, we now obtain

$$\Big((4+\kappa_1 d_2)||A^{1/2}\hat{u}||^2 - \eta^*\kappa_1 d_1||A^{1/2}\hat{v}||^2 - \mu\kappa_2 d_3||A_0^{1/2}\hat{w}||^2 + \mu\eta^*\kappa_2 d_4||A_0^{1/2}\hat{z}||^2\Big)$$
$$+i\beta\Big((4+\kappa_1 d_2)||\hat{u}||^2 - \eta^*\kappa_1 d_1||\hat{v}||^2 - \mu\kappa_2 d_3||\hat{w}||^2 + \mu\eta^*\kappa_2 d_4||\hat{z}||^2\Big)$$
$$+\hat{\lambda}\int\Big((4+\kappa_1 d_2)\phi\overline{\hat{u}} - \eta^*\kappa_1 d_1\xi\overline{\hat{v}} - \mu\kappa_2 d_3\rho\overline{\hat{w}} + \mu\eta^*\kappa_2 d_4\zeta\overline{\hat{z}}\Big) = 0$$

and from (25), we have

(26)
$$\begin{cases} (4+\kappa_1 d_2)||A^{1/2}\hat{u}||^2 - \eta^*\kappa_1 d_1||A^{1/2}\hat{v}||^2 - \mu\kappa_2 d_3||A_0^{1/2}\hat{w}||^2 + \mu\eta^*\kappa_2 d_4||A_0^{1/2}\hat{z}||^2 = 0 \\ (4+\kappa_1 d_2)||\hat{u}||^2 - \eta^*\kappa_1 d_1||\hat{v}||^2 - \mu\kappa_2 d_3||\hat{w}||^2 + \mu\eta^*\kappa_2 d_4||\hat{z}||^2 = 0. \end{cases}$$

Multiplying (17) by $\overline{\phi}$ and (18) by $\overline{\xi}$, we then get

$$||A^{1/2}\phi||^2 + i\beta||\phi||^2 = -\mu\overline{\phi}(\eta^*) \quad\text{and}\quad ||A^{1/2}\xi||^2 + i\beta||\xi||^2 = \frac{\mu}{\eta^*}\,\overline{\xi}(\eta^*)$$

and applying (22) to the above equation, we have

$$||A^{1/2}\phi||^2 = (\eta^*)^2||A^{1/2}\xi||^2 \quad\text{and}\quad ||\phi||^2 = (\eta^*)^2||\xi||^2. \tag{27}$$

Now, multiplying (12) by $2i\beta\overline{\hat{u}}$ and (17) by $\hat{\lambda}\overline{\hat{u}}$ and subtracting the resultants from each other, we now obtain

$$2i\beta\big(||A^{1/2}\hat{u}||^2 - (\eta^*)^2||A^{1/2}\hat{v}||^2\big) - 2\beta^2\big(||\hat{u}||^2 - (\eta^*)^2||\hat{v}||^2\big) + \hat{\lambda}\big(||\phi||^2 - (\eta^*)^2||\xi||^2\big).$$

Applying (27) to the above equation, we have

$$||A^{1/2}\hat{u}||^2 - (\eta^*)^2||A^{1/2}\hat{v}||^2 = 0 \quad\text{and}\quad ||\hat{u}||^2 - (\eta^*)^2||\hat{v}||^2 = 0$$

and thus (26) implies:

$$\big(4+\kappa_1 d_2 - \frac{\kappa_1 d_1}{\eta^*}\big)||\hat{u}||^2 + \mu\kappa_2\big(\frac{d_4}{\eta^*} - d_3\big)||\hat{w}||^2 = 0.$$

Since $d_2 - \frac{d_1}{\eta^*} > 0$ and $\frac{d_4}{\eta^*} - d_3 > 0$, we have $\hat{u} = 0$ and $\hat{w} = 0$, and so, $\hat{v} = 0, \hat{z} = 0$, $\hat{\lambda} = 0$. □

THEOREM 3.3. *Under the same condition as in Theorem 3.2,* $(0,0,0,0,\eta^*,\tau^*)$ *satisfies the transversality condition. Hence, this is a Hopf point for (7).*

Proof. By implicit differentiation of $E(\psi_0(\tau), v_0(\tau), w_0(\tau), z_0(\tau), \lambda(\tau), \tau) = 0$, we find that

$$D_{(u,v,w,z,\lambda)}E(\psi_0, v_0, w_0, z_0, i\beta, \tau^*)(\psi_0'(\tau^*), v_0'(\tau^*), w_0'(\tau^*), z_0'(\tau^*), \lambda'(\tau^*))$$

$$= \begin{pmatrix} \mu G(\cdot,\eta^*)S(\eta^*) \\ -\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)S(\eta^*) \\ J(\cdot,\eta^*)S(\eta^*) \\ -\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)S(\eta^*) \\ S(\eta^*) \end{pmatrix},$$

where $S(\eta^*) = (4+\kappa_1 d_2)(\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1 d_1(v_0(\eta^*) + \hat{\gamma}'(\eta^*)) - \kappa_2 d_3(w_0(\eta^*) + \alpha'(\eta^*)) - \kappa_2 d_4(z_0(\eta^*) + \hat{\alpha}'(\eta^*))$. This means that the functions $\tilde{u} := \psi_0'(\tau^*)$, $\tilde{v} := v_0'(\tau^*)$, $\tilde{w} := w_0'(\tau^*)$, $\tilde{z} := z_0'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

$$(28)\quad \begin{cases} (A+i\beta)\tilde{u} + \tilde{\lambda}\psi_0 - \tau^*\mu G(\cdot,\eta^*)\widetilde{S}(\eta^*) = \mu G(\cdot,\eta^*)S(\eta^*), \\ (A+i\beta)\tilde{v} + \tilde{\lambda}\xi_0 + \tau^*\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)\widetilde{S}(\eta^*) = -\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)S(\eta^*) \\ (A_0+i\beta)\tilde{w} + \tilde{\lambda}\rho_0 - \tau^* J(\cdot,\eta^*)\widetilde{S}(\eta^*) = J(\cdot,\eta^*)S(\eta^*), \\ (A_0+i\beta)\tilde{z} + \tilde{\lambda}\zeta_0 + \tau^*\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)\widetilde{S}(\eta^*) = -\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)S(\eta^*) \\ \tilde{\lambda} - \tau^*\big((4+\kappa_1 d_2)\tilde{u}(\eta^*) + \kappa_1 d_1\tilde{v}(\eta^*) - \kappa_2 d_3\tilde{w}(\eta^*) - \kappa_2 d_4\tilde{z}(\eta^*)\big) = S(\eta^*). \end{cases}$$

By letting $\phi := \psi_0 - \mu G(\cdot,\eta^*)$, $\xi = v_0 + \frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)$, $\rho = w_0 - J(\cdot,\eta^*)$ and $\zeta = z_0 - \hat{J}(\cdot,\eta^*)$ as before, we obtain

$$(29)\qquad (A+i\beta)\tilde{u} + \tilde{\lambda}\phi = 0,$$

$$(30)\qquad (A+i\beta)\tilde{v} + \tilde{\lambda}\xi = 0,$$

$$(31)\qquad (A_0+i\beta)\tilde{w} + \tilde{\lambda}\rho = 0,$$

$$(32)\qquad (A_0+i\beta)\tilde{z} + \tilde{\lambda}\zeta = 0,$$

$$(33)\qquad \tilde{\lambda} - \tau^*\big((4+\kappa_1 d_2)\tilde{u}(\eta^*) + \kappa_1 d_1\tilde{v}(\eta^*) - \kappa_2 d_3\tilde{w}(\eta^*) - \kappa_2 d_4\tilde{z}(\eta^*)\big) = \tfrac{i\beta}{\tau^*}.$$

Multiplying (29) by $(4+\kappa_1 d_2)\overline{\phi}$, (30) by $-\eta^*\kappa_1 d_1\overline{\xi}$, (31) by $-\mu\kappa_2 d_3\overline{\rho}$ and (31) by $\mu\eta^*\kappa_2 d_4\overline{\zeta}$ and adding the resultants to each, we now obtain

$$\begin{aligned} &-(4+\kappa_1 d_2)\mu\tilde{u}(\eta^*) - \kappa_1 d_1\mu\,\tilde{v}(\eta^*) - \mu\kappa_2 d_3\,\tilde{w}(\eta^*) + \mu\eta^*\kappa_2 d_4\tilde{z}(\eta^*) \\ &+\tilde{\lambda}((4+\kappa_1 d_2)||\phi||^2 - \eta^*\kappa_1 d_1||\xi||^2 - \mu\kappa_2 d_3||\rho||^2 + \mu\eta^*\kappa_2 d_4||\zeta||^2) \\ &+2i\beta\textstyle\int\big((4+\kappa_1 d_2)\tilde{u}\overline{\phi} - \eta^*\kappa_1 d_1\tilde{v}\overline{\xi} - \mu\kappa_2 d_3\tilde{w}\,\overline{\rho} + \mu\eta^*\kappa_2 d_4\tilde{z}\,\overline{\zeta}\big) = 0. \end{aligned}$$

From (24) and (33), the above equation implies that

$$(34)\quad i\beta\frac{\mu}{(\tau^*)^2} + 2i\beta\int\Big((4+\kappa_1 d_2)\tilde{u}\overline{\phi} - \eta^*\kappa_1 d_1\tilde{v}\overline{\xi} - \mu\kappa_2 d_3\tilde{w}\,\overline{\rho} + \mu\eta^*\kappa_2 d_4\tilde{z}\,\overline{\zeta}\Big) = 0.$$

Multiplying (29) by $(4+\kappa_1 d_2)\overline{\tilde{u}}$, (30) by $-\eta^*\kappa d_1\overline{\tilde{v}}$, (31) by $-\mu\kappa_2 d_3\overline{\tilde{w}}$ and (32) by $\mu\eta^*\kappa_2 d_4\overline{\tilde{z}}$ and adding the resultants to each, we now obtain

$$\begin{aligned}&(4+\kappa_1 d_2)||A^{1/2}\tilde{u}||^2-\eta^*\kappa d_1||A^{1/2}\tilde{v}||^2-\mu\kappa_2 d_3||A_0^{1/2}\tilde{w}||^2+\mu\eta^*\kappa_2 d_4||A_0^{1/2}\tilde{z}||^2\\ &+i\beta\Big((4+\kappa_1 d_2)||\tilde{u}||^2-\eta^*\kappa d_1||\tilde{v}||^2-\mu\kappa_2 d_3||\tilde{w}||^2+\mu\eta^*\kappa_2 d_4||\tilde{z}||^2\Big)\\ &+\tilde{\lambda}\int\big(4+\kappa_1 d_2)\overline{\tilde{u}}\phi-\eta^*\kappa_1 d_1\overline{\tilde{v}}\xi-\mu\kappa_2 d_3\overline{\tilde{w}}\rho+\mu\eta^*\kappa_2 d_4\overline{\tilde{z}}\zeta\big)=0.\end{aligned}$$

From (34), we have

$$\begin{aligned}\tilde{\lambda}\,\tfrac{\mu}{2(\tau^*)^2}&=(4+\kappa_1 d_2)||A^{1/2}\tilde{u}||^2-\eta^*\kappa d_1||A^{1/2}\tilde{v}||^2-\mu\kappa_2 d_3||A_0^{1/2}\tilde{w}||^2+\mu\eta^*\kappa_2 d_4||A_0^{1/2}\tilde{z}||^2\\ &\quad+i\beta\Big((4+\kappa_1 d_2)||\tilde{u}||^2-\eta^*\kappa d_1||\tilde{v}||^2-\mu\kappa_2 d_3||\tilde{w}||^2+\mu\eta^*\kappa_2 d_4||\tilde{z}||^2\Big).\end{aligned}$$

The real part of the above is given by

$$\frac{\mu}{2(\tau^*)^2}\mathrm{Re}\tilde{\lambda}=(4+\kappa_1 d_2)||A^{1/2}\tilde{u}||^2-\eta^*\kappa d_1||A^{1/2}\tilde{v}||^2-\mu\kappa_2 d_3||A_0^{1/2}\tilde{w}||^2+\mu\eta^*\kappa_2 d_4||A_0^{1/2}\tilde{z}||^2. \tag{35}$$

Now, multiplying (29) by $2i\beta\overline{\tilde{u}}$ and (30) by $\tilde{\lambda}\overline{\tilde{u}}$ and subtracting resultants from each other, we now obtain

$$||A^{1/2}\tilde{u}||^2-(\eta^*)^2||A^{1/2}\tilde{v}||^2=0 \quad\text{and}\quad ||\tilde{u}||^2-(\eta^*)^2||\tilde{v}||^2=0$$

by (27). Thus (35) implies that

$$\frac{\mu}{2(\tau^*)^2}\mathrm{Re}\tilde{\lambda}=\big(4+\kappa_1 d_2-\frac{\kappa_1 d_1}{\eta^*}\big)||A^{1/2}\tilde{u}||^2+\kappa_2(\frac{d_4}{\eta^*}-d_3)||A_0^{1/2}\tilde{w}||^2$$

which is positive since $d_2>\frac{d_1}{\eta^*}>0$ and $0>\frac{d_4}{\eta^*}>d_3$. We have $\mathrm{Re}\lambda'(\tau^*)>0$ for $\beta>0$, and thus, by the Hopf-bifurcation theorem in [1], there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^*. □

We shall show that there exists a unique $\tau^*>0$ such that $(0,0,0,0,\eta^*,\tau^*)$ is a Hopf point; thus τ^* is the origin of a branch of nontrivial periodic orbits.

Lemma 3.4. *Under the same condition as in Theorem 3.2, let G_β and $\hat{G}_\beta$ be Green functions of the differential operator $A+i\beta$ satisfying (17) and (18), respectively. Then, the expression $(4+\kappa_1 d_2)\,\mathrm{Re}\,(G_\beta(\eta^*,\eta^*))-\frac{\kappa_1 d_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*))$ and $-d_3(\mathrm{Re}(J_\beta(\eta^*,\eta^*))+\frac{d_4}{\eta^*}(\mathrm{Re}(\hat{J}_\beta(\eta^*,\eta^*))$ are strictly decreasing in $\beta\in\mathbb{R}^+$ with*

$$\mathrm{Re}\,G_0(\eta^*,\eta^*)=G(\eta^*,\eta^*),\quad \lim_{\beta\to\infty}\mathrm{Re}\,G_\beta(\eta^*,\eta^*)=0\,.$$

Moreover,

$$-(4+\kappa_1 d_2)\,\mathrm{Im}(G_\beta(\eta^*,\eta^*))+\tfrac{\kappa_1 d_1}{\eta^*}\mathrm{Im}(\hat{G}_\beta(\eta^*,\eta^*))+\kappa_2 d_3\mathrm{Im}(J_\beta(\eta^*,\eta^*))-\tfrac{\kappa_2 d_4}{\eta^*}\mathrm{Im}(\hat{J}_\beta(\eta^*,\eta^*))>0$$

for $\beta>0$.

Proof. First, we have $(A+i\beta)^{-1}=(A-i\beta)(A^2+\beta^2)^{-1}$, so if $L(\beta):=\mathrm{Re}\,(A+i\beta)^{-1}$, then $L(\beta)=A(A^2+\beta^2)^{-1}$. Moreover, $L(\beta)\longrightarrow A^{-1}$ as $\beta\to 0$ and $L(\beta)\longrightarrow 0$ as $\beta\to\infty$, which results in the corresponding limiting behavior for $\mathrm{Re}\,\big(G_\beta(\eta^*,\eta^*)\big)$.

Now to show that $\beta \mapsto ((4+\kappa_1 d_2)\,\mathrm{Re}\,(G_\beta(\eta^*,\eta^*)) - \frac{\kappa_1 d_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*))$ is strictly decreasing, define $h(\beta)(r) := (4+\kappa_1 d_2)\,G_\beta(r,\eta^*) - \frac{\kappa_1 d_1}{\eta^*}\,\hat{G}_\beta(r,\eta^*) - d_2\,G(r,\eta^*) + \frac{\kappa_1 d_1}{\eta^*}\,\hat{G}(r,\eta^*)$. Then (in the weak sense initially)

$$(A+i\beta)h(\beta) = -i\beta(4+\kappa_1 d_2)G(\cdot,\eta^*) + i\beta\tfrac{\kappa_1 d_1}{\eta^*}\hat{G}(\eta^*,\eta^*). \tag{36}$$

As a result $h(\beta) \in D(A)_{\mathbf{C}}$ and $h : \mathbb{R}^+ \to D(A)_{\mathbf{C}}$ is differentiable with $ih(\beta) + (A+i\beta)h'(\beta) = -i\,G(\cdot,\eta^*)$, therefore

$$(A+i\beta)h'(\beta) = -i\big((4+\kappa_1 d_2)\,G_\beta(\cdot,\eta^*) - \tfrac{\kappa_1 d_1}{\eta^*}\hat{G}_\beta(\cdot,\eta^*)\big).$$

Thus, we get

(37)

$$\begin{aligned} -i\,(4+\kappa_1 d_2 - \tfrac{\kappa_1 d_1}{\eta^*})\overline{h'(\beta)(\eta^*)} &= \textstyle\int (A+i\beta)^2 h'(\beta)\overline{h'(\beta)(r)}\,dr \\ &= \textstyle\int (A+i\beta)h'(\beta)\cdot(A+i\beta)\overline{h'(\beta)}\,dr \\ &= \textstyle\int |Ah'(\beta)|^2 - \beta^2|h'(\beta)|^2\,dr + 2i\beta\int Ah'(\beta)\overline{h'(\beta)}\,dr\,. \end{aligned}$$

From (37) it follows that

$$-(4+\kappa_1 d_2\, - \tfrac{\kappa_1 d_1}{\eta^*})\mathrm{Re}\,(h'(\beta)(\eta^*)) = 2\beta\textstyle\int |A^{1/2}h'(\beta)|^2 > 0$$

and thus $\mathrm{Re}\,(h'(\beta)(\eta^*)) < 0$ if $d_2\, > \frac{d_1}{\eta^*}$.

In order to show $(4+\kappa_1 d_2)\,\mathrm{Im}(G_\beta(\eta^*,\eta^*)) - \frac{\kappa_1 d_1}{\eta^*}\,\mathrm{Im}(\,\hat{G}_\beta(\eta^*,\eta^*)) < 0$ for $\beta > 0$, we multiply (36) by $\overline{h(\beta)(r)}$ and integrate the resulting equation, then we have:

$$-i\,\beta(4+\kappa_1 d_2 - \tfrac{\kappa_1 d_1}{\eta^*})\overline{h(\beta)(\eta^*)} = \textstyle\int A(A+i\beta)h(\beta)(r)\overline{h(\beta)(r)}\,dr = \int |Ah(\beta)|^2 + i\beta\int A|h(\beta)|^2,$$

which implies that $-\beta(4+\kappa_1 d_2 - \frac{\kappa_1 d_1}{\eta^*})\mathrm{Im}h(\beta)(\eta^*) = \int |Ah(\beta)|^2 > 0$. Since $d_2 > \frac{d_1}{\eta^*}$, we have $\mathrm{Im}h(\beta)(\eta^*) < 0$ for $\beta > 0$.

We define $k(\beta)(r) := d_3\,J_\beta(r,\eta^*) - \frac{d_4}{\eta^*}\,\hat{J}_\beta(r,\eta^*) - d_3\,J(r,\eta^*) + \frac{d_4}{\eta^*}\,\hat{J}(r,\eta^*)$. Then

$$\begin{cases} (-d_3 + \frac{d_4}{\eta^*})\mathrm{Re}\,(k'(\beta)(\eta^*)) = 2\beta||A_0^{1/2}k'(\beta)||^2 > 0 \\ \beta(-d_3 + \frac{d_4}{\eta^*})\mathrm{Im}k(\beta)(\eta^*) = ||A_0 k(\beta)||^2 > 0. \end{cases}$$

If $0 > \frac{d_4}{\eta^*} > d_3$, we have $\mathrm{Re}k'(\beta)(\eta^*) > 0$ and $\mathrm{Im}k(\beta)(\eta^*) > 0$ for $\beta > 0$. Thus, $-(4+\kappa_1 d_2\,\mathrm{Im}G_\beta(\eta^*,\eta^*) + \frac{\kappa_1 d_1}{\eta^*}\,\mathrm{Im}\,\hat{G}_\beta(\eta^*,\eta^*) + \kappa_2 d_3\mathrm{Im}(J_\beta(\eta^*,\eta^*)) - \frac{\kappa_2 d_4}{\eta^*}\mathrm{Im}(\hat{J}_\beta(\eta^*,\eta^*)) > 0$ for $\beta > 0$ if $d_2 > \frac{d_1}{\eta^*}$ and $\frac{d_4}{\eta^*} > d_3$. Similarly, $(4+\kappa_1 d_2)\,\mathrm{Re}\,(G_\beta(\eta^*,\eta^*)) - \frac{\kappa_1 d_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*)) - \kappa_2 d_3\mathrm{Re}(J_\beta(\eta^*,\eta^*)) + \frac{\kappa_2 d_4}{\eta^*}\mathrm{Re}(\hat{J}_\beta(\eta^*,\eta^*))$ is a strictly decreasing function of $\beta > 0$ since $d_2 > \frac{d_1}{\eta^*}$ and $\frac{d_4}{\eta^*} > d_3$. □

THEOREM 3.5. *Under the same condition as in Theorem 3.2, for a unique critical point $\tau^* > 0$, there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (9) with $\beta > 0$.*

Proof. We only need to show that the function $(u,v,w,z,\beta,\tau) \mapsto E(u,v,w,z,i\beta,\tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving the system of equations (9) with $\lambda = i\beta$, $u = V - \mu G(\cdot,\eta^*)$, $v = p + \frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)$, $w = W - J(\cdot,\eta^*)$ and

$z = q + \frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)$,

$$\begin{cases} (A+i\beta)V = -\mu\,\delta_{\eta^*}, \\ (A+i\beta)p = \frac{\mu}{\eta^*}\,\delta_{\eta^*}, \\ (A_0+i\beta)W = -\delta_{\eta^*}, \\ (A_0+i\beta)q = \frac{1}{\eta^*}\delta_{\eta^*}, \\ \frac{i\beta}{\tau^*} = (4+\kappa_1 d_2)(V(\eta^*)+\mu G(\eta^*,\eta^*)+\gamma'(\eta^*)) - 4\nu \; + \kappa_1 d_1(p(\eta^*) - \frac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*)+\hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2 d_3(W(\eta^*)+J(\eta^*,\eta^*)+\alpha'(\eta^*)) - \kappa_2 d_4(q(\eta^*) - \frac{1}{\eta^*}\hat{J}(\eta^*,\eta^*)+\hat{\alpha}'(\eta^*)). \end{cases}$$

The real and imaginary parts of the above equation are given by

(38)
$$\begin{aligned} \frac{\beta}{\tau^*} = \;& (4+\kappa_1 d_2)\mathrm{Im}(-\mu G_\beta(\eta^*,\eta^*)) + \frac{\mu\kappa_1 d_1}{\eta^*}\mathrm{Im}(\hat{G}_\beta(\eta^*,\eta^*)) + \kappa_2 d_3 \mathrm{Im}(J_\beta(\eta^*,\eta^*)) \\ & -\frac{\kappa_2 d_4}{\eta^*}\mathrm{Im}(\hat{J}_\beta(\eta^*,\eta^*)) \end{aligned}$$

and

$$\begin{aligned} 0 = \;& (4+\kappa_1 d_2)(\mathrm{Re}(-\mu G_\beta(\eta^*,\eta^*)) + \mu G(\eta^*,\eta^*)+\gamma'(\eta^*)) - 4\nu \\ & +\kappa_1 d_1\big(\mathrm{Re}(\tfrac{\mu}{\eta^*}\hat{G}_\beta(\eta^*,\eta^*)) - \tfrac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*)+\hat{\gamma}'(\eta^*)\big) \\ & -\kappa_2 d_3(\mathrm{Re}(-J_\beta(\eta^*,\eta^*)) + J(\eta^*,\eta^*)+\alpha'(\eta^*)) \\ & -\kappa_2 d_4(\mathrm{Re}(\tfrac{1}{\eta^*}\hat{J}_\beta(\eta^*,\eta^*)) - \tfrac{1}{\eta^*}\hat{J}(\eta^*,\eta^*)+\hat{\alpha}'(\eta^*)). \end{aligned}$$

There is a critical point τ^* provided the existence of β since the right hand side of (38) is positive by Lemma 3.4.

We now define

$$\begin{aligned} T(\beta) = \;& (4+\kappa_1 d_2)(\mathrm{Re}(-\mu G_\beta(\eta^*,\eta^*)) + \mu G(\eta^*,\eta^*)+\gamma'(\eta^*)) - 4\nu \; + \kappa_1 d_1\big(\mathrm{Re}(\tfrac{\mu}{\eta^*}\hat{G}_\beta(\eta^*,\eta^*)) \\ & -\tfrac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*)+\hat{\gamma}'(\eta^*)\big) - \kappa_2 d_3(\mathrm{Re}(-J_\beta(\eta^*,\eta^*)) + J(\eta^*,\eta^*)+\alpha'(\eta^*)) \\ & -\kappa_2 d_4(\mathrm{Re}(\tfrac{1}{\eta^*}\hat{J}_\beta(\eta^*,\eta^*)) - \tfrac{1}{\eta^*}\hat{J}(\eta^*,\eta^*)+\hat{\alpha}'(\eta^*)). \end{aligned}$$

Using Lemma 3.4, we have $T'(\beta) > 0$ for $\beta > 0$ and

$$T(0) \;= (4+\kappa_1 d_2)\gamma'(\eta^*) - 4\nu \; + \kappa_1 d_1 \hat{\gamma}'(\eta^*) - \kappa_2 d_3 \alpha'(\eta^*) - \kappa_2 d_4 \hat{\alpha}'(\eta^*)$$

and thus $T(0) < 0$ if $0 < \frac{1}{2} - a_0 + \nu s_0 < \frac{2}{3}\nu + \gamma(\frac{1}{3})$. Moreover,

$$\begin{aligned} \lim_{\beta\to\infty} T(\beta) \;& = (4+\kappa_1 d_2 - \tfrac{\kappa_1 d_1}{\eta^*})(\mu G(\eta^*,\eta^*)+\gamma'(\eta^*)) - 4\nu + \kappa_1 d_1 \hat{\gamma}'(\eta^*) \\ & +\kappa_2(\tfrac{d_4}{\eta^*} - d_3)(J(\eta^*,\eta^*)+\alpha'(\eta^*)) - \kappa_2 d_4 \hat{\alpha}'(\eta^*) \\ & = (\kappa_1 d_2 - \tfrac{\kappa_1 d_1}{\eta^*} - \kappa_1 d_1))(\mu G(\eta^*,\eta^*)+\gamma'(\eta^*)) + 4(\mu G(\eta^*,\eta^*)+\gamma'(\eta^*) - \nu) \\ & +\kappa_1 d_1 \tfrac{2\mu}{\sqrt{1+\mu}\,\sinh\sqrt{1+\mu}}\Big(\sinh \tfrac{\sqrt{1+\mu}(1-\eta^*)}{2}\Big)^2 \cosh(\sqrt{1+\mu}\,\eta^*) \\ & +\kappa_2(\tfrac{d_4}{\eta^*} - d_3 + d_4)(J(\eta^*,\eta^*)+\alpha'(\eta^*)) - \kappa_2 d_4 \tfrac{2}{\sinh 1}\Big(\sinh \tfrac{(1-\eta^*)}{2}\Big)^2 \cosh \eta^* \end{aligned}$$

Since $\mu G(\eta^*,\eta^*)+\gamma'(\eta^*) > 0, J(\eta^*,\eta^*)+\alpha'(\eta^*) > 0, \hat{\gamma}'(\eta^*) < 0, \hat{\alpha}'(\eta^*) < 0$ for $\eta^* \in (1/3,1)$, $\lim_{\beta\to\infty} T(\beta) > 0$ under the assumptions $d_2 > (\frac{1}{\eta^*}+1)d_1 > 0,\; 0 > (\frac{1}{\eta^*}+1)d_4 > d_3$ and $\nu < \mu G(\eta^*,\eta^*)+\gamma'(\eta^*)$ thus, there exists a unique $\beta > 0$. □

The following theorem summarizes the results above.

THEOREM 3.6. *Suppose that* $0 < \frac{1}{2} - a_0 + \nu s_0 < \frac{2}{3}\nu + \gamma(\frac{1}{3})$. *Then (7) and (3) has at least one stationary solution* $(u^*, v^*, w^*, z^*, \eta^*)$ *where* $u^* = v^* = w^* = z^* = 0$, $\eta^* \in (\frac{1}{3}, 1)$ *and* $(V^*, p^*, W^*, q^*, \eta^*)$ *for all* τ, *respectively. Moreover, assume that* $d_2 > (\frac{1}{\eta^*} + 1)d_1$ *and* $(\frac{1}{\eta^*} + 1)d_4 > d_3$, *where* $d_1 = \chi'(\gamma(\eta^*)) > 0$, $d_2 = \chi''(\gamma(\eta^*))\hat{\gamma}(\eta^*) > 0, d_3 = \xi''(\alpha(\eta^*))\hat{\alpha}(\eta^*) < 0$ *and* $d_4 = \xi'(\alpha(\eta^*)) < 0$ *and the global coupling constant* ν *satisfies that* $\nu < \mu G(\eta^*, \eta^*) + \gamma'(\eta^*)$. *Then there exists a unique* τ^* *such that the linearization* $-\widetilde{A} + \tau^* B$ *has a purely imaginary pair of eigenvalues. The point* $(0, 0, 0, 0, \eta^*, \tau^*)$ *is then a Hopf point for (7), and there exists a* C^0*-curve of nontrivial periodic orbits for (7) and (3), bifurcating from* $(0, 0, 0, 0, \eta^*, \tau^*)$ *and* $(V^*, p^*, W^*, q^*, \eta^*, \tau^*)$, *respectively.*

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