

ORIENTED TRANSFORMATIONS ON A FINITE CHAIN: ANOTHER DESCRIPTION

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ABSTRACT. Following the new description of an oriented full transformation on a finite chain given recently by Higgins and Vernitski in [4], in this short note we present a refinement of this description which is extendable to partial transformations and to injective partial transformations.

1. Introduction

For $n \in \mathbb{N}$, let Ω_n be a set with n elements, e.g. $\Omega_n = \{1, 2, \dots, n\}$. As usual, denote by \mathcal{PT}_n the monoid (under composition) of all partial transformations on Ω_n , by \mathcal{T}_n the submonoid of \mathcal{PT}_n of all full transformations on Ω_n , by \mathcal{I}_n the *symmetric inverse monoid* on Ω_n , i.e., the inverse submonoid of \mathcal{PT}_n of all partial permutations on Ω_n , and by \mathcal{S}_n the *symmetric group* on Ω_n , i.e., the subgroup of \mathcal{PT}_n of all permutations on Ω_n .

Next, suppose that Ω_n is a chain, e.g. $\Omega_n = \{1 < 2 < \dots < n\}$. Let $s = (a_1, a_2, \dots, a_t)$ be a sequence of t ($t \geq 0$) elements from the chain Ω_n . We say that s is *cyclic* [*anti-cyclic*] if there exists no more than one index $i \in \{1, \dots, t\}$ such that $a_i > a_{i+1}$ [$a_i < a_{i+1}$], where a_{t+1} denotes a_1 . Notice that, the sequence s is cyclic [*anti-cyclic*] if and only if s is empty or there exists $i \in \{0, 1, \dots, t-1\}$ such that $a_{i+1} \leq a_{i+2} \leq \dots \leq a_t \leq a_1 \leq \dots \leq a_i$ [$a_{i+1} \geq a_{i+2} \geq \dots \geq a_t \geq a_1 \geq \dots \geq a_i$] (the index $i \in \{0, 1, \dots, t-1\}$ is unique unless s is constant and $t \geq 2$). We also say that s is *oriented* if s is cyclic or s is anti-cyclic. Given a partial transformation $\alpha \in \mathcal{PT}_n$ such that $\text{Dom}(\alpha) = \{a_1 < \dots < a_t\}$, with $t \geq 0$, we say that α is *orientation-preserving* [*orientation-reversing*, *oriented*] if the sequence of its images $(a_1\alpha, \dots, a_t\alpha)$ is cyclic [*anti-cyclic*, *oriented*]. We denote by \mathcal{POP}_n the submonoid of \mathcal{PT}_n of all orientation-preserving partial transformations and by \mathcal{POR}_n the submonoid of

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\mathcal{PT}_n of all oriented partial transformations. Let \mathcal{OP}_n be the submonoid of \mathcal{PT}_n of all orientation-preserving full transformations and \mathcal{OR}_n be the submonoid of \mathcal{PT}_n of all oriented full transformations, i.e., $\mathcal{OP}_n = \mathcal{POP}_n \cap \mathcal{T}_n$ and $\mathcal{OR}_n = \mathcal{POR}_n \cap \mathcal{T}_n$. Let us also consider the injective partial counterparts of \mathcal{OP}_n and \mathcal{OR}_n , i.e., the inverse submonoids $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$ and $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$ of \mathcal{PT}_n .

The notion of an orientation-preserving full transformation was introduced by McAlister in [6] and, independently, by Catarino and Higgins in [1]. The partial and injective partial versions of this concept were first considered by the author in [2]. Since then, many articles have been published by several authors involving semigroups of oriented transformations. In particular, the author, with various different co-authorships, has more than a dozen published papers on these semigroups.

Recently Higgins and Vernitski presented in [4] a new description of an oriented transformation. From this description, in this short note, we derive yet another description of an oriented transformation which is, on the one hand and in a certain sense, simpler and, on the other hand, extendable to partial transformations and to injective partial transformations.

2. The description

Let us consider the following permutations of Ω_n of order n and 2, respectively:

$$g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

It is clear that g is an orientation-preserving transformation and h is an orientation-reversing transformation. Moreover, for $n \geq 3$, g together with h generate the well-known *dihedral group* \mathcal{D}_{2n} of order $2n$ (considered as a subgroup of \mathcal{S}_n):

$$\mathcal{D}_{2n} = \{1, g, g^2, \dots, g^{n-1}, h, hg, hg^2, \dots, hg^{n-1}\}.$$

Let \mathcal{C}_n be the *cyclic group* of order n generated by g , i.e.,

$$\mathcal{C}_n = \{1, g, g^2, \dots, g^{n-1}\}.$$

Notice that \mathcal{C}_n is the group of units of \mathcal{OP}_n , \mathcal{POPI}_n and \mathcal{POP}_n and, on the other hand, for $n \geq 3$, \mathcal{D}_{2n} is the group of units of \mathcal{OR}_n , \mathcal{PORI}_n and \mathcal{POR}_n (for $n \in \{1, 2\}$, their group of units is also \mathcal{C}_n).

Now, let $s = (a_1, a_2, \dots, a_t)$ be a sequence of t elements from the chain Ω_n , with $t \geq 3$. It is easy to show that:

- s is cyclic [anti-cyclic] if and only if, for all $\sigma \in \mathcal{C}_t$, $(a_{1\sigma}, a_{2\sigma}, \dots, a_{t\sigma})$ is cyclic [anti-cyclic];
- s is cyclic [anti-cyclic] if and only if, for all $\sigma \in \mathcal{D}_{2t} \setminus \mathcal{C}_t$, $(a_{1\sigma}, a_{2\sigma}, \dots, a_{t\sigma})$ is anti-cyclic [cyclic];
- s is oriented if and only if, for all $\sigma \in \mathcal{D}_{2t}$, $(a_{1\sigma}, a_{2\sigma}, \dots, a_{t\sigma})$ is oriented;

- s is cyclic [anti-cyclic, oriented] if and only if there exists $\sigma \in \mathcal{C}_t$ [$\sigma \in \mathcal{D}_{2t} \setminus \mathcal{C}_t$, $\sigma \in \mathcal{D}_{2t}$] such that $a_{1\sigma} \leq a_{2\sigma} \leq \dots \leq a_{t\sigma}$.

Next, we recall the following recent descriptions of \mathcal{OP}_n and \mathcal{OR}_n proved by Higgins and Vernitski in [4, Theorem 3 and Theorem 7] (see also [5]):

Proposition 2.1. *Let $\alpha \in \mathcal{T}_n$. Then*

- (1) $\alpha \in \mathcal{OP}_n$ if and only if, for every triple (a_1, a_2, a_3) of elements of Ω_n , (a_1, a_2, a_3) and $(a_1\alpha, a_2\alpha, a_3\alpha)$ are both cyclic or both anti-cyclic;
- (2) $\alpha \in \mathcal{OR}_n$ if and only if, for every oriented quadruple (a_1, a_2, a_3, a_4) of elements of Ω_n , the quadruple $(a_1\alpha, a_2\alpha, a_3\alpha, a_4\alpha)$ is also oriented.

Observe that it is easy to show that any triple of elements of Ω_n is oriented. Furthermore, given a triple (a_1, a_2, a_3) of elements of Ω_n , (a_1, a_2, a_3) is cyclic if and only if (a_3, a_2, a_1) is anti-cyclic. This allows us to easily derive from Property 1 of Proposition 2.1 the following simpler characterization of \mathcal{OP}_n .

Corollary 2.2. *Let $\alpha \in \mathcal{T}_n$. Then $\alpha \in \mathcal{OP}_n$ if and only if, for every cyclic triple (a_1, a_2, a_3) of elements of Ω_n , $(a_1\alpha, a_2\alpha, a_3\alpha)$ is also cyclic.*

In fact, we can further simplify both characterizations given in Proposition 2.1:

Corollary 2.3. *Let $\alpha \in \mathcal{T}_n$. Then $\alpha \in \mathcal{OP}_n$ [$\alpha \in \mathcal{OR}_n$] if and only if, for every non-decreasing triple [quadruple] (a_1, a_2, a_3, a_4) of elements of Ω_n , the triple [quadruple] $(a_1\alpha, a_2\alpha, a_3\alpha, a_4\alpha)$ is cyclic [oriented].*

Proof. Since any non-decreasing sequence of Ω_n is cyclic, for both properties, it remains to prove the converse implications.

Suppose that, for every non-decreasing triple [quadruple] (a_1, a_2, a_3, a_4) of elements of Ω_n , the triple [quadruple] $(a_1\alpha, a_2\alpha, a_3\alpha, a_4\alpha)$ is cyclic [oriented]. Let (a_1, a_2, a_3, a_4) be a cyclic triple [an oriented quadruple] of elements of Ω_n . Take $\sigma \in \mathcal{C}_3$ [$\sigma \in \mathcal{D}_{2,4}$] such that $a_{1\sigma} \leq a_{2\sigma} \leq a_{3\sigma}$ [$\leq a_{4\sigma}$]. Then, by hypothesis, $(a_{1\sigma}\alpha, a_{2\sigma}\alpha, a_{3\sigma}\alpha, a_{4\sigma}\alpha)$ is cyclic [oriented]. Let $a'_i = a_i\alpha$ for $i = 1, 2, 3, 4$. Then $(a'_{1\sigma}, a'_{2\sigma}, a'_{3\sigma}, a'_{4\sigma})$ is cyclic [oriented] and so, since $\sigma^{-1} \in \mathcal{C}_3$ [$\sigma^{-1} \in \mathcal{D}_{2,4}$], by the above observation, $(a_1\alpha, a_2\alpha, a_3\alpha, a_4\alpha) = (a'_{1\sigma}, a'_{2\sigma}, a'_{3\sigma}, a'_{4\sigma}) = (a'_{(1\sigma)\sigma^{-1}}, a'_{(2\sigma)\sigma^{-1}}, a'_{(3\sigma)\sigma^{-1}}, a'_{(4\sigma)\sigma^{-1}})$ is also cyclic [oriented]. Thus, by Corollary 2.2 [Proposition 2.1], we get $\alpha \in \mathcal{OP}_n$ [$\alpha \in \mathcal{OR}_n$], as required. \square

As observed above, any triple of elements of Ω_n is oriented. From this fact, it is easy to deduce that any quadruples of elements of Ω_n of the form (a, a, b, c) , (a, b, b, c) and (a, b, c, c) are also oriented. It is also clear that triples of Ω_n of the form (a, a, b) and (a, b, b) are cyclic. Therefore, it is easy to check that we can replace non-decreasing triples and quadruples by (strictly) increasing triples and quadruples, respectively, in the statements of Corollary 2.3 and, in this way, obtaining the following characterizations of \mathcal{OP}_n and \mathcal{OR}_n , where by *width* of a partial transformation we mean, as usual, the number of elements in its domain:

Theorem 2.4. *Let $\alpha \in \mathcal{T}_n$. Then $\alpha \in \mathcal{OP}_n$ [$\alpha \in \mathcal{OR}_n$] if and only if every restriction of α of width three [four] belongs to \mathcal{POP}_n [\mathcal{POR}_n].*

Next, we aim to extend this last result to partial transformations.

Let $\alpha \in \mathcal{PT}_n$. For our purpose, it is enough to consider transformations of width greater than or equal to three. Thus, let us suppose that $\text{Dom}(\alpha) = \{i_1 < i_2 < \dots < i_k\}$, with $k \geq 3$. Then, we define $\bar{\alpha} \in \mathcal{T}_n$ by

$$i\bar{\alpha} = \begin{cases} i_1\alpha & \text{if } 1 \leq i \leq i_2 - 1, \\ i_\ell\alpha & \text{if } i_\ell \leq i < i_{\ell+1}, \ell = 2, 3, \dots, k - 1, \\ i_k\alpha & \text{if } i_k \leq i \leq n. \end{cases}$$

The following lemma is easy to check:

Lemma 2.5. *Let $\alpha \in \mathcal{PT}_n$ such that $|\text{Dom}(\alpha)| \geq 3$. Then $\alpha \in \mathcal{POP}_n$ [$\alpha \in \mathcal{POR}_n$] if and only if $\bar{\alpha} \in \mathcal{OP}_n$ [$\bar{\alpha} \in \mathcal{OR}_n$].*

Now, take $\alpha \in \mathcal{PT}_n$ such that every restriction of α of width three [four] belongs to \mathcal{POP}_n [\mathcal{POR}_n].

Our objective is to prove that $\alpha \in \mathcal{POP}_n$ [$\alpha \in \mathcal{POR}_n$]. Take $\text{Dom}(\alpha) = \{i_1 < i_2 < \dots < i_k\}$.

If $|\text{Dom}(\alpha)| \leq 2$ [≤ 3], then it is clear that $\alpha \in \mathcal{POP}_n$ [$\alpha \in \mathcal{POR}_n$]. Therefore, we are going to consider $|\text{Dom}(\alpha)| \geq 3$ [≥ 4].

Define $I_1 = \{1, \dots, i_2 - 1\}$, $I_\ell = \{i_\ell, \dots, i_{\ell+1} - 1\}$ for $\ell = 2, \dots, k - 1$, and $I_k = \{i_k, \dots, n\}$.

Let $X = \{a_1 < a_2 < a_3 [< a_4]\}$ be a subset of $\text{Dom}(\alpha)$ with three [four] elements. Then, we aim to show that $\bar{\alpha}|_X \in \mathcal{POP}_n$ [$\bar{\alpha}|_X \in \mathcal{POR}_n$].

Take $r_1, r_2, r_3, r_4 \in \{1, 2, \dots, k\}$ such that $a_i \in I_{r_i}$ for $i = 1, 2, 3, 4$. Since $a_1 < a_2 < a_3 [< a_4]$, then $r_1 \leq r_2 \leq r_3 [\leq r_4]$ and

$$\bar{\alpha}|_X = \begin{pmatrix} a_1 & a_2 & a_3 & \begin{bmatrix} a_4 \\ \end{bmatrix} \\ i_{r_1}\alpha & i_{r_2}\alpha & i_{r_3}\alpha & \begin{bmatrix} i_{r_4}\alpha \\ \end{bmatrix} \end{pmatrix}.$$

If $r_1 = r_2$ or $r_2 = r_3$ [or $r_3 = r_4$], then $i_{r_1}\alpha = i_{r_2}\alpha$ or $i_{r_2}\alpha = i_{r_3}\alpha$ [or $i_{r_3}\alpha = i_{r_4}\alpha$], whence $(i_{r_1}\alpha, i_{r_2}\alpha, i_{r_3}\alpha, i_{r_4}\alpha)$ is cyclic [oriented] and so $\bar{\alpha}|_X \in \mathcal{POP}_n$ [$\bar{\alpha}|_X \in \mathcal{POR}_n$].

Otherwise, we have $r_1 < r_2 < r_3 [< r_4]$ and thus, by hypothesis,

$$\alpha|_{\{i_{r_1}, i_{r_2}, i_{r_3}\}} \in \mathcal{POP}_n \quad [\alpha|_{\{i_{r_1}, i_{r_2}, i_{r_3}, i_{r_4}\}} \in \mathcal{POR}_n].$$

Hence $(i_{r_1}\alpha, i_{r_2}\alpha, i_{r_3}\alpha, i_{r_4}\alpha)$ is cyclic [oriented] and so, also in this case, we obtain $\bar{\alpha}|_X \in \mathcal{POP}_n$ [$\bar{\alpha}|_X \in \mathcal{POR}_n$].

Now, by applying Theorem 2.4, we get $\bar{\alpha} \in \mathcal{OP}_n$ [$\bar{\alpha} \in \mathcal{OR}_n$] and then, by Lemma 2.5, it follows that $\alpha \in \mathcal{POP}_n$ [$\alpha \in \mathcal{POR}_n$].

Thus, we may conclude the following characterizations of \mathcal{POP}_n and \mathcal{POR}_n :

Theorem 2.6. *Let $\alpha \in \mathcal{PT}_n$. Then $\alpha \in \mathcal{POP}_n$ [$\alpha \in \mathcal{POR}_n$] if and only if every restriction of α of width three [four] belongs to \mathcal{POP}_n [\mathcal{POR}_n].*

And, as an immediate corollary of the previous result, we have:

Theorem 2.7. *Let $\alpha \in \mathcal{I}_n$. Then $\alpha \in \mathcal{POPI}_n$ [$\alpha \in \mathcal{PORI}_n$] if and only if every restriction of α of width three [four] belongs to \mathcal{POPI}_n [\mathcal{PORI}_n].*

3. An application

Let $n \geq 3$. Consider the *cycle graph* $C_n = (\{1, 2, \dots, n\}, \{\{i, i + 1\} \mid i = 1, 2, \dots, n - 1\} \cup \{\{1, n\}\})$ with n vertices. Denote by $d(x, y)$ the (*geodesic distance*) between two vertices x and y of C_n , i.e., the number of edges in a shortest path between x and y . Notice that $d(x, y) = \min\{|x - y|, n - |x - y|\}$ and so $0 \leq d(x, y) \leq \frac{n}{2}$, for all $x, y \in \{1, 2, \dots, n\}$. Let us consider the monoid \mathcal{DPC}_n of all *partial isometries* (or *distance preserving partial transformations*) of C_n , i.e., $\mathcal{DPC}_n = \{\alpha \in \mathcal{PT}_n \mid d(x\alpha, y\alpha) = d(x, y) \text{ for all } x, y \in \text{Dom}(\alpha)\}$. This monoid was studied by the author together with Paulista in [3]. Observe that it is easy to show that \mathcal{DPC}_n is an inverse submonoid of the symmetric inverse monoid \mathcal{I}_n . Furthermore, \mathcal{DPC}_n is also a submonoid of \mathcal{PORI}_n . However, this last inclusion is not trivial. Although it has already been proved in the above mentioned paper, we aim to present here an alternative proof of this property by making use of Theorem 2.7:

Proposition 3.1. *The monoid \mathcal{DPC}_n is contained in \mathcal{PORI}_n .*

Proof. By Theorem 2.7, it suffices to show that all transformations of \mathcal{DPC}_n of width 4 belong to \mathcal{PORI}_n (notice that any restriction of an element of \mathcal{DPC}_n is obviously an element of \mathcal{DPC}_n).

Let

$$\alpha = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{pmatrix} \in \mathcal{DPC}_n,$$

with $i_1 < i_2 < i_3 < i_4$. Let $\alpha' = g^{i_1-1}\alpha g^{n-j_1+1} \in \mathcal{DPC}_n$. Then $1\alpha' = 1g^{i_1-1}\alpha g^{n-j_1+1} = i_1\alpha g^{n-j_1+1} = j_1g^{n-j_1+1} = 1$ and, since $\alpha = g^{n-i_1+1}\alpha' g^{j_1-1}$ and $g \in \mathcal{PORI}_n$, we also have $\alpha \in \mathcal{PORI}_n$ if and only if $\alpha' \in \mathcal{PORI}_n$. Therefore, we can reduce the proof to transformations of \mathcal{DPC}_n of width 4 of the form

$$\alpha = \begin{pmatrix} 1 & i_2 & i_3 & i_4 \\ 1 & j_2 & j_3 & j_4 \end{pmatrix},$$

with $i_2 < i_3 < i_4$. Clearly, in this case, $\alpha \notin \mathcal{PORI}_n$ if and only if either $(j_2 < j_3 \text{ and } j_3 > j_4)$ or $(j_2 > j_3 \text{ and } j_3 < j_4)$. Let

$$\alpha' = \alpha h g = \begin{pmatrix} 1 & i_2 & i_3 & i_4 \\ 1 & n - j_2 + 2 & n - j_3 + 2 & n - j_4 + 2 \end{pmatrix}.$$

Since $g, h \in \mathcal{PORI}_n$ and $\alpha = \alpha' g^{n-1} h$, we have $\alpha \in \mathcal{PORI}_n$ if and only if $\alpha' \in \mathcal{PORI}_n$. On the other hand, we also have $j_2 < j_3$ and $j_3 > j_4$ if and only if $n - j_2 + 2 > n - j_3 + 2$ and $n - j_3 + 2 < n - j_4 + 2$ and, dually, $j_2 > j_3$ and $j_3 < j_4$ if and only if $n - j_2 + 2 < n - j_3 + 2$ and $n - j_3 + 2 > n - j_4 + 2$. Thus,

we can further reduce the proof to transformations of \mathcal{DPC}_n of width 4 of the form

$$\alpha = \begin{pmatrix} 1 & i_2 & i_3 & i_4 \\ 1 & j_2 & j_3 & j_4 \end{pmatrix},$$

with $i_2 < i_3 < i_4$ and $j_2 < j_3$.

In order to obtain a contradiction, suppose that $\alpha \notin \mathcal{PORI}_n$. Then, we must also have $j_3 > j_4$.

Let $p = 2, 3, 4$. Then, from the equality $d(1, i_p) = d(1, j_p)$, we obtain

$$j_p = i_p,$$

if either $i_p - 1 \leq \frac{n}{2}$ and $j_p - 1 \leq \frac{n}{2}$ or $i_p - 1 > \frac{n}{2}$ and $j_p - 1 > \frac{n}{2}$, and

$$j_p = n - i_p + 2,$$

if either $i_p - 1 \leq \frac{n}{2}$ and $j_p - 1 > \frac{n}{2}$ or $i_p - 1 > \frac{n}{2}$ and $j_p - 1 \leq \frac{n}{2}$.

Now, we proceed by considering all possible cases for j_2, j_3 and j_4 .

Case 1: $j_2 = i_2$.

Case 1.1: $j_2 = i_2$ and $j_3 = i_3$.

Since $j_4 < j_3$, we can not have $j_4 = i_4$ and so $j_4 = n - i_4 + 2$, i.e.,

$$\alpha = \begin{pmatrix} 1 & i_2 & i_3 & i_4 \\ 1 & i_2 & i_3 & n - i_4 + 2 \end{pmatrix}.$$

Notice that $n - i_4 + 2 < i_3$.

First, admit that $i_4 - i_3 \leq \frac{n}{2}$, whence $d(i_3, i_4) = i_4 - i_3$. If $d(n - i_4 + 2, i_3) = n - i_3 + (n - i_4 + 2)$, then $i_4 - i_3 = n - i_3 + n - i_4 + 2$ and so $i_4 = n + 1$, which is a contradiction. Hence, $d(n - i_4 + 2, i_3) = i_3 - (n - i_4 + 2)$ and so $i_4 - i_3 = i_3 - n + i_4 - 2$, which implies that $i_3 = \frac{n+2}{2}$.

On the other hand, admit that $i_4 - i_3 > \frac{n}{2}$, whence $d(i_3, i_4) = n - i_4 + i_3$. If $d(n - i_4 + 2, i_3) = i_3 - (n - i_4 + 2)$, then $n - i_4 + i_3 = i_3 - n + i_4 - 2$ and so $i_4 = n + 1$, which is a contradiction. Hence, $d(n - i_4 + 2, i_3) = n - i_3 + (n - i_4 + 2)$ and so $n - i_4 + i_3 = n - i_3 + n - i_4 + 2$, which implies that $i_3 = \frac{n+2}{2}$.

Therefore, in either case, we have $i_3 = \frac{n+2}{2}$.

Next, suppose that $i_4 - i_2 \leq \frac{n}{2}$, whence $d(i_2, i_4) = i_4 - i_2$. If $d(i_2, n - i_4 + 2) = n - |(n - i_4 + 2) - i_2|$, then $i_2 = 1$ or $i_4 = n + 1$, which is a contradiction in both cases. Hence, $d(i_2, n - i_4 + 2) = |(n - i_4 + 2) - i_2|$ and so $i_4 = \frac{n+2}{2}$ or $i_2 = \frac{n+2}{2}$, which is again a contradiction (since $i_3 = \frac{n+2}{2}$).

Therefore, $i_4 - i_2 > \frac{n}{2}$ and so $d(i_2, i_4) = n - i_4 + i_2$. If $d(i_2, n - i_4 + 2) = |(n - i_4 + 2) - i_2|$, then $i_2 = 1$ or $i_4 = n + 1$, which is a contradiction in both cases. Hence, $d(i_2, n - i_4 + 2) = n - |(n - i_4 + 2) - i_2|$ and so $i_4 = \frac{n+2}{2}$ or $i_2 = \frac{n+2}{2}$, which is again a contradiction (since $i_3 = \frac{n+2}{2}$).

Thus, Case 1.1 cannot occur.

Case 1.2: $j_2 = i_2$ and $j_3 = n - i_3 + 2$.

We begin by observing that, since $i_2 < n - i_3 + 2$, from the equality $d(i_2, i_3) = d(i_2, n - i_3 + 2)$, we have $i_2 = 1$ or $i_3 = \frac{n+2}{2}$, whence $i_3 = \frac{n+2}{2}$.

If $j_4 = n - i_4 + 2$, then from the equality $d(i_2, i_4) = d(i_2, n - i_4 + 2)$, by the same calculations made above, we obtain $i_2 = \frac{n+2}{2}$ or $i_4 = \frac{n+2}{2}$, which is a contradiction (since $i_3 = \frac{n+2}{2}$). Therefore, $j_4 = i_4 > i_3 = \frac{n+2}{2} = n - i_3 + 2 = j_3$, which is again a contradiction.

Thus, Case 1.2 cannot occur either and so Case 1 does not occur.

Case 2: $j_2 = n - i_2 + 2$.

As $i_2 < i_3$, then $n - i_3 + 2 < n - i_2 + 2$ and so $j_3 = i_3$ (since $j_2 < j_3$). Now, since $n - i_2 + 2 < i_3$, from the equality $d(i_2, i_3) = d(n - i_2 + 2, i_3)$, we have $i_3 = n + 1$ or $i_2 = \frac{n+2}{2}$, whence $i_2 = \frac{n+2}{2} = n - i_2 + 2 = j_2$. Therefore, this case is a particular instance of Case 1 and so cannot occur either.

Thus, we conclude that $\alpha \in \mathcal{PORL}_n$, as required. \square

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