Commun. Korean Math. Soc. **38** (2023), No. 3, pp. 865–880
https://doi.org/10.4134/CKMS.c220145
pISSN: 1225-1763 / eISSN: 2234-3024

$\star$-CONFORMAL RICCI SOLITONS ON ALMOST COKÄHLER MANIFOLDS

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Abstract. The main intention of the current paper is to characterize certain properties of $\star$-conformal Ricci solitons on non-coKähler (κ,μ)-almost coKähler manifolds. At first, we find that there does not exist $\star$-conformal Ricci soliton if the potential vector field is the Reeb vector field θ. We also prove that the non-coKähler (κ,μ)-almost coKähler manifolds admit $\star$-conformal Ricci solitons if the potential vector field is the infinitesimal contact transformation. It is also studied that there does not exist $\star$-conformal gradient Ricci solitons on the said manifolds. An example has been constructed to verify the obtained results.

1. Introduction

The famous theory of Ricci flow was coined by R. S. Hamilton [16] in order to solve the well-known Poincare conjecture. A Ricci flow is described by the following pseudo-parabolic partial differential equations:

$$\frac{\partial g}{\partial t} = -2\mathcal{S}_{ij},$$

$$g_{ij}(t_0) = g.$$

A fixed point of the above system is called a Ricci soliton which is given by

$$\mathcal{L}_{\mathcal{V}} g + 2\mathcal{S} = 2\lambda g,$$

$\mathcal{L}$ being the Lie-derivative operator and λ is a constant. A Ricci soliton is called expanding or steady or shrinking if λ is negative or zero or positive, respectively. A Ricci soliton is called an almost Ricci soliton when λ is a smooth function. The notion of almost Ricci soliton was introduced by Pigola *et al.* [20]. The theory of Ricci soliton in the context of contact geometry was first studied by R. Sharma [25]. To study about Ricci solitons, reader can see the papers [3, 13, 18, 19, 24, 26, 29–31].

Received May 22, 2022; Revised December 18, 2022; Accepted May 9, 2023.
2020 *Mathematics Subject Classification.* Primary 53C15, 53D25.
Key words and phrases. Almost coKähler manifolds, (κ,μ)-nullity distribution, infinitesimal contact transformation, $\star$-Ricci tensors, Ricci solitons, conformal Ricci solitons.

In 2005, A. E. Fischer [14] presented the idea of conformal Ricci flow on Riemannian manifolds by the following partial differential equation:

$$\frac{\partial g}{\partial t} + 2(\mathcal{S} + \frac{g}{m}) = -qg \tag{1}$$

and $r = -1$, where r, $\mathcal{S}$, g, $q(\geq 0)$ and m are, respectively, scalar curvature, Ricci tensor of type $(0, 2)$, Riemannian metric, time-dependent non-dynamical scalar field, and dimension of the manifold. A conformal Ricci flow is a type of classical Ricci flow that modifies the unit volume constraint of the equation to the scalar curvature constraint. The name conformal is used here for the role of conformal geometry in constructing the scalar curvature constraint.

Let $\bar{g}_{ij}$ be a metric which evolves under conformal Ricci flow. Hence we can write

$$\bar{g}_{ij} = \sigma(t)\phi_t^* g_{ij}, \tag{2}$$

where σ being a scaling function and ϕ_t^* is the 1-parameter group of transformation induced by the flow. Let at $t = t_0$, $\sigma(t) = 1$ and ϕ_t^* be an identity transformation. Using (2) in (1), we get

$$\sigma'(t)\phi_t^* g_{ij} + \sigma(t)\frac{d}{dt}(\phi_t^* g_{ij}) = -2\mathcal{S}_{ij} - (q + \frac{2}{m})\sigma(t)\phi_t^* g_{ij}.$$

Putting $t = t_0$ and taking $\sigma'(t_0) = -2\lambda$, we get from the above equation

$$\mathcal{L}_{\mathcal{V}} g_{ij} + 2\mathcal{S}_{ij} = [2\lambda - (q + \frac{2}{m})]g_{ij}, \tag{3}$$

where $\mathcal{V}$ is the vector field induced by the 1-parameter group of transformation ϕ_t^* and $\mathcal{L}_{\mathcal{V}}$ denotes the Lie-derivative operator along the potential vector field (shortly, **PVF**) $\mathcal{V}$. In co-ordinate free notation the above equation is

$$\mathcal{L}_{\mathcal{V}} g + 2\mathcal{S} = [2\lambda - (q + \frac{2}{m})]g. \tag{4}$$

The concept of conformal Ricci solitons was presented in 2015 and studied by N. Basu and A. Bhattacharyya [2] which are the self-similar solution of the conformal Ricci flow equation (1). In [11–13], authors have derived so many characteristics of conformal Ricci solitons on different types of contact and para-contact manifolds. A conformal Ricci soliton is expanding or steady or shrinking according as λ is negative or zero or positive. Conformal Ricci solitons are special type of almost Ricci solitons. Here the unit volume constraint is replaced by scalar curvature constraint.

In [27], Tachibana defined $\star$-Ricci tensor in almost Kählerian manifolds. The $\star$-Ricci tensor $\mathcal{S}^\star$ is defined by

$$\mathcal{S}^\star(V_1, V_2) = \frac{1}{2} trace\{V_3 \to R(V_1, \phi V_2)\phi V_3\}$$

for every vector fields V_1, V_2 and V_3 on the manifold. $\star$-Ricci tensor on several sorts of manifolds has been researched in articles [7, 15, 18, 22, 23].

On contact or para-contact manifolds, ⋆-conformal Ricci solitons are defined by replacing the Ricci tensor with the ⋆-Ricci tensor in (4). Thus the ⋆-conformal Ricci soliton is given by

$$\mathcal{L}_{\mathcal{V}} g + 2\mathcal{S}^\star = [2\lambda - (q + \frac{2}{m})]g, \tag{5}$$

where the symbols in the above equation bear their usual meaning. ⋆-conformal Ricci solitons have been studied in the papers [10, 17, 21].

A ⋆-conformal Ricci soliton is named as a ⋆-conformal gradient Ricci soliton if the **PVF** $\mathcal{V}$ is the gradient of some smooth function ψ on the manifold $\mathcal{M}$. Thus, the ⋆-conformal gradient Ricci soliton is given by

$$\nabla^2\psi + \mathcal{S}^\star = (\lambda - \frac{1}{2}(q + \frac{2}{m}))g, \tag{6}$$

where $\nabla^2\psi$ is the Hessian of ψ.

The paper is organized as follows: After the introduction, we relate (κ, μ)-almost coKähler manifolds and give some essential formulas in Section 2. In Section 3, we deduce some characteristics of ⋆-conformal Ricci solitons on (κ, μ)-almost coKähler manifolds. Section 4 is devoted to study ⋆-conformal gradient Ricci solitons on the said manifolds. In the last part, we provide an example to validate our generated results.

2. (κ, μ)-almost coKähler manifolds

Let $\mathcal{M}$ be a $(2m+1)$-dimensional smooth manifold endowed with an almost contact metric structure $(\phi, \theta, \varrho, g)$, where ϕ is a $(1,1)$-tensor field, θ is a vector field, ϱ is a 1-form and g is the Riemannian metric on $\mathcal{M}$ such that [5, 9]

$$\phi^2(V_1) = -V_1 + \varrho(V_1)\theta, \quad \varrho(\theta) = 1. \tag{7}$$

As a consequence, we get the following:

$$\phi\theta = 0, \quad g(V_1, \theta) = \varrho(V_1), \quad \varrho(\phi V_1) = 0,$$
$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \varrho(V_1)\varrho(V_2),$$
$$g(\phi V_1, V_2) = -g(V_1, \phi V_2),$$

for every vector fields V_1, $V_2 \in \chi(\mathcal{M})$, the set of all vector fields on $\mathcal{M}$. An almost contact metric manifold (shortly, **ACMM**) is a differentiable manifold $\mathcal{M}$ of dimension $(2m+1)$ with an almost contact metric structure. The 2-form Φ on **ACMM**s is defined by

$$\Phi(V_1, V_2) = g(V_1, \phi V_2)$$

for every V_1, $V_2 \in \chi(\mathcal{M})$. An almost contact metric manifold is also known as an almost coKähler manifold (shortly, **ACM**) if both ϱ and Φ are closed, that is, $d\Phi = 0$ and $d\varrho = 0$. According to Blair [4], an (almost) coKähler manifold and an (almost) cosymplectic manifold are same.

Let $\mathcal{M}$ be an **ACM** of dimension $(2m + 1)$. Assume the two operators h and l which are defined by $h = \frac{1}{2}\mathcal{L}_\theta\phi$ and $l = R(\cdot, \theta)\theta$, where R denotes the curvature tensor and $\mathcal{L}$ is the Lie differentiation operator. These operators satisfy the following [9]:

$$h\theta = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi = -\phi h,$$

$$\nabla_{V_1}\theta = h\phi V_1, \tag{8}$$

$$\phi l \phi - l = 2h^2,$$

for every vector field V_1, and 'tr' stands for trace. In an **ACM**, the 1-form ϱ is closed, that is

$$(\nabla_{V_1}\varrho)V_2 - (\nabla_{V_2}\varrho)V_1 = 0$$

for every vector fields V_1, $V_2 \in \chi(\mathcal{M})$.

The almost coKähler structure is integrable if and only if

$$(\nabla_{V_1}\phi)V_2 = g(hV_1, V_2)\theta - \varrho(V_2)hV_1 \tag{9}$$

for any vector fields V_1, $V_2 \in \chi(\mathcal{M})$.

Blair et al. [6] established the concept of (κ, μ)-nullity distribution on contact metric manifolds. A contact metric manifold $\mathcal{M}$ of dimension $(2m+1)$ whose curvature tensor satisfies

$$R(V_1, V_2)\theta = \kappa[\varrho(V_2)V_1 - \varrho(V_1)V_2] + \mu[\varrho(V_2)hV_1 - \varrho(V_1)hV_2]$$

for every vector fields V_1, V_2 on $\mathcal{M}$, and $\kappa, \mu \in \mathbb{R}$ is known as (κ, μ)-contact metric manifold and it is said that θ belongs to the (κ, μ)-nullity distribution. When κ, μ are smooth functions, the manifold is called a generalized (κ, μ)-contact metric manifold.

An **ACM** $\mathcal{M}$ of dimension $(2m+1)$ is said to be a (κ, μ)-**ACM** if θ satisfies the equation

$$R(V_1, V_2)\theta = \kappa[\varrho(V_2)V_1 - \varrho(V_1)V_2] + \mu[\varrho(V_2)hV_1 - \varrho(V_1)hV_2] \tag{10}$$

for every vector fields V_1, V_2 on $\mathcal{M}$ and κ, μ are real numbers.

A (κ, μ)-**ACM** of dimension $(2m+1)$ satisfies the following curvature properties [9]:

$$h^2 V_1 = \kappa\phi^2 V_1, \tag{11}$$

$$\mathcal{S}(V_1, \theta) = 2m\kappa\varrho(V_1),$$

$$Q\theta = 2m\kappa\theta$$

for every vector field $V_1 \in \chi(\mathcal{M})$, $\mathcal{S}$ being the Ricci tensor of type (0,2) and Q being Ricci operator. From the equation (11), we can easily derive that $\kappa \leq 0$ and the manifold is coKähler if and only if $\kappa = 0$.

Lemma 2.1 ([1]). *In a non-coKähler* (κ, μ)-**ACM** *of dimension* $(2m+1)$*, the following relations hold*

$$QV_1 = \mu hV_1 + 2m\kappa\varrho(V_1)\theta, \tag{12}$$

$$\begin{aligned}(\nabla_{V_1}h)V_2 - (\nabla_{V_2}h)V_1 = {} & \kappa[\varrho(V_2)\phi V_1 - \varrho(V_1)\phi V_2 + 2g(\phi V_1, V_2)\theta] \\ & + \mu[\varrho(V_2)\phi hV_1 - \varrho(V_1)\phi hV_2],\end{aligned} \tag{13}$$

$$(\nabla_{V_1} h\phi)V_2 - (\nabla_{V_2} h\phi)V_1 = \kappa(\varrho(V_2)V_1 - \varrho(V_1)V_2) + \mu(\varrho(V_2)hV_1 - \varrho(V_1)hV_2) \tag{14}$$

for every vector fields V_1, $V_2 \in \chi(\mathcal{M})$.

Lemma 2.2 ([8])**.** *The* $\star$*-Ricci tensor and* $\star$*-scalar curvature on a* $(2m+1)$*-dimensional* (κ, μ)-**ACM** $\mathcal{M}$ *are, respectively, given by*

$$\mathcal{S}^\star(V_2, V_3) = -\kappa(g(V_2, V_3) - \varrho(V_2)\varrho(V_3)), \tag{15}$$

$$r^\star = -2m\kappa \tag{16}$$

for every vector fields V_2, V_3 *on* $\mathcal{M}$.

According to Blair ([5, p. 72]) and Tanno [28], we give the following definition.

Definition. A vector field V on an **ACMM** $\mathcal{M}$ is called an infinitesimal contact transformation if it satisfies

$$\mathcal{L}_{\mathcal{V}}\varrho = f\varrho$$

for some smooth function f on $\mathcal{M}$. If $f = 0$, then the vector field V is called a strict infinitesimal contact transformation.

3. ⋆-conformal Ricci solitons on (κ, μ)-almost coKähler manifolds

In the present section, we study ⋆-conformal Ricci solitons on non-coKähler (κ, μ)-**ACM**s.

Let $\mathcal{M}$ be a non-coKähler (κ, μ)-**ACM** of dimension $(2m+1)$ which admits ⋆-conformal Ricci soliton. Assume the **PVF** $\mathcal{V}$ is pointwise collinear with θ, that is, $V = \rho\theta$, where ρ is a smooth function on the manifold. Then, from (5), we have

$$\rho g(\nabla_{V_1}\theta, V_2) + (V_1\rho)\varrho(V_2) + \rho g(\nabla_{V_2}\theta, V_1) + (V_2\rho)\varrho(V_1) + 2\mathcal{S}^\star(V_1, V_2) = (2\lambda - (q + \frac{2}{2m+1}))g(V_1, V_2). \tag{17}$$

Using (8), (15) and (16) in (17), we infer

$$2\rho g(h\phi V_1, V_2) + (V_1\rho)\varrho(V_2) + (V_2\rho)\varrho(V_1) = (2\lambda + 2\kappa - (q + \frac{2}{2m+1}))g(V_1, V_2) - 2\kappa\varrho(V_1)\varrho(V_2). \tag{18}$$

Putting $V_2 = \theta$ in (18), we have

$$(V_1\rho) + (\theta\rho)\varrho(V_1) = (2\lambda - (q + \frac{2}{2m+1}))\varrho(V_1). \tag{19}$$

Again, putting $V_1 = \theta$ in (19), we get

$$(\theta\rho) = \frac{1}{2}(2\lambda - (q + \frac{2}{2m+1})). \tag{20}$$

Let $\{e_1, e_2, \ldots, e_{2m+1}\}$ be an orthonormal ϕ-basis, with $e_{2m+1} = \theta$, of the tangent space at each point of the manifold. Contracting V_1 and V_2 of (18) with respect to the above basis, we get

$$(\theta\rho) = (\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))(2m+1) - \kappa. \tag{21}$$

Comparing (20) and (21), we infer

$$\lambda = \frac{q}{2} + \frac{1}{2m+1} - \kappa.$$

As a result, we may establish the following theorem.

Theorem 3.1. *If the metric of a non-coKähler (κ, μ)-**ACM** admits $\star$-conformal Ricci soliton with **PVF** is pointwise collinear with the Reeb vector field θ, then $\lambda = \frac{q}{2} + \frac{1}{2m+1} - \kappa$.*

Using (20) in (19), we obtain

$$(V_1\rho) = \frac{1}{2}(2\lambda - (q + \frac{2}{2m+1}))\varrho(V_1). \tag{22}$$

Applying (22) in (18), we obtain

$$\rho g(h\phi V_1, V_2) = (\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))(g(V_1, V_2) - \varrho(V_1)\varrho(V_2)). \tag{23}$$

Replacing V_1 by ϕV_1 in (23), we get

$$\rho g(hV_1, V_2) = -(\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))g(\phi V_1, V_2).$$

Again, replacing V_1 by hV_1 in the above equation, we obtain

$$g(h\phi V_1, V_2) = -\frac{\rho\kappa}{(\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))}(g(V_1, V_2) - \varrho(V_1)\varrho(V_2)). \tag{24}$$

Using(24) in (23), we get

$$\Big(\frac{\rho^2\kappa}{(\lambda+\kappa-(\frac{q}{2}+\frac{1}{2m+1}))} + (\lambda+\kappa-(\frac{q}{2}+\frac{1}{2m+1}))\Big)(g(V_1, V_2) - \varrho(V_1)\varrho(V_2)) = 0,$$

which is true for any vector fields V_1, V_2. Thus, from above, we get

$$\rho^2 = -\frac{(\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))^2}{\kappa},$$

which implies that ρ is a constant. Thus we can state the following.

Theorem 3.2. *If a non-coKähler (κ, μ)-**ACM** admits $\star$-conformal Ricci soliton and the **PVF** $\mathcal{V}$ is pointwise collinear with the Reeb vector field θ, then $\mathcal{V}$ is a constant multiple of θ.*

If the **PVF** is the Reeb vector field θ, then from (18), we can write

$$g(h\phi V_1, V_2) = (\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))(g(V_1, V_2) - \kappa\varrho(V_1)\varrho(V_2)),$$

which implies

$$h\phi V_1 = (\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))V_1 - \kappa\varrho(V_1)\theta. \tag{25}$$

Operating both sides of (25) by ϕ and using (7), we obtain

$$hV_1 = (\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))\phi V_1. \tag{26}$$

Again, operating both sides of (26) by h and using (11), we get

$$\kappa\phi^2 V_1 = (\lambda + \kappa - (\frac{q}{2} + \frac{1}{2m+1}))h\phi V_1. \tag{27}$$

Tracing the above equation, and using $tr\phi^2 = -2m$ and $tr(h\phi) = 0$, we infer that $\kappa = 0$, which is a contradiction.

As a result, we may establish the following theorem.

Theorem 3.3. *There does not exist ⋆-conformal Ricci soliton on a $(2m+1)$-dimensional non-coKähler (κ, μ)-**ACM** if the **PVF** is the Reeb vector field θ.*

Let the metric g of a non-coKähler (κ, μ)-**ACM** be a ⋆-conformal Ricci soliton. Then with the help of (5), (15) and (16), we get

$$(\mathcal{L}_{\mathcal{V}} g)(V_2, V_3) = (2\lambda + 2\kappa - (q + \frac{2}{2m+1}))g(V_2, V_3) - 2\kappa\varrho(V_2)\varrho(V_3). \tag{28}$$

Differentiating the equation (28) covariantly with respect to V_1, we obtain

$$(\nabla_{V_1}\mathcal{L}_{\mathcal{V}} g)(V_2, V_3) = -2\kappa[(\nabla_{V_1}\varrho)(V_2)\varrho(V_3) + \varrho(V_2)(\nabla_{V_1}\varrho)(V_3)]. \tag{29}$$

From (8), we have

$$(\nabla_{V_1}\varrho)(V_2) = g(h\phi V_1, V_2). \tag{30}$$

Using (30) in (29), we obtain

$$(\nabla_{V_1}\mathcal{L}_{\mathcal{V}} g)(V_2, V_3) = -2\kappa[g(h\phi V_1, V_2)\varrho(V_3) + g(h\phi V_1, V_3)\varrho(V_2)]. \tag{31}$$

Using formula for commutativity of Lie derivative and covariant derivative (for details see Yano [32, p. 23]), we have

$$\begin{aligned}&(\mathcal{L}_{\mathcal{V}}\nabla_{V_1} g - \nabla_{V_1}\mathcal{L}_{\mathcal{V}} g - \nabla_{[V,V_1]} g)(V_2, V_3)\\ =& -g((\mathcal{L}_{\mathcal{V}}\nabla)(V_1, V_2), V_3) - g((\mathcal{L}_{\mathcal{V}}\nabla)(V_1, V_3), V_2).\end{aligned}$$

Because of the parallelism of the metric tensor g, the above equation reduces to

$$(\nabla_{V_1}\mathcal{L}_{\mathcal{V}} g)(V_2, V_3) = g((\mathcal{L}_{\mathcal{V}}\nabla)(V_1, V_2), V_3) + g((\mathcal{L}_{\mathcal{V}}\nabla)(V_1, V_3), V_2).$$

From the above equation, we have

$$2g((\mathcal{L}_{\mathcal{V}}\nabla)(V_1,V_2),V_3) = (\nabla_{V_1}\mathcal{L}_{\mathcal{V}}g)(V_2,V_3) + (\nabla_{V_2}\mathcal{L}_{\mathcal{V}}g)(V_3,V_1) - (\nabla_{V_3}\mathcal{L}_{\mathcal{V}}g)(V_1,V_2). \tag{32}$$

Applying (31) in (32), we get

$$g((\mathcal{L}_{\mathcal{V}}\nabla)(V_1,V_2),V_3) = -2\kappa g(h\phi V_1,V_2)\varrho(V_3),$$

from which we obtain

$$(\mathcal{L}_{\mathcal{V}}\nabla)(V_1,V_2) = -2\kappa g(h\phi V_1,V_2)\theta.$$

Differentiating the above equation covariantly and using (8), we infer

$$(\nabla_{V_1}\mathcal{L}_{\mathcal{V}}\nabla)(V_2,V_3) = -2\kappa g((\nabla_{V_1}h\phi)V_2,V_3)\theta - 2\kappa g(h\phi V_2,V_3)h\phi V_1. \tag{33}$$

According to Yano ([32, p. 23]), we get

$$(\mathcal{L}_{\mathcal{V}}R)(V_1,V_2)V_3 = (\nabla_{V_1}\mathcal{L}_{\mathcal{V}}\nabla)(V_2,V_3) - (\nabla_{V_2}\mathcal{L}_{\mathcal{V}}\nabla)(V_1,V_3) \tag{34}$$

for any vector fields V_1, V_2, V_3.

Substituting (33) in (34), we obtain

$$\begin{aligned}(\mathcal{L}_{\mathcal{V}}R)(V_1,V_2)V_3 = & -2\kappa[g((\nabla_{V_1}h\phi)V_2 - (\nabla_{V_2}h\phi)V_1,V_3)\theta \\ & + g(h\phi V_2,V_3)h\phi V_1 - g(h\phi V_1,V_3)h\phi V_2].\end{aligned}$$

Using (14) in the above equation, we obtain

$$\begin{aligned}(\mathcal{L}_{\mathcal{V}}R)(V_1,V_2)V_3 = & -2\kappa[\kappa(g(V_1,V_3)\varrho(V_2)\theta - g(V_2,V_3)\varrho(V_1)\theta) \\ & + \mu(g(hV_1,V_3)\varrho(V_2)\theta - g(hV_2,V_3)\varrho(V_1)\theta) \\ & + g(h\phi V_2,V_3)h\phi V_1 - g(h\phi V_1,V_3)h\phi V_2].\end{aligned}$$

Contracting the above equation over V_1, we obtain

$$(\mathcal{L}_{\mathcal{V}}\mathcal{S})(V_2,V_3) = 2\kappa\mu g(hV_2,V_3). \tag{35}$$

From (12), we can write

$$\mathcal{S}(V_2,V_3) = \mu g(hV_2,V_3) + 2m\kappa\varrho(V_2)\varrho(V_3). \tag{36}$$

Taking Lie derivative of (36) with respect to $\mathcal{V}$ and using (28), we obtain

$$\begin{aligned}(\mathcal{L}_{\mathcal{V}}\mathcal{S})(V_2,V_3) = & \mu((2\lambda + 2\kappa - (q + \frac{2}{2m+1}))g(hV_2,V_3) \\ & + g((\mathcal{L}_{\mathcal{V}}h)V_2,V_3)) + 2m\kappa((\mathcal{L}_{\mathcal{V}}\varrho)(V_2)\varrho(V_3) \\ & + \varrho(V_2)(\mathcal{L}_{\mathcal{V}}\varrho)(V_3)).\end{aligned} \tag{37}$$

Now, with the help of (28), we get

$$\begin{aligned}(\mathcal{L}_{\mathcal{V}}\varrho)(V_2) &= (\mathcal{L}_{\mathcal{V}}g)(V_2,\theta) + g(V_2,\mathcal{L}_{\mathcal{V}}\theta) \\ &= (2\lambda - (q + \frac{2}{2m+1}))\varrho(V_2) + g(V_2,\mathcal{L}_{\mathcal{V}}\theta).\end{aligned} \tag{38}$$

Using (38) in (37), we have

$$
\begin{aligned}
&(\mathcal{L}_{\mathcal{V}}\mathcal{S})(V_2, V_3)\\
&= \mu((2\lambda + 2\kappa - (q + \frac{2}{2m+1}))g(hV_2, V_3)\\
&\quad + g((\mathcal{L}_{\mathcal{V}}h)V_2, V_3)) + 2m\kappa(2(2\lambda - (q + \frac{2}{2m+1}))\varrho(V_2)\varrho(V_3)\\
&\quad + g(V_2, \mathcal{L}_{\mathcal{V}}\theta)\varrho(V_3) + g(V_3, \mathcal{L}_{\mathcal{V}}\theta)\varrho(V_2)).
\end{aligned}
\tag{39}
$$

Comparing (35) and (39), we obtain

$$
\begin{aligned}
&\mu((2\lambda + 2\kappa - (q + \frac{2}{2m+1}))g(hV_2, V_3)\\
&+g((\mathcal{L}_{\mathcal{V}}h)V_2, V_3)) + 2m\kappa(2(2\lambda + 2\kappa - (q + \frac{2}{2m+1}))\varrho(V_2)\varrho(V_3)\\
&+g(V_2, \mathcal{L}_{\mathcal{V}}\theta)\varrho(V_3) + g(V_3, \mathcal{L}_{\mathcal{V}}\theta)\varrho(V_2)) = 2\kappa\mu g(hV_2, V_3).
\end{aligned}
$$

Let $\{e_1, e_2, \ldots, e_n, e_{m+1}, e_{m+2}, \ldots, e_{2m}, e_{2m+1}\}$ be an orthonormal ϕ-basis, with $e_{2m+1} = \theta$, of the tangent space at each point of the manifold, where $he_i = \sqrt{-\kappa}e_i$. Contracting V_2 and V_3 with respect to the above basis, we get

$$
\varrho(\mathcal{L}_{\mathcal{V}}\theta) = -(2\lambda - (q + \frac{2}{2m+1})).
\tag{40}
$$

Again, putting $V_2 = V_3 = \theta$ in (28), we obtain

$$
\varrho(\mathcal{L}_{\mathcal{V}}\theta) = -\frac{1}{2}(2\lambda - (q + \frac{2}{2m+1})).
\tag{41}
$$

Comparing (40) and (41), we have

$$
\lambda = \frac{q}{2} + \frac{1}{2m+1}.
$$

Thus we can state the following.

Theorem 3.4. *If the metric of a non-coKähler (κ, μ)-**ACM** is a ⋆-conformal Ricci soliton, then $\lambda = \frac{q}{2} + \frac{1}{2m+1}$.*

As $q \geq 0$, from the above discussion, we may establish the following corollary.

Corollary 3.5. *On a non-coKähler (κ, μ)-**ACM**, a ⋆-conformal Ricci soliton is shrinking.*

Applying $V_3 = \theta$ in (28), we obtain

$$
(\mathcal{L}_{\mathcal{V}}g)(V_2, \theta) = (2\lambda - (q + \frac{2}{2m+1}))\varrho(V_2).
\tag{42}
$$

Again, replacing V_2 by θ in the above equation, we get

$$
g(\mathcal{L}_{\mathcal{V}}\theta, \theta) = -\frac{1}{2}(2\lambda - (q + \frac{2}{2m+1})),
$$

which implies

$$\mathcal{L}_{\mathcal{V}}\theta = -\frac{1}{2}(2\lambda - (q + \frac{2}{2m+1}))\theta. \tag{43}$$

Taking Lie derivative of $\varrho(V_2) = g(V_2, \theta)$ with respect to $\mathcal{V}$, we have

$$(\mathcal{L}_{\mathcal{V}}\varrho)(V_2) = (\mathcal{L}_{\mathcal{V}}g)(V_2, \theta) + g(V_2, \mathcal{L}_{\mathcal{V}}\theta). \tag{44}$$

Using (42) and (43) in (44), we obtain

$$(\mathcal{L}_{\mathcal{V}}\varrho)(V_2) = \frac{1}{2}(2\lambda - (q + \frac{2}{2m+1}))\varrho(V_2),$$

it follows that $\mathcal{V}$ is an infinitesimal contact transformation. As a result, we may establish the following theorem.

Theorem 3.6. *If a non-coKähler (κ, μ)-**ACM** of dimension $2m+1$ admits a $\star$-conformal Ricci soliton, then the **PVF** is an infinitesimal contact transformation.*

As a consequence of the above theorem, we can state the following corollary

Corollary 3.7. *If the **PVF** of a $\star$-conformal Ricci soliton is a strict infinitesimal contact transformation, then $\lambda = \frac{q}{2} + \frac{1}{2m+1}$.*

4. $\star$-conformal gradient Ricci solitons on (κ, μ)-almost coKähler manifolds

In this section we study $\star$-conformal gradient Ricci solitons on non-coKähler (κ, μ)-**ACM**s.

Let us consider a non-coKähler (κ, μ)-**ACM** $\mathcal{M}$ of dimension $2m+1$ which admits a $\star$-conformal gradient Ricci soliton. Then, from (6), we have

$$\nabla_{V_1} D\psi = (\lambda - \frac{1}{2}(q + \frac{2}{2m+1}))V_1 - Q^{\star}V_1, \tag{45}$$

where $\psi : \mathcal{M} \to \mathbb{R}$ is a smooth function on $\mathcal{M}$.

Using (15) and (16) in (45), we obtain

$$\nabla_{V_1} D\psi = (\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1}))V_1 - \kappa\varrho(V_1)\theta. \tag{46}$$

Differentiating (46) covariantly with respect to V_2, we get

$$\begin{aligned} \nabla_{V_2}\nabla_{V_1} D\psi = {} & (\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1}))\nabla_{V_2} V_1 \\ & - \kappa[\nabla_{V_2}\varrho(V_1)\theta + \varrho(V_1)\nabla_{V_2}\theta]. \end{aligned} \tag{47}$$

Interchanging V_1 and V_2 in (47), we obtain

$$\begin{aligned} \nabla_{V_1}\nabla_{V_2} D\psi = {} & (\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1}))\nabla_{V_1} V_2 \\ & - \kappa[\nabla_{V_1}\varrho(V_2)\theta + \varrho(V_2)\nabla_{V_1}\theta]. \end{aligned} \tag{48}$$

Also, from (46), we get

$$\nabla_{[V_1,V_2]}D\psi = (\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1}))[V_1, V_2] - \kappa\varrho([V_1, V_2])\theta. \tag{49}$$

Using (8), (47)-(49), we obtain

$$R(V_1, V_2)D\psi = -\kappa[\varrho(V_2)h\phi V_1 - \varrho(V_1)h\phi V_2]. \tag{50}$$

Consider the inner product with θ in (50), we infer

$$g(R(V_1, V_2)D\psi, \theta) = 0. \tag{51}$$

Taking inner product of (10) with $D\psi$, we obtain

$$\begin{aligned} g(R(V_1, V_2)\theta, D\psi) = \kappa[(V_1\psi)\varrho(V_2) - (V_2\psi)\varrho(V_1)] \\ + \mu[(hV_1\psi)\varrho(V_2) - (hV_2\psi)\varrho(V_1)]. \end{aligned} \tag{52}$$

As $g(R(V_1, V_2)V_3, V_4) = -g(R(V_1, V_2)V_4, V_3)$ for every vector fields V_1, V_2, V_3 and V_4 on the manifold, from (51) and (52), we get

$$\kappa[(V_1\psi)\varrho(V_2) - (V_2\psi)\varrho(V_1)] + \mu[(hV_1\psi)\varrho(V_2) - (hV_2\psi)\varrho(V_1)] = 0.$$

Putting $V_2 = \theta$ in the above equation, we obtain

$$\kappa[(V_1\psi) - (\theta\psi)\varrho(V_1)] + \mu(hV_1\psi) = 0. \tag{53}$$

Replacing V_1 by hV_1 in (53), we get

$$(hV_1\psi) = \mu[(V_1\psi) - (\theta\psi)\varrho(V_1)]. \tag{54}$$

Therefore, from (53) and (54), we get

$$(\kappa + \mu^2)[(V_1\psi) - (\theta\psi)\varrho(V_1)] = 0,$$

Thus we get either $\kappa = -\mu^2$ or $D\psi = (\theta\psi)\theta$.

When $D\psi = (\theta\psi)\theta$, from (46), we obtain

$$V_1(\theta\psi)\theta + (\theta\psi)h\phi V_1 = (\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1}))V_1 - \kappa\varrho(V_1)\theta. \tag{55}$$

Consider the inner product of (55) with V_2, we obtain

$$\begin{aligned} & V_1(\theta\psi)\varrho(V_2) + (\theta\psi)g(h\phi V_1, V_2) \\ & = (\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1}))g(V_1, V_2) - \kappa\varrho(V_1)\varrho(V_2). \end{aligned} \tag{56}$$

Putting $V_2 = \theta$ in the above equation, we obtain

$$V_1(\theta\psi) = (\lambda - \frac{1}{2}(q + \frac{2}{2m+1}))\varrho(V_1). \tag{57}$$

From (56) and (57), we obtain

$$(\theta\psi)g(h\phi V_1, V_2) = (\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1}))(g(V_1, V_2) - \varrho(V_1)\varrho(V_2)). \tag{58}$$

Contracting V_1 and V_2 in the above equation and using $tr(\phi h) = 0$, we obtain

$$(\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1})) = 0. \tag{59}$$

Using (59) in (58), we obtain

$$(\theta\psi)g(h\phi V_1, V_2) = 0,$$

which gives $(\theta\psi) = 0$. Thus, from the relation $D\psi = (\theta\psi)\theta$, we get $D\psi = 0$, i.e., $\mathcal{V} = 0$.

Thus we can state the following.

Theorem 4.1. *Let $\mathcal{M}$ be a $(2m+1)$-dimensional non-coKähler (κ, μ)-**ACM**. If $\mathcal{M}$ admits a $\star$-conformal gradient Ricci soliton, then either $\mu^2 = -\kappa$ or the soliton is trivial.*

Applying $D\psi = 0$ in (46), we obtain

$$(\lambda + \kappa - \frac{1}{2}(q + \frac{2}{2m+1}))g(V_1, V_2) - \kappa\varrho(V_1)\varrho(V_2) = 0. \tag{60}$$

Contracting the above equation, we have

$$\lambda = \frac{q}{2} + \frac{1}{2m+1} - \frac{2m\kappa}{2m+1}. \tag{61}$$

Again, putting $V_1 = V_2 = \theta$ in (60), we have

$$\lambda = \frac{1}{2}(q + \frac{2}{2m+1}). \tag{62}$$

Comparing (61) and (62), we infer that $\kappa = 0$. As a result, we may establish the following theorem.

Theorem 4.2. *There does not exist $\star$-conformal gradient Ricci solitons on non-coKähler (κ, μ)-**ACM**s.*

5. Example

Let us consider the manifold $\mathcal{M} = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ of dimension 3, where (x, y, z) are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields

$$\zeta_1 = e^{\frac{z}{2}}\frac{\partial}{\partial x} - e^{-\frac{z}{2}}\frac{\partial}{\partial y}, \quad \zeta_2 = e^{\frac{z}{2}}\frac{\partial}{\partial x} + e^{-\frac{z}{2}}\frac{\partial}{\partial y}, \quad \zeta_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of $\mathcal{M}$. We get the following by direct computations:

$$[\zeta_1, \zeta_2] = 0, \quad [\zeta_1, \zeta_3] = -\frac{1}{2}\zeta_2, \quad [\zeta_2, \zeta_3] = -\frac{1}{2}\zeta_1.$$

Let the metric tensor g be defined by

$$g(\zeta_1, \zeta_1) = g(\zeta_2, \zeta_2) = g(\zeta_3, \zeta_3) = 1$$

and $g(\zeta_i, \zeta_j) = 0$ for every $i \neq j$; $i, j = 1, 2, 3$.

The 1-form ϱ is defined by $\varrho(V_1) = g(V_1, \zeta_3)$ for every V_1 on $\mathcal{M}$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\zeta_1) = -\zeta_2, \quad \phi(\zeta_2) = \zeta_1, \quad \phi(\zeta_3) = 0.$$

Then we find that

$$\eta(\zeta_3) = 1, \quad \phi^2 V_1 = -V_1 + \eta(V_1)\zeta_3,$$
$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2)$$

for every vector fields V_1, V_2 on $\mathcal{M}$. Thus $(\phi, \zeta_3, \varrho, g)$ defines an almost contact metric structure.

For the components of Levi-Civita connection ∇ with respect to the metric g on M, we can write

$$2g(\nabla_{V_1} V_2, V_3) = V_1 g(V_2, V_3) + V_2 g(V_3, V_1) - V_3 g(V_1, V_2) - g(V_1, [V_2, V_3]) - g(V_2, [V_1, V_3]) + g(V_3, [V_1, V_2]),$$

which is known as Koszul's formula.

By Koszul's formula, we get the following expressions:

$$\nabla_{\zeta_1}\zeta_1 = 0, \quad \nabla_{\zeta_1}\zeta_2 = \frac{1}{2}\zeta_3, \quad \nabla_{\zeta_1}\zeta_3 = -\frac{1}{2}\zeta_2,$$
$$\nabla_{\zeta_2}\zeta_2 = 0, \quad \nabla_{\zeta_2}\zeta_1 = \frac{1}{2}\zeta_3, \quad \nabla_{\zeta_2}\zeta_3 = -\frac{1}{2}\zeta_1,$$
$$\nabla_{\zeta_3}\zeta_1 = 0, \quad \nabla_{\zeta_3}\zeta_2 = 0, \quad \nabla_{\zeta_3}\zeta_3 = 0.$$

The above data indicates that the given manifold is an **ACM** with $h\zeta_1 = -\frac{1}{2}\zeta_1$, $h\zeta_2 = \frac{1}{2}\zeta_2$ and $h\zeta_3 = 0$.

Using the formula $R(V_1, V_2)V_3 = \nabla_{V_1}\nabla_{V_2}V_3 - \nabla_{V_2}\nabla_{V_1}V_3 - \nabla_{[V_1,V_2]}V_3$, we get

$$R(\zeta_1, \zeta_2)\zeta_1 = -\frac{1}{4}\zeta_2, \quad R(\zeta_1, \zeta_2)\zeta_2 = \frac{1}{4}\zeta_1, \quad R(\zeta_2, \zeta_3)\zeta_3 = -\frac{1}{4}\zeta_2,$$
$$R(\zeta_3, \zeta_2)\zeta_2 = -\frac{1}{4}\zeta_3, \quad R(\zeta_1, \zeta_3)\zeta_3 = -\frac{1}{4}\zeta_1, \quad R(\zeta_3, \zeta_1)\zeta_1 = -\frac{1}{4}\zeta_3,$$
$$R(\zeta_1, \zeta_2)\zeta_3 = 0, \quad R(\zeta_2, \zeta_3)\zeta_1 = 0, \quad R(\zeta_3, \zeta_1)\zeta_2 = 0.$$

We may deduce from the foregoing that the manifold is a (κ, μ)-**ACM** with $\kappa = -\frac{1}{4}$ and $\mu = 0$.

From the expressions of curvature tensor, we get

$$S(\zeta_1, \zeta_1) = 0, \quad S(\zeta_2, \zeta_2) = 0, \quad S(\zeta_3, \zeta_3) = -1$$

and $S(\zeta_i, \zeta_j) = 0$ for every $i \neq j$; $i, j = 1, 2, 3$.

Let r be the scalar curvature. Then from above

$$r = S(\zeta_1, \zeta_1) + S(\zeta_2, \zeta_2) + S(\zeta_3, \zeta_3) = -1.$$

With the help of the curvature tensor, the components of $\star$-Ricci tensor are given by

$$S^\star(\zeta_1, \zeta_1) = \frac{1}{4}, \quad S^\star(\zeta_2, \zeta_2) = \frac{1}{4}, \quad S^\star(\zeta_3, \zeta_3) = 0 \tag{63}$$

and $S^\star(\zeta_i, \zeta_j) = 0$ for every $i \neq j$; $i, j = 1, 2, 3$.

Therefore, the $\star$-scalar curvature $r^\star$ is given by

$$r^\star = \frac{1}{2}.$$

Let the **PVF** $\mathcal{V}$ be the Reeb vector field ζ_3. Then

$$(\mathcal{L}_{\zeta_3} g)(\zeta_i, \zeta_i) = 0$$

for every $i = 1, 2, 3$. Thus, from the equation (28), we get the following two equations

$$2\lambda + 2\kappa - (q + \frac{2}{3}) = 0$$

and

$$2\lambda - (q + \frac{2}{3}) = 0$$

which gives $\kappa = 0$. But the value of κ of the given manifold is $-\frac{1}{4}$, which proves that the manifold $\mathcal{M}$ does not admit $\star$-conformal Ricci solitons if the **PVF** is the Reeb vector field ζ_3. This verifies Theorem 3.3.

Let us imagine that $\mathcal{V} = xe^{-\frac{z}{2}}\zeta_1 + ye^{\frac{z}{2}}\zeta_2 + \frac{5}{4}z\zeta_3$. Then it is easy to see that the equation (5) holds good. Thus the given manifold admits $\star$-conformal Ricci solitons. Also $\lambda = \frac{19}{12} + \frac{q}{2} > 0$, i.e., the soliton is shrinking, that verifies Corollary 3.5. Also $(\mathcal{L}_{\mathcal{V}}\varrho)(V_1) = \frac{5}{4}\varrho(V_1)$ for every vector field V_1 on $\mathcal{M}$. This indicates $\mathcal{V}$ is an infinitesimal contact transformation. Hence Theorem 3.6 is verified.

Let the potential vector field $\mathcal{V}(= xe^{-\frac{z}{2}}\zeta_1 + ye^{\frac{z}{2}}\zeta_2 + \frac{5}{4}z\zeta_3)$ be the gradient of a smooth function ψ. Then we can obtain the following partial differential equations:

$$\frac{\partial\psi}{\partial x} = \frac{1}{2}(xe^{-z} + y),$$
$$\frac{\partial\psi}{\partial y} = \frac{1}{2}(ye^{z} - x),$$
$$\frac{\partial\psi}{\partial z} = \frac{5}{4}z.$$

From the above set of equations, one obtains

$$\frac{\partial^2\psi}{\partial x\partial y} \neq \frac{\partial^2\psi}{\partial y\partial x}.$$

Thus $\mathcal{V}$ can not be the gradient of any smooth function. Hence Theorem 4.2 is verified.

Acknowledgement. The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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