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RESULTS CONCERNING SEMI-SYMMETRIC METRIC F-CONNECTIONS ON THE HSU-B MANIFOLDS

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Abstract. In this paper, we firstly construct a Hsu-B manifold and give some basic results related to it. Then, we address a semi-symmetric metric F-connection on the Hsu-B manifold and obtain the curvature tensor fields of such connection, and study properties of its curvature tensor and torsion tensor fields.

1. Introduction

In [6], for any $(1,1)$-type tensor field F and vector field X, Hsu constructed a new structure on a differentiable manifold of class C^∞ such that $F^2(X) = a^r I(X)$, where r and a are, respectively, an integer and complex number, and I denotes the identity operator. According to this new structure,

- if $a = -1$ and r is an odd number, then it is an almost complex structure;
- if $a = 1$ or $r = 0$ $(a \neq 0)$, then it is an almost product structure;
- if $a = 0$ $(r \neq 0)$, then it is an almost tangent structure;
- if $r = 2$, then it is a GF-structure and
  - for $a \neq 0$, it is a π-structure;
  - for $a = \pm i$, it is an almost complex structure;
  - for $a = 1$, it is an almost product structure;
  - for $a = 0$, it is an almost tangent structure.

Hsu examined the integrability of the new structure F [7]. Then, Singh defined a general algebraic Hsu-structure F and explored its integrability conditions [14]. Nivas and Geeta introduced a semi-symmetric connection that provides $\overline{\nabla} g \neq 0$ (non-metric properties) on a manifold with a generalised Hsu-structure and investigated the properties of this connection [8].

In this paper, we firstly introduce the Hsu-B manifold which is a manifold endowed with a Hsu structure F and a Riemannian metric g. Then, we give some basic results to prove our main results. Secondly, by following the method

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in [5], we define a semi-symmetric metric Hsu F-connection on the Hsu-B manifold and examine the curvature, Ricci, torsion tensor fields and scalar curvature of this connection. Finally, we study the Einstein manifold with regard to the semi-symmetric metric Hsu F-connection and investigate the condition for the Hsu-B manifold to be an Einstein Hsu-B manifold.

2. Some basic concepts and results

In this section, we shall give some basic definitions and results without proof. Let M_n be an n-dimensional ($n = 2k$) manifold. Throughout this article, all tensor fields, linear connections and manifolds will always be regarded as differentiable of class C^∞. The set of (p, q)-type tensor fields will also be denoted $\Im^p_q(M_n)$.

Let F be an almost Hsu structure on M_n such that $F^2(X) = a^r I(X)$ for any $X \in \Im^1_0(M_n)$ and g be a Riemannian metric on M_n. If

$$g(FX, Y) = g(X, FY) \tag{2.1}$$

or equivalently

$$g(FX, FY) = a^r g(X, Y),$$

then we will call the triplet (M_n, F, g) an almost Hsu-B manifold. The condition (2.1) means that F is self-adjoint with respect to the Riemannian metric g. The Riemannian metric g is also called B-metric, pure metric, anti-Hermitian metric or Norden metric [1,2,4,13]. The triplet (M_n, F, g) is a Hsu-B manifold if and only if $\nabla F = 0$, where ∇ is the Levi-Civita connection of g [7].

We will give the definition of the so-called Tachibana operator [12,15]. This definition will be given specifically for tensor fields of types $(0, q)$ and $(1, q)$ because we will work with these type of tensor fields in this paper. For the operators applied to the tensor fields of type (p, q), we refer to [13]. Let K and P be tensor fields of types $(0, q)$ and $(1, q)$, respectively. If

$$\begin{aligned} K(FX_1, X_2, \ldots, X_q) &= K(X_1, FX_2, \ldots, X_q) \\ &= \cdots = K(X_1, X_2, \ldots, FX_q) \end{aligned}$$

and

$$\begin{aligned} F(P(X_1, X_2, \ldots, X_q)) &= P(FX_1, X_2, \ldots, X_q) \\ &= \cdots = P(X_1, X_2, \ldots, FX_q), \end{aligned}$$

where $X_1, X_2, \ldots, X_q \in \Im^1_0(M_n)$ and $F \in \Im^1_1(M_n)$, then the tensor field K of type $(0, q)$ (resp. the tensor field P of type $(1, q)$) is called a pure tensor field with respect to F. For a vector field V, the Tachibana operators applied to the pure tensor fields of types $(0, q)$ and $(1, q)$ are, respectively, as follows:

$$\begin{aligned} &(\phi_F K)(V, X_1, X_2, \ldots, X_q) \\ &= (FV)(K(X_1, X_2, \ldots, X_q)) - V(K(FX_1, X_2, \ldots, X_q)) \end{aligned} \tag{2.2}$$

$$+\sum_{i=1}^{q} K(X_1,\dots,(L_{X_i}F)V,\dots,X_q)$$

and

$$(\phi_F P)\,(V,X_1,X_2,\dots,X_q) = -\left(L_{P(X_1,X_2,\dots,X_q)}F\right)V + \sum_{i=1}^{q} P\,(X_1,\dots,(L_{X_i}F)V,\dots,X_q)\,, \tag{2.3}$$

where L_X shows the Lie differentiation with respect to X. For any $(1,1)$-tensor field F, if the pure tensor field K satisfies $\phi_F K = 0$, then it is called a ϕ-tensor field. Especially, if F is a complex structure ($F^2(X) = -I(X)$), then K is called a holomorphic tensor field [15] and if F is a product structure ($F^2(X) = I(X)$), K is called a decomposable tensor field [12].

The following propositions can be proven by following the method used in [13]. Because of this, we omit them.

Proposition 2.1. *Let (M_n, g, F) be an almost Hsu-B manifold. Then, $\phi_F g = 0$ is equivalent to $\nabla F = 0$, i.e., the triplet (M_n, g, F) is a Hsu-B manifold, where ∇ is the Levi-Civita connection of g.*

As is known, a torsion-free F-connection ($\nabla F = 0$) is always pure [12]. Based on this fact, we easily say that in case of a Hsu-B manifold, the Levi-Civita connection is an F-connection with zero torsion, so it is clearly pure with respect to F. The pureness of the Levi-Civita connection immediately gives the pureness of the Riemannian curvature tensor field. With respect to the Levi-Civita connection ∇, the equations (2.2) and (2.3) become, respectively, the following simpler forms:

$$(\phi_F K)(V,X_1,X_2,\dots,X_q) = (\nabla_{FV}K)(X_1,X_2,\dots,X_q) - (\nabla_V K)(FX_1,X_2,\dots,X_q)$$

and

$$(\phi_F P)(V,X_1,X_2,\dots,X_q) = (\nabla_{FV}P)(X_1,X_2,\dots,X_q) - F[(\nabla_V P)(X_1,X_2,\dots,X_q)]. \tag{2.4}$$

Proposition 2.2. *The Riemannian curvature tensor field R of the Hsu-B manifold (M_n, g, F) is pure according to F and also a ϕ-tensor field, i.e., $\phi_F R = 0$.*

3. Main results

The section deals with the geometry of a semi-symmetric metric F-connection on a Hsu-B manifold. Firstly, we will define this connection and then study the properties of its torsion and curvature tensor fields. Lastly, we will construct Einstein manifolds in this setting.

In [5], Hayden defined a linear connection with non-zero torsion. Then, Yano constructed a semi-symmetric metric connection and studied its properties [16]. A semi-symmetric metric connection is a connection such that $\overline{\nabla} g = 0$ and its torsion tensor field is: $T(X,Y) = p(Y)X - p(X)Y$, where p is a 1-form field [17]. Prvanovic also created a new type of semi-symmetric metric connections on a locally decomposable Riemannian manifold equipped with an almost product structure [10]. In addition, this connection provides the condition $\overline{\nabla} F = 0$. Therefore, Prvanovic called the new type of connection a semi-symmetric metric F-connection and investigated its curvature tensor properties [9,11]. Besides, Gezer and Karaman formed the golden semi-symmetric metric F-connections and examined some properties of these connections [3].

Given any a linear connection ${}^{Hsu}\nabla$ with torsion tensor T, if its torsion tensor satisfies the condition

$$T(X,Y) = p(Y)(X) - p(X)(Y) + a^{-r}\left[p(FY)(FX) - p(FX)(FY)\right], \tag{3.1}$$

then we will call this connection a semi-symmetric Hsu connection, where $p \in \Im_1^0(M_n)$ and $X, Y \in \Im_0^1(M_n)$. By using the method of Hayden [5], for $g(U,Y) = p(Y)$, $U \in \Im_0^1(M_n)$, we obtain

$$\begin{aligned} {}^{Hsu}\nabla_X Y = \nabla_X Y &+ p(Y)(X) - g(X,Y)(U) \\ &+ a^{-r}\left[p(FY)(FX) - g(FX,Y)(FU)\right]. \end{aligned} \tag{3.2}$$

The connection (3.2) satisfies the following equations:

$${}^{Hsu}\nabla g = 0 \quad \text{and} \quad {}^{Hsu}\nabla F = 0,$$

i.e., this connection is both metric and F-connection. Throughout the paper, we will call this connection a semi-symmetric metric Hsu F-connection and will denote it $S.S.M.$-Hsu F-connection.

The torsion tensor field T given by (3.2) is pure with respect to the Hsu structure F:

$$T(FX,Y) = T(X,FY) = F(T(X,Y)).$$

Also, note that if the torsion tensor field of any F-connection is pure, then the connection is pure [12]. Thus, we can easily say that the connection (3.2) is pure with respect to F:

$${}^{Hsu}\nabla_{FX} Y = {}^{Hsu}\nabla_X FY = F\,{}^{Hsu}\nabla_X Y.$$

Theorem 3.1. *Let (M_n, g, F) be a Hsu-B manifold. If the 1-form p is a ϕ-tensor field, then the torsion tensor field T of the connection (3.2) is a ϕ-tensor field and the following equation always holds:*

$$\begin{aligned} (\nabla_{FV} T)(X,Y) &= (\nabla_V T)(FX,Y) \\ &= (\nabla_V T)(X,FY) = F(\nabla_V T)(X,Y), \end{aligned}$$

i.e., the covariant derivatives of the tensor field T are pure with respect to the Hsu structure F.

Proof. Using (2.4) and the torsion tensor field T of the connection (3.2), we obtain

$$(\phi_F T)(V,X,Y) = (\nabla_{FV}T)(X,Y) - (\nabla_V T)(FX,Y). \tag{3.3}$$

Substituting (3.1) into (3.3), we get

$$\begin{aligned}
&(\phi_F T)(V,X,Y)\\
&= [(\nabla_{FV}p)(Y) - (\nabla_V p)(FY)](X) - [(\nabla_{FV}p)(X) - (\nabla_V p)(FX)](Y)\\
&\quad + [a^{-r}(\nabla_{FV}p)(FY) - (\nabla_V p)(Y)](FX)\\
&\quad - [a^{-r}(\nabla_{FV}p)(FX) - (\nabla_V p)(X)](FY).
\end{aligned}$$

Also, for the 1-form p, the following condition is valid:

$$(\phi_F p)(V,X) = (\nabla_{FV}p)(X) - (\nabla_V p)(FX).$$

Finally, it is obvious that if $\phi_F p = 0$, then $\phi_F T = 0$. Furthermore, since the torsion tensor field T satisfies the condition $(\nabla_V T)(FX,Y) = (\nabla_V T)(X,FY) = F(\nabla_V T)(X,Y)$, we can write

$$\begin{aligned}
(\nabla_{FV}T)(X,Y) &= (\nabla_V T)(FX,Y)\\
&= (\nabla_V T)(X,FY) = F(\nabla_V T)(X,Y),
\end{aligned}$$

which completes the proof. □

From now on, we will assume a special case of the $S.S.M.$-Hsu F-connection when $\phi_F p = 0$, i.e., $(\nabla_{FV}p)(X) - (\nabla_V p)(FX) = 0$.

Now, we turn our attention to the curvature tensor fields of this connection. The curvature tensor field of the $S.S.M.$-Hsu F-connection is defined as:

$${}^{Hsu}R(X,Y,Z) = {}^{Hsu}\nabla_X\, {}^{Hsu}\nabla_Y Z - {}^{Hsu}\nabla_Y\, {}^{Hsu}\nabla_X Z - {}^{Hsu}\nabla_{[X,Y]} Z.$$

Considering ${}^{Hsu}R(X,Y,Z,W) = g({}^{Hsu}R(X,Y,Z),W)$, the curvature tensor field ${}^{Hsu}R$ of the connection (3.2) is of the form:

$$\begin{aligned}
&{}^{Hsu}R(X,Y,Z,W)\\
&= R(X,Y,Z,W) + g(Y,W)A(X,Z) - g(X,W)A(Y,Z)\\
&\quad + g(X,Z)A(Y,W) - g(Y,Z)A(X,W)\\
&\quad + a^{-r}[g(FY,W)A(X,FZ) - g(FX,W)A(Y,FZ)\\
&\quad + g(FX,Z)A(Y,FW) - g(FY,Z)A(X,FW)],
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
A(X,Z) &= (\nabla_X p)(Z) - p(X)p(Z) + \frac{1}{2}p(U)g(X,Z)\\
&\quad - a^{-r}p(FX)p(FZ) + \frac{1}{2}a^{-r}p(FU)g(FX,Z).
\end{aligned} \tag{3.5}$$

It is easy to see that the curvature tensor field ${}^{Hsu}R$ satisfies

$${}^{Hsu}R(X,Y,W,Z) = -{}^{Hsu}R(X,Y,Z,W) = {}^{Hsu}R(Y,X,Z,W).$$

Also, from

$$A(X,Y) - A(Y,X) = (\nabla_X p)(Y) - (\nabla_Y p)(X)$$

and by applying the exterior differential operator to the 1-form, we obtain

$$\begin{aligned} 2(dp)(X,Y) &= X(p(Y)) - Y(p(X)) - p([X,Y]) \\ &= (\nabla_X p)Y + p(\nabla_X Y) - (\nabla_Y p)X - p(\nabla_Y X) - p([X,Y]) \\ &= (\nabla_X p)Y - (\nabla_Y p)X + p(\nabla_X Y - \nabla_Y X - [X,Y]) \\ &= (\nabla_X p)(Y) - (\nabla_Y p)(X). \end{aligned}$$

From the last two equations, we can write

$$\begin{aligned} A(X,Y) - A(Y,X) &= (\nabla_X p)(Y) - (\nabla_Y p)(X) \\ &= 2(dp)(X,Y). \end{aligned}$$

Corollary 3.2. *The tensor field A given by* (3.5) *is symmetric if and only if the* 1*-form p is closed.*

The tensor field A given by (3.5) is pure with respect to the Hsu structure F, i.e.,

$$A(X,FY) - A(FX,Y) = [(\nabla_X p)(FY) - (\nabla_{FX} p)(Y)] = 0.$$

Then we can give the following useful lemma.

Lemma 3.3. *Let (M_n, g, F) be a Hsu-B manifold. If the* 1*-form p is a ϕ-tensor field, then the tensor field A given by* (3.5) *on the Hsu-B manifold (M_n, g, F) is a ϕ-tensor field, i.e., $\phi_F A = 0$ and the following condition holds*:

$$(\nabla_{FX} A)(X,Y) = (\nabla_X A)(FX,Y) = (\nabla_X A)(X,FY).$$

Proof. By applying the Tachibana operator to the tensor field A, we find

$$(\phi_F A)(V,X,Y) = (\nabla_{FV} A)(X,Y) - (\nabla_V A)(FX,Y). \tag{3.6}$$

Substituting (3.5) in (3.6), we have

$$(\phi_F A)(V,X,Y) = \left[(\nabla_{FV}\nabla_X p)(Y) - (\nabla_V \nabla_{FX} p)(Y)\right].$$

Besides, the Ricci identity for the 1-form p is as follows:

$$(\nabla_{FV}\nabla_X p)(Y) = (\nabla_X \nabla_{FV} p)(Y) - \frac{1}{2} p(R(FV,X,Y))$$

and

$$\begin{aligned} (\nabla_V \nabla_{FX} p)(Y) &= (\nabla_V \nabla_X p)(FY) \\ &= (\nabla_X \nabla_V p)(FY) - \frac{1}{2} p(R(V,X,FY)). \end{aligned}$$

From the last two equations, we can write

$$(\phi_F A)(V,X,Y) = -\frac{1}{2} p(R(FV,X,Y) - R(V,X,FY)) = 0$$

and

$$(\nabla_{FX} A)(X,Y) = (\nabla_X A)(FX,Y) = (\nabla_X A)(X,FY). \qquad \square$$

From Lemma 3.3, it immediately follows that

$$\begin{aligned}{}^{Hsu}R(FX,Y,Z,W) &= {}^{Hsu}R(X,FY,Z,W)\\ &= {}^{Hsu}R(X,Y,FZ,W) = {}^{Hsu}R(X,Y,Z,FW),\end{aligned}$$

i.e., the curvature tensor field ${}^{Hsu}R$ is pure with respect to the Hsu structure F.

Theorem 3.4. *Let (M_n, g, F) be a Hsu-B manifold. If the 1-form p is a ϕ-tensor field, then the curvature tensor field ${}^{Hsu}R$ of the connection (3.2) satisfies $\phi_F {}^{Hsu}R = 0$, i.e., the tensor field ${}^{Hsu}R$ is a ϕ-tensor field and following condition always holds*:

$$\begin{aligned}&(\nabla_{FV} {}^{Hsu}R)(X,Y,Z,W)\\ &= (\nabla_V {}^{Hsu}R)(FX,Y,Z,W) = (\nabla_V {}^{Hsu}R)(X,FY,Z,W)\\ &= (\nabla_V {}^{Hsu}R)(X,Y,FZ,W) = (\nabla_V {}^{Hsu}R)(X,Y,Z,FW).\end{aligned}$$

Proof. By applying the Tachibana operator, we get

$$\begin{aligned}(\phi_F {}^{Hsu}R)(V,X,Y,Z,W) &= (\nabla_{FV} {}^{Hsu}R)(X,Y,Z,W)\\ &\quad - (\nabla_V {}^{Hsu}R)(FX,Y,Z,W).\end{aligned}$$

From (3.4), standard calculations give the following

$$\begin{aligned}&(\phi_F {}^{Hsu}R)(V,X,Y,Z,W)\\ &= (\nabla_{FV}R)(X,Y,Z,W) - (\nabla_V R)(FX,Y,Z,W)\\ &\quad + [(\nabla_{FV}A)(X,Z) - (\nabla_V A)(FX,Z)]g(Y,W)\\ &\quad - [(\nabla_{FV}A)(Y,Z) - (\nabla_V A)(FY,Z)]g(X,W)\\ &\quad + [(\nabla_{FV}A)(Y,W) - (\nabla_V A)(FY,W)]g(X,Z)\\ &\quad - [(\nabla_{FV}A)(X,W) - (\nabla_V A)(FX,W)]g(Y,Z)\\ &\quad + a^{-r}[(\nabla_{FV}A)(FX,Z) - a^r(\nabla_V A)(X,Z)]g(FY,W)\\ &\quad - a^{-r}[(\nabla_{FV}A)(FY,Z) - a^r(\nabla_V A)(Y,Z)]g(FX,W)\\ &\quad + a^{-r}[(\nabla_{FV}A)(FY,W) - a^r(\nabla_V A)(Y,W)]g(FX,Z)\\ &\quad - a^{-r}[(\nabla_{FV}A)(FX,W) - a^r(\nabla_V A)(X,W)]g(FY,Z).\end{aligned}$$

From Lemma 3.3 and Proposition 2.2, we obtain

$$\begin{aligned}(\phi_F {}^{Hsu}R)(V,X,Y,Z,W) &= (\nabla_{FV}R)(X,Y,Z,W) - (\nabla_V R)(FX,Y,Z,W)\\ &= (\phi_F R)(V,X,Y,Z,W)\\ &= 0.\end{aligned}$$

We also get the following result with a simple calculation:

$$\begin{aligned}&(\nabla_{FV} {}^{Hsu}R)(X,Y,Z,W)\\ &= (\nabla_V {}^{Hsu}R)(FX,Y,Z,W) = (\nabla_V {}^{Hsu}R)(X,FY,Z,W)\end{aligned}$$

$$= (\nabla_V {}^{Hsu}R)(X, Y, FZ, W) = (\nabla_V {}^{Hsu}R)(X, Y, Z, FW). \qquad \square$$

The Ricci tensor field of the $S.S.M.$-Hsu F-connection given by (3.2) is as follows:

$$\begin{aligned}\sum_{i=1}^{n} {}^{Hsu}R(E_i, Y, Z, E_i) = {}& {}^{Hsu}R(Y, Z) \\ = {}& R(Y, Z) + (4 - n)A(Y, Z) - g(Y, Z)(trA) \\ & - a^{-r}[A(Y, FZ)(trF) + g(FY, Z)(tr\theta)],\end{aligned}$$

where $\{E_i\}, i = 1, \ldots, n$, are orthonormal vector fields on M_n and trA and $tr\theta$ are defined by

$$\begin{aligned}trA &= \sum_{i=1}^{n} A(E_i, E_i) \\ &= \sum_{i=1}^{n} (\nabla_{E_i} p)(E_i) + \frac{(n-4)}{2} p(U) + \frac{1}{2} a^{-r}(trF)p(FU),\end{aligned}$$

$$\begin{aligned}tr\theta &= \sum_{i=1}^{n} (A \circ F)(E_i, E_i) = \sum_{i=1}^{n} A(E_i, FE_i) \\ &= \sum_{i=1}^{n} (\nabla_{E_i} p)(FE_i) + \frac{(n-4)}{2} p(FU) + \frac{1}{2}(trF)p(U).\end{aligned}$$

Besides, we get

$$\begin{aligned}{}^{Hsu}R(Y, Z) - {}^{Hsu}R(Y, Z) &= (n - 4)[A(Y, Z) - A(Z, Y)] \\ &\quad + a^{-r}(trF)[A(FY, Z) - A(Z, FY)] \\ &= 2(n - 4)(dp)(Z, Y) + 2a^{-r}(trF)(dp)(FY, Z).\end{aligned}$$

Then, we can say that ${}^{Hsu}R(Y, Z) - {}^{Hsu}R(Z, Y) = 0$, i.e., the Ricci tensor field is symmetric, if $dp = 0$. The scalar curvature tensor of the connection (3.2) is characterized by

$$\begin{aligned}\sum_{i=1}^{n} {}^{Hsu}R(E_i, E_i) &= {}^{Hsu}\tau \\ &= \tau + 2(2 - n)(trA) - 2a^{-r}(trF)(tr\theta).\end{aligned} \tag{3.7}$$

If the Ricci tensor field $R(X, Y)$ of a Riemannian manifold satisfies the equation $R(X, Y) = \lambda g(X, Y)$, then the Riemannian manifold is called an Einstein manifold, where λ is a scalar function. Let the Ricci tensor field of the $S.S.M.$-Hsu F-connection satisfy the following equation:

$$\underset{(X,Y)}{sym} {}^{Hsu}R(X, Y) = \mu g(X, Y). \tag{3.8}$$

Then, the Hsu-B manifold (M_n, g, F) with the $S.S.M.$-Hsu F-connection may be called an Einstein Hsu-B manifold, where μ is a scalar function and $\underset{(X,Y)}{sym}$ is the symmetric part of the Ricci tensor field of the $S.S.M.$-Hsu F-connection. It is clear that the condition ${}^H\tau = \mu n$ is satisfied on the Einstein Hsu-B manifold. Thus, we can write following theorem.

Theorem 3.5. *If the Hsu-B manifold (M_n, g, F) with the S.S.M.-Hsu F-connection is an Einstein Hsu-B manifold, then the following condition holds*:

$$\mu - \lambda = \alpha \sum_{i=1}^{n} (\nabla_{E_i} p)(E_i) - \beta \sum_{i=1}^{n} (\nabla_{E_i} p)(FE_i) - \gamma p(U) - \epsilon p(FU),$$

where

$$\alpha = \frac{2(2-n)}{n}, \ \beta = \frac{2a^{-r}(trF)}{n},$$
$$\gamma = \frac{(n-2)(n-4) + a^{-r}(trF)^2}{n}, \ \epsilon = \frac{2(n-3)a^{-r}(trF)}{n},$$

and λ is a scalar function that is due to the Einstein property of the Riemannian manifold, i.e., $R(X,Y) = \lambda g(X,Y)$.

Proof. From (3.8) and $R(X,Y) = \lambda g(X,Y)$, we get

$$\begin{cases} {}^{Hsu}\tau &= \mu n, \\ \tau &= \lambda n. \end{cases} \tag{3.9}$$

From (3.9) and (3.8), we have

$$(\mu - \lambda)n = 2(2-n)(trA) - 2a^{-r}(trF)(tr\theta).$$

Finally, substituting trA and $tr\theta$ in the last equation, and with simple calculations we reach the end of the proof. □

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