Commun. Korean Math. Soc. **38** (2023), No. 3, pp. 807–819
https://doi.org/10.4134/CKMS.c220217
pISSN: 1225-1763 / eISSN: 2234-3024

A NOTE ON CERTAIN TRANSFORMATION FORMULAS RELATED TO APPELL, HORN AND KAMPÉ DE FÉRIET FUNCTIONS

Asmaa Orabi Mohammed, Medhat Ahmed Rakha, and Arjun K. Rathie

Abstract. In 2019, Mathur and Solanki [7, 8] obtained a few transformation formulas for Appell, Horn and the Kampé de Fériet functions. Unfortunately, some of the results are well-known and very old results in literature while others are erroneous. Thus the aim of this note is to provide the results in corrected forms and some of the results have been written in more compact form.

1. Introduction and results required

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by [2, 9, 11]

$$ {}_pF_q\left[\begin{matrix} a_1, & \ldots, & a_p \\ & & \\ b_1, & \ldots, & b_q \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{1} $$

where $(a)_n$ is the well-known Pochhammer's symbol defined by

$$ (a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases} $$

In terms of gamma functions, we have

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. $$

As usual, p and q are non-negative integers and the parameters a_j $(j = 1, \ldots, p)$ and b_j $(j = 1, \ldots, q)$ can have arbitrary complex values with zero or negative integer values of b_j excluded. The sum (1) converges for $|z| < \infty$ $(p \leq q)$,

Received July 20, 2022; Revised October 10, 2022; Accepted November 15, 2022.
2020 *Mathematics Subject Classification.* Primary 33B15, 33C05, 33C15; Secondary 33C65, 33C20.
Key words and phrases. Hypergeometric function, Kummer's second theorem, contiguous results, Appell's series, Horn's function, Kampé de Fériet function.

$|z| < 1$ $(p = q+1)$ and $|z| = 1$ $(p = q+1, Re(s) > 0)$, where s is the parametric excess defined by

$$s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j.$$

For more details about generalized hypergeometric function ${}_pF_q(z)$, one may refer [2, 9, 11].

On the other hand, by making use of the Kummer's second theorem [9] viz.

$$e^{-\frac{t}{2}}\ {}_1F_1\left[\begin{matrix}\lambda \\ 2\lambda\end{matrix};t\right] = {}_0F_1\left[\begin{matrix}- \\ \lambda+\frac{1}{2}\end{matrix};\frac{t^2}{16}\right], \tag{2}$$

and two results closely related to (2) obtained earlier by Rathie and Nagar [10], also recorded in Kim et al. [6] viz.

$$e^{-\frac{t}{2}}\ {}_1F_1\left[\begin{matrix}\lambda \\ 2\lambda+1\end{matrix};t\right] = {}_0F_1\left[\begin{matrix}- \\ \lambda+\frac{1}{2}\end{matrix};\frac{t^2}{16}\right] - \frac{t}{2(2\lambda+1)}\ {}_0F_1\left[\begin{matrix}- \\ \lambda+\frac{3}{2}\end{matrix};\frac{t^2}{16}\right], \tag{3}$$

and

$$e^{-\frac{t}{2}}\ {}_1F_1\left[\begin{matrix}\lambda \\ 2\lambda-1\end{matrix};t\right] = {}_0F_1\left[\begin{matrix}- \\ \lambda-\frac{1}{2}\end{matrix};\frac{t^2}{16}\right] + \frac{t}{2(2\lambda-1)}\ {}_0F_1\left[\begin{matrix}- \\ \lambda+\frac{1}{2}\end{matrix};\frac{t^2}{16}\right], \tag{4}$$

in the following integral representation of the Appell's series F_2 recorded, for example, in [11, eqn. (29), p. 282] viz.

$$F_2[\lambda,\mu_1,\mu_2;\nu_1,\nu_2;x_1,x_2] = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-t}t^{\lambda-1}\ {}_1F_1\left[\begin{matrix}\mu_1 \\ \nu_1\end{matrix};x_1t\right]\ {}_1F_1\left[\begin{matrix}\mu_2 \\ \nu_2\end{matrix};x_2t\right]dt, \tag{5}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < 1$.

Very recently, Mathur and Solanki [8] obtained the following three transformation formulas between Appell's series F_2 and the Kampé de Fériet function written here in a slightly different form viz.

$$F_2[\lambda,\mu,\nu;2\mu,2\nu;2x_1,2x_2] = (1-x_1-x_2)^{-\lambda}$$

$$\times\ F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu+\frac{1}{2}; \quad \nu+\frac{1}{2}\end{array}\right], \tag{6}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < \frac{1}{2}$,

$$F_2[\lambda, \mu, \nu; 2\mu+1, 2\nu+1; 2x_1, 2x_2] = (1-x_1-x_2)^{-\lambda}$$

$$\times\left\{ F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{y}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu+\frac{1}{2}; \quad \nu+\frac{1}{2}\end{array}\right]\right.$$

$$-\frac{x_1}{2\mu+1} F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu+\frac{3}{2}; \quad \nu+\frac{1}{2}\end{array}\right]$$

$$-\frac{x_2}{2\nu+1} F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu+\frac{1}{2}; \quad \nu+\frac{3}{2}\end{array}\right]$$

$$+\frac{x_1 x_2}{(2\mu+1)(2\nu+1)}$$

$$\left.\times\ F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu+\frac{3}{2}; \quad \nu+\frac{3}{2}\end{array}\right]\right\}, \tag{7}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < \frac{1}{2}$,
and

$$F_2[\lambda, \mu, \nu; 2\mu-1, 2\nu-1; 2x_1, 2x_2] = (1-x_1-x_2)^{-\lambda}$$

$$\times\left\{ F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu-\frac{1}{2}; \quad \nu-\frac{1}{2}\end{array}\right]\right.$$

$$+\frac{x_1}{2\mu-1} F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu+\frac{1}{2}; \quad \nu-\frac{1}{2}\end{array}\right]$$

$$+\frac{x_2}{2\nu-1} F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu-\frac{1}{2}; \quad \nu+\frac{1}{2}\end{array}\right]$$

$$+\frac{x_1 x_2}{(2\mu-1)(2\nu-1)}$$

$$\left.\times\ F^{2:0;0}_{0:1;1}\left[\begin{array}{l}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : \quad -; \quad - \\ \qquad\qquad\qquad\qquad ; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ \quad - : \quad \mu+\frac{1}{2}; \quad \nu+\frac{1}{2}\end{array}\right]\right\}, \tag{8}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < \frac{1}{2}$.

In the above three results, F_2 is the well-known Appell's double hypergeometric series defined by [1]:

$$F_2[a, \mu_1, \mu_2; \nu_1, \nu_2; x_1, x_2] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(\mu_1)_m(\mu_2)_n x_1^m x_2^n}{(\nu_1)_m(\nu_2)_n m!n!},$$

provided $|x_1| + |x_2| < 1$.

Also, the definition of Appell's double hypergeomtric series F_4 is defined by [1]

$$F_4[\alpha, \beta; \gamma, \gamma^{'}; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n} x^m y^n}{(\gamma)_m(\gamma^{'})_n m!n!}, \tag{9}$$

provided $\sqrt{x} + \sqrt{y} < 1$.

The four Appell's double hypergeometric series F_1, F_2, F_3 and F_4 was generalized by Kampé de Fériet function [4, 5] defined in more general form and recorded in [8] in the following manner:

$$F_{l:m;n}^{p:q;k}\left[\begin{array}{cccc} (\alpha_p): & (\beta_q); & (\gamma_k) & \\ & & & ;x,\ y \\ (\lambda_l): & (\mu_m); & (\eta_n) & \end{array}\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_{r+s}\prod_{j=1}^{q}(\beta_j)_r\prod_{j=1}^{k}(\gamma_j)_s x^r y^s}{\prod_{j=1}^{l}(\lambda_j)_{r+s}\prod_{j=1}^{m}(\mu_j)_r\prod_{j=1}^{n}(\eta_j)_s s!r!},$$

where, for convergence

(i) $p + q < l + m + 1$, $p + k < l + n + 1$, $|x| < \infty$, $|y| < \infty$, or
(ii) $p + q = l + m + 1$, $p + k = l + n + 1$, and

$$\begin{cases} |x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-l}} < 1, & \text{if } p > l, \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq l. \end{cases}$$

Moreover, the notation $F_{l:m;n}^{p:q;k}$ for a generalized double hypergeometric series of superior order applies successfully to the Appell's series F_1, F_2, F_3 and F_4. Thus, for example, we have

$$F_1 = F_{1:0;0}^{1:1;0},$$
$$F_2 = F_{0:1;1}^{1:1;1},$$
$$F_3 = F_{1:0;0}^{0:2;2},$$

and

$$F_4 = F_{0:1;1}^{2:0;0}.$$

For more details about this function, we refer the standard text [11].

Similarly, by making use of (2) to (4) in (5), in another paper, Mathur and Solanki [8] have established the following three transformation formulas between the Appell's function F_2 and the Horn's function H_4 viz.

$$\begin{aligned} &F_2[\lambda, \mu_1, \mu_2; 2\mu_1, \nu; 4x_1, x_2] \\ &= (1-2x_1)^{-\lambda}\ H_4\left[\lambda, \mu_2; \mu_1 + \frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right], \end{aligned} \tag{10}$$

$$\begin{aligned} &F_2[\lambda, \mu_1, \mu_2; 2\mu_1 + 1, \nu; 4x_1, x_2] \\ &= (1-2x_1)^{-\lambda}\left\{ H_4\left[\lambda, \mu_2; \mu_1 + \frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right] \right. \\ &\left. \quad -\frac{2x_1}{2\mu_1+1}\ H_4\left[\lambda, \mu_2; \mu_1 + \frac{3}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right]\right\}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} &F_2[\lambda, \mu_1, \nu; 2\mu_1 - 1, 2\nu - 1; 2x_1, 2x_2] \\ &= (1-2x_1)^{-\lambda}\left\{ H_4\left[\lambda, \mu_2; \mu_1 - \frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right] \right. \\ &\left. \quad -\frac{2x_1}{2\mu_1-1}\ H_4\left[\lambda, \mu_2; \mu_1 + \frac{1}{2}, c; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right]\right\}. \end{aligned} \tag{12}$$

In the above three results, F_2 is the well-known Appell's function defined by (8), and the Horn's function H_4 is defined by [4]

$$H_4\left[\alpha, \beta; \gamma, \delta; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n x^m y^n}{(\gamma)_m(\delta)_n m! n!}, \tag{13}$$

provided $2\sqrt{|x|} + |y| < 1$.

For more details about these functions F_2 and H_4, we refer to the standard text [4] and [11].

Unfortunately, the results (7), (8), (11) and (12) are erroneous and the result (10) is the well-known result due to Erdélyi [3]. Thus the aim of this note is to provide the corrected forms of these results. We also put the results (6) to (8) in more compact forms.

2. Main results

In this section, we shall establish the following five results:

$$\begin{aligned} &F_2[\lambda, \mu, \nu; 2\mu, 2\nu; 2x_1, 2x_2] = (1-x_1-x_2)^{-\lambda} \\ &\times\ F_4\left[\frac{1}{2}\lambda, \frac{1}{2}\lambda + \frac{1}{2}; \mu + \frac{1}{2}, \nu + \frac{1}{2}; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2\right], \end{aligned} \tag{14}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < \frac{1}{2}$,

$$F_2[\lambda, \mu, \nu; 2\mu + 1, 2\nu + 1; 2x_1, 2x_2] = (1 - x_1 - x_2)^{-\lambda}$$
$$\times \left\{ F_4\left[\frac{1}{2}\lambda, \frac{1}{2}\lambda + \frac{1}{2}; \mu + \frac{1}{2}, \nu + \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right.$$
$$- \frac{\lambda x_1}{(2\mu + 1)(1 - x_1 - x_2)}$$
$$\times F_4\left[\frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda + 1; \mu + \frac{3}{2}, \nu + \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]$$
$$- \frac{\lambda x_2}{(2\nu + 1)(1 - x_1 - x_2)}$$
$$\times F_4\left[\frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda + 1; \mu + \frac{1}{2}, \nu + \frac{3}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]$$
$$+ \frac{\lambda(\lambda + 1)x_1 x_2}{(2\mu + 1)(2\nu + 1)(1 - x_1 - x_2)^2}$$
$$\left. \times F_4\left[\frac{1}{2}\lambda + 1, \frac{1}{2}\lambda + \frac{3}{2}; \mu + \frac{3}{2}, \nu + \frac{3}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right\}, \tag{15}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < \frac{1}{2}$,

$$F_2[\lambda, \mu, \nu; 2\mu - 1, 2\nu - 1; 2x_1, 2x_2] = (1 - x_1 - x_2)^{-\lambda}$$
$$\times \left\{ F_4\left[\frac{1}{2}\lambda, \frac{1}{2}\lambda + \frac{1}{2}; \mu - \frac{1}{2}, \nu - \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right.$$
$$+ \frac{\lambda x_1}{(2\mu - 1)(1 - x_1 - x_2)}$$
$$\times F_4\left[\frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda + 1; \mu + \frac{1}{2}, \nu - \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]$$
$$+ \frac{\lambda x_2}{(\nu - 1)(1 - x_1 - x_2)}$$
$$\times F_4\left[\frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda + 1; \mu - \frac{1}{2}, \nu + \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]$$
$$+ \frac{\lambda(\lambda + 1)x_1 x_2}{(2\mu - 1)(2\nu - 1)(1 - x_1 - x_2)^2}$$
$$\left. \times F_4\left[\frac{1}{2}\lambda + 1, \frac{1}{2}\lambda + \frac{3}{2}; \mu + \frac{1}{2}, \nu + \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right\}, \tag{16}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < \frac{1}{2}$,

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1 + 1, \nu; 4x_1, x_2]$$

$$= (1-2x_1)^{-\lambda}\Bigg\{ H_4\left[\lambda, \mu_2; \mu_1+\frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right]$$

$$-\frac{2\lambda x_1}{(2\mu_1+1)(1-2x_1)}\ H_4\left[\lambda+1, \mu_2; \mu_1+\frac{3}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right]\Bigg\}, \tag{17}$$

provided $Re(\lambda) > 0$ and $Re(4x_1 + x_2) < 1$, and

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1 - 1, \nu; 4x_1, x_2]$$

$$= (1-2x_1)^{-\lambda}\Bigg\{ H_4\left[\lambda, \mu_2; \mu_1-\frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right]$$

$$+\frac{2\lambda x_1}{(2\mu_1-1)(1-2x_1)}\ H_4\left[\lambda+1, \mu_2; \mu_1+\frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_2}\right)^2, \frac{x_2}{1-2x_1}\right]\Bigg\}, \tag{18}$$

provided $Re(\lambda) > 0$ and $Re(4x_1 + x_2) < 1$.

Proof. The derivations of the results (14) to (16) are quite simple. So we shall establish only one of the results, say (15). Rest can be proven on similar lines.

In order to derive the results (15), we proceed as follows. In the integral representation (5) of F_2, if we take $\nu_1 = 2\mu + 1$, $\nu_2 = 2\nu + 1$, and replace x_1, x_2 by $2x_1$ and $2x_2$, respectively, we have

$$F_2[\lambda, \mu, \nu; 2\mu+1, 2\nu+1; 2x_1, 2x_2]$$

$$= \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-t}t^{\lambda-1}\ {}_1F_1\left[\begin{matrix}\mu \\ \\ 2\mu+1\end{matrix}; 2x_1t\right]\ {}_1F_1\left[\begin{matrix}\nu \\ \\ 2\nu+1\end{matrix}; 2x_2t\right]dt,$$

which is valid for $Re(\lambda) > 0$ and $Re(x_1 + x_2) < 1$.

Using the result (3), it takes the following form:

$$F_2[\lambda, \mu, \nu; 2\mu+1, 2\nu+1; 2x_1, 2x_2] = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(1-x_1-x_2)t}t^{\lambda-1}$$

$$\times\Bigg\{ {}_0F_1\left[\begin{matrix}- \\ \\ \mu+\frac{1}{2}\end{matrix}; \tfrac{x_1^2t^2}{4}\right]\ {}_0F_1\left[\begin{matrix}- \\ \\ \nu+\frac{1}{2}\end{matrix}; \tfrac{x_2^2t^2}{4}\right]$$

$$-\frac{x_1t}{(2\mu+1)}\ {}_0F_1\left[\begin{matrix}- \\ \\ \mu+\frac{3}{2}\end{matrix}; \tfrac{x_1^2t^2}{4}\right]\ {}_0F_1\left[\begin{matrix}- \\ \\ \nu+\frac{1}{2}\end{matrix}; \tfrac{x_2^2t^2}{4}\right]$$

$$-\frac{x_2t}{(2\nu+1)}\ {}_0F_1\left[\begin{matrix}- \\ \\ \mu+\frac{1}{2}\end{matrix}; \tfrac{x_1^2t^2}{4}\right]\ {}_0F_1\left[\begin{matrix}- \\ \\ \nu+\frac{3}{2}\end{matrix}; \tfrac{x_2^2t^2}{4}\right]$$

$$+\frac{x_1x_2t^2}{(2\mu+1)(2\nu+1)}\ {}_0F_1\left[\begin{matrix}- \\ \\ \mu+\frac{3}{2}\end{matrix}; \tfrac{x_1^2t^2}{4}\right]\ {}_0F_1\left[\begin{matrix}- \\ \\ \nu+\frac{3}{2}\end{matrix}; \tfrac{x_2^2t^2}{4}\right]\Bigg\}\,dt.$$

Therefore

$$\begin{aligned} & F_2[\lambda,\mu,\nu;2\mu+1,2\nu+1;2x_1,2x_2] \\ & = A - \frac{x_1}{(2\mu+1)}B - \frac{x_2}{(2\nu+1)}C + \frac{x_1x_2}{(2\mu+1)(2\nu+1)}D. \end{aligned} \tag{19}$$

Now, we shall evaluate $A,\ B,\ C$ and D one by one.

$$A = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(1-x_1-x_2)t}t^{\lambda-1}\ {}_0F_1\left[\begin{matrix} - \\ \mu+\frac{1}{2}\end{matrix};\frac{x_1^2t^2}{4}\right]\ {}_0F_1\left[\begin{matrix} - \\ \nu+\frac{1}{2}\end{matrix};\frac{x_2^2t^2}{4}\right].$$

Expressing both ${}_0F_1$ functions as series, we have, after some simplification,

$$A = \sum_{m=0}^\infty\sum_{n=0}^\infty \frac{x_1^{2m}x_2^{2n}}{\left(\mu+\frac{1}{2}\right)_m\left(\nu+\frac{1}{2}\right)_n 2^{2m+2n}m!n!}\ \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(1-x_1-x_2)t}t^{\lambda+2m+2n-1}dt.$$

Evaluating the Gamma integral and using the results $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ and $(\lambda)_{2n} = 2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda}{2}+\frac{1}{2}\right)_n$, we have

$$A = (1-x_1-x_2)^{-\lambda}\ \sum_{m=0}^\infty\sum_{n=0}^\infty \frac{\left(\frac{\lambda}{2}\right)_{m+n}\left(\frac{\lambda}{2}+\frac{1}{2}\right)_{m+n}x_1^{2m}x_2^{2n}}{\left(\mu+\frac{1}{2}\right)_m\left(\nu+\frac{1}{2}\right)_n(1-x_1-x_2)^{2m+2n}m!n!}.$$

Finally, using the definition (9), we have

$$A = (1-x_1-x_2)^{-\lambda}\ F_4\left[\frac{1}{2}\lambda,\frac{1}{2}\lambda+\frac{1}{2};\mu+\frac{1}{2},\nu+\frac{1}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right].$$

In exactly the same manner, it is not difficult to see that

$$\begin{aligned} B = {} & \lambda\ (1-x_1-x_2)^{-\lambda-1} \\ & \times F_4\left[\frac{1}{2}\lambda+\frac{1}{2},\frac{1}{2}\lambda+1;\mu+\frac{3}{2},\nu+\frac{1}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right], \end{aligned}$$

$$\begin{aligned} C = {} & \lambda\ (1-x_1-x_2)^{-\lambda-1} \\ & \times F_4\left[\frac{1}{2}\lambda+\frac{1}{2},\frac{1}{2}\lambda+1;\mu+\frac{1}{2},\nu+\frac{3}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right], \end{aligned}$$

and

$$\begin{aligned} D = {} & \lambda(\lambda+1)\ (1-x_1-x_2)^{-\lambda-2} \\ & \times F_4\left[\frac{1}{2}\lambda+1,\frac{1}{2}\lambda+\frac{3}{2};\mu+\frac{3}{2},\nu+\frac{3}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]. \end{aligned}$$

Upon substituting the values of $A,\ B,\ C$ and D in (19), we easily arrive at the right hand side of (15). This completes the proof of the result (15). In exactly the same manner, the results (14) and (16) can be established.

Now, in order to derive the result (17), we proceed as follows. In the integral representation (5), if we set $\nu_1 = 2\mu_1 + 1$, $\nu_2 = \nu$ and replace x_1 by $4x_1$, we have

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1 + 1, \nu; 4x_1, x_2]$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t} t^{\lambda-1} {}_1F_1\left[\begin{matrix} \mu_1 \\ 2\mu_1 + 1 \end{matrix}; 4x_1 t\right] {}_1F_1\left[\begin{matrix} \mu_2 \\ \nu \end{matrix}; x_2 t\right] dt,$$

which is valid for $Re(\lambda) > 0$ and $Re(4x_1 + x_2) < 1$.

Now, in first ${}_1F_1$, if we use the result (3), we have, after some simplification,

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1 + 1, \nu; 4x_1, x_2] = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-2x_1)t} t^{\lambda-1}$$
$$\times \left\{ {}_0F_1\left[\begin{matrix} - \\ \mu_1 + \frac{1}{2} \end{matrix}; x_1^2 t^2\right] - \frac{2x_1 t}{(2\mu_1 + 1)} {}_0F_1\left[\begin{matrix} - \\ \mu_1 + \frac{3}{2} \end{matrix}; x_1^2 t^2\right] \right\}$$
$$\times \; {}_1F_1\left[\begin{matrix} \mu_2 \\ \nu \end{matrix}; x_2 t\right] dt.$$

Thus

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1 + 1, 2\nu; 4x_1, x_2] = A - \frac{2x_1}{(2\mu_1 + 1)} B, \tag{20}$$

where

$$A = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-2x_1)t} t^{\lambda-1} {}_0F_1\left[\begin{matrix} - \\ \mu_1 + \frac{1}{2} \end{matrix}; x_1^2 t^2\right] {}_1F_1\left[\begin{matrix} \mu_2 \\ \nu \end{matrix}; x_2 t\right] dt,$$

and

$$B = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-2x_1)t} t^{\lambda} {}_0F_1\left[\begin{matrix} - \\ \mu_1 + \frac{3}{2} \end{matrix}; x_1^2 t^2\right] {}_1F_1\left[\begin{matrix} \mu_2 \\ \nu \end{matrix}; x_2 t\right] dt.$$

Now, we shall evaluate A and B separately.

For evaluating A, expressing both functions ${}_1F_1$ and ${}_0F_1$ as series and after some algebra, we have

$$A = \frac{1}{\Gamma(\lambda)} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(\mu_2)_n \, x_1^{2m} x_2^n}{\left(\mu_1 + \frac{1}{2}\right)_m (\nu)_n \, m! n!} \int_0^\infty e^{-(1-2x_1)t} t^{\lambda+2m+n-1} dt.$$

Evaluating gamma integral and after some simplification, we have

$$A = (1 - 2x_1)^{-\lambda} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(\lambda)_{2m+n} (\mu_2)_n}{\left(\mu_1 + \frac{1}{2}\right)_m (\nu)_n \, m! n!} \left(\frac{x_1}{1 - 2x_1}\right)^{2m} \left(\frac{x_2}{1 - 2x_1}\right)^n.$$

Using the definition (9), we finally have

$$A = (1-2x_1)^{-\lambda}\ H_4\left[\lambda, \mu_2; \mu_1 + \frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right]. \tag{21}$$

In exactly the same manner, it is not difficult to show that

$$B = \lambda(1-2x_1)^{-\lambda-1}\ H_4\left[\lambda, \mu_2; \mu_1 + \frac{3}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \frac{x_2}{1-2x_1}\right]. \tag{22}$$

Upon substituting the values of A and B from (21) and (22), we easily arrive at the right hand side of (17). This completes the proof of the result (17).

In exactly the same manner, the result (18) can be established. So we prefer to omit the details. □

We conclude this section by remarking that the results (17) and (18) are the corrected forms of the results (7) and (8) obtained earlier by Mathur and Solanki [8].

3. Further observations

(i) In view of the relation

$$F^{2:0;0}_{0:1;1}\left[\begin{matrix} \frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ - : & \mu+\frac{1}{2}; & \nu+\frac{1}{2} \end{matrix}\right]$$

$$= F_4\left[\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2}; \mu+\frac{1}{2}, \nu+\frac{1}{2}; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2\right], \tag{23}$$

the result (14) can be written with the form

$$F_2[\lambda, \mu, \nu; 2\mu; 2\nu; 2x_1, 2x_2] = (1-x_1-x_2)^{-\lambda}$$
$$\times F^{2:0;0}_{0:1;1}\left[\begin{matrix} \frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ - : & \mu+\frac{1}{2}; & \nu+\frac{1}{2} \end{matrix}\right],$$

provided $Re(\lambda) > 0$ and $Re(x_1+x_2) < \frac{1}{2}$.

(ii) In view of the relation (23), the result (15) can be written in the form

$$F_2[\lambda, \mu, \nu; 2\mu+1; 2\nu+1; 2x_1, 2x_2] = (1-x_1-x_2)^{-\lambda}$$
$$\times \Bigg\{ F^{2:0;0}_{0:1;1}\left[\begin{matrix} \frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ - : & \mu+\frac{1}{2}; & \nu+\frac{1}{2} \end{matrix}\right]$$
$$- \frac{\lambda x_1}{(2\mu+1)(1-x_1-x_2)}$$

$$\times F^{2:0;0}_{0:1;1}\left[\begin{matrix}\frac{1}{2}\lambda+\frac{1}{2},\frac{1}{2}\lambda+1: & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2,\ \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ -: & \mu+\frac{3}{2}; & \nu+\frac{1}{2}\end{matrix}\right]$$

$$-\frac{\lambda x_1}{(2\nu+1)(1-x_1-x_2)}$$

$$\times F^{2:0;0}_{0:1;1}\left[\begin{matrix}\frac{1}{2}\lambda+\frac{1}{2},\frac{1}{2}\lambda+1: & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2,\ \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ -: & \mu+\frac{1}{2}; & \nu+\frac{3}{2}\end{matrix}\right]$$

$$+\frac{\lambda(\lambda+1)x_1x_2}{(2\mu+1)(2\nu+1)(1-x_1-x_2)^2}$$

$$\left.\times F^{2:0;0}_{0:1;1}\left[\begin{matrix}\frac{1}{2}\lambda+1,\frac{1}{2}\lambda+\frac{3}{2}: & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2,\ \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ -: & \mu+\frac{3}{2}; & \nu+\frac{3}{2}\end{matrix}\right]\right\},$$

provided $Re(\lambda)>0$ and $Re(x_1+x_2)<\frac{1}{2}$, which is the corrected form of the result (11) obtained by Mathur and Solanki [8].

(iii) In view of the relation (23), the result (16) can be written in the form

$$F_2[\lambda,\mu,\nu;2\mu-1,2\nu-1;2x_1,2x_2]=(1-x_1-x_2)^{-\lambda}$$

$$\times\left\{F^{2:0;0}_{0:1;1}\left[\begin{matrix}\frac{1}{2}\lambda,\frac{1}{2}\lambda+\frac{1}{2}: & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2,\ \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ -: & \mu-\frac{1}{2}; & \nu-\frac{1}{2}\end{matrix}\right]\right.$$

$$+\frac{\lambda x_1}{(2\mu-1)(1-x_1-x_2)}$$

$$\times F^{2:0;0}_{0:1;1}\left[\begin{matrix}\frac{1}{2}\lambda+\frac{1}{2},\frac{1}{2}\lambda+1: & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2,\ \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ -: & \mu+\frac{1}{2}; & \nu-\frac{1}{2}\end{matrix}\right]$$

$$+\frac{\lambda x_1}{(2\mu-1)(1-x_1-x_2)}$$

$$\times F^{2:0;0}_{0:1;1}\left[\begin{matrix}\frac{1}{2}\lambda+\frac{1}{2},\frac{1}{2}\lambda+1: & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2,\ \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ -: & \mu-\frac{1}{2}; & \nu+\frac{1}{2}\end{matrix}\right]$$

$$+\frac{\lambda(\lambda+1)x_1x_2}{(2\mu-1)(2\nu-1)(1-x_1-x_2)^2}$$

$$\left.\times F^{2:0;0}_{0:1;1}\left[\begin{matrix}\frac{1}{2}\lambda+1,\frac{1}{2}\lambda+\frac{3}{2}: & -; & - \\ & & & ;\left(\frac{x_1}{1-x_1-x_2}\right)^2,\ \left(\frac{x_2}{1-x_1-x_2}\right)^2 \\ -: & \mu+\frac{1}{2}; & \nu+\frac{1}{2}\end{matrix}\right]\right\},$$

provided $Re(\lambda)>0$ and $Re(x_1+x_2)<\frac{1}{2}$, which is the corrected form of the result (12) obtained by Mathur and Solanki [8].

Moreover, in the results (6), (17) and (18), if we replace x_1 by $\frac{1}{2}x_1$, we get the results in the following more compact forms:

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1, \nu; 2x_1, x_2]$$
$$= (1-x_1)^{-\lambda}\ H_4\left[\lambda, \mu_2; \mu_1+\frac{1}{2}, \nu; \frac{1}{4}\left(\frac{x_1}{1-x_1}\right)^2, \frac{x_2}{1-x_1}\right],$$

provided $Re(\lambda) > 0$ and $Re(2x_1 + x_2) < 1$,

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1+1, \nu; 2x_1, x_2]$$
$$= (1-x_1)^{-\lambda}\left\{ H_4\left[\lambda, \mu_2; \mu_1+\frac{1}{2}, \nu; \frac{1}{4}\left(\frac{x_1}{1-x_1}\right)^2, \frac{x_2}{1-x_1}\right]\right.$$
$$\left. -\frac{\lambda x_1}{(2\mu_1+1)(1-x_1)}\ H_4\left[\lambda+1, \mu_2; \mu_1+\frac{3}{2}, \nu; \frac{1}{4}\left(\frac{x_1}{1-x_1}\right)^2, \frac{x_2}{1-2x_1}\right]\right\},$$

provided $Re(\lambda) > 0$ and $Re(2x_1 + x_2) < 1$, and

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1-1, \nu; 2x_1, x_2]$$
$$= (1-x_2)^{-\lambda}\left\{ H_4\left[\lambda, \mu_2; \mu_1-\frac{1}{2}, \nu; \frac{1}{4}\left(\frac{x_1}{1-x_1}\right)^2, \frac{x_2}{1-x_1}\right]\right.$$
$$\left. +\frac{\lambda x_1}{(2\mu_1-1)(1-x_1)}\ H_4\left[\lambda+1, \mu_2; \mu_1+\frac{1}{2}, \nu; \frac{1}{4}\left(\frac{x_1}{1-x_1}\right)^2, \frac{x_2}{1-2x_1}\right]\right\},$$

provided $Re(\lambda) > 0$ and $Re(2x_1 + x_2) < 1$.

Concluding remark

In this note, we have established a few corrected forms of the results due to Mathur and Solanki on Appell, Horn and Kampé de Fériet functions. Further results in the most general forms are under investigations and will form a part of the subsequent paper in this direction.

Acknowledgment. The authors are grateful to the learned referee for making certain useful suggestions which led to better presentation of the paper.

References

[1] P. Appell, *Sur les fonctions hypérgéometriques de plusieurs variables*, Mémoir Sci. Math., Gauthier-Villars, Paris, 1925.

[2] W. N. Bailey, *On the sum of a terminating* ${}_3F_2(1)$, Quart. J. Math. Oxford Ser. (2) **4** (1953), 237–240. https://doi.org/10.1093/qmath/4.1.237

[3] A. Erdélyi, *Transformations of hypergeometric functions of two variables*, Proc. Roy. Soc. Edinburgh Sect. A **62** (1948), 378–385.

[4] J. Horn, *Hypergeometrische Funktionen zweier Veränderlichen*, Math. Ann. **105** (1931), no. 1, 381–407. https://doi.org/10.1007/BF01455825

[5] J. Kampé de Fériet, *Les fonctions hypergéométriques d'ordre supérieura deux variables*, CR Acad. Sci. Paris **173** (1921), 401–404.

[6] Y. S. Kim, M. A. Rakha, and A. K. Rathie, *Generalization of Kummer's second theorem with applications*, Comput. Math. Math. Phys. **50** (2010), no. 3, 387–402; translated from Zh. Vychisl. Mat. Mat. Fiz. **50** (2010), no. 3, 407–422. https://doi.org/10.1134/S0965542510030024

[7] R. Mathur and N. S. Solanki, *Transformation formulas of Appell series F_2 with contiguous extensions*, J. Gujrat Research Society **21** (2019), no. 1, 336–340.

[8] R. Mathur and N. S. Solanki, *On the transformation formulas of Appell hypergeometric function F_2*, J. Gujrat Research Society **21** (2019), no. 16, 2590–2596.

[9] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

[10] A. K. Rathie and V. Nagar, *On Kummer's second theorem involving product of generalized hypergeometric series*, Matematiche (Catania) **50** (1995), no. 1, 35–38.

[11] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric series*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1985.

Asmaa Orabi Mohammed
Department of Mathematics
Faculty of Science
Suez Canal University
El-Sheik Zayed 41522, Ismailia - Egypt
Email address: asmaa.orabi@science.suez.edu.eg

Medhat Ahmed Rakha
Department of Mathematics
Faculty of Science
Suez Canal University
El-Sheik Zayed 41522, Ismailia - Egypt
Email address: medhat_rakha@science.suez.edu.eg

Arjun K. Rathie
Department of Mathematics
Vedant College of Engineering & Technology
Rajasthan Technical University
Village: Tulsi, Post: Jakhmund, District: BUNDI-323021
Rajasthan State - India
Email address: arjunkumarrathie@gmail.com