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ON NONNIL-SFT RINGS

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Abstract. The purpose of this paper is to introduce a new class of rings containing the class of SFT-rings and contained in the class of rings with Noetherian prime spectrum. Let A be a commutative ring with unit and I be an ideal of A. We say that I is SFT if there exist an integer $k \geq 1$ and a finitely generated ideal $F \subseteq I$ of A such that $x^k \in F$ for every $x \in I$. The ring A is said to be nonnil-SFT, if each nonnil-ideal (i.e., not contained in the nilradical of A) is SFT. We investigate the nonnil-SFT variant of some well known theorems on SFT-rings. Also we study the transfer of this property to Nagata's idealization and the amalgamation algebra along an ideal. Many examples are given. In fact, using the amalgamation construction, we give an infinite family of nonnil-SFT rings which are not SFT.

Introduction

In this paper, all rings are commutative with identity and the dimension of a ring means its Krull dimension. Let A be a ring. We shall denote by $\mathrm{Nil}(A)$ the nilradical of A and $I \subset J$ means I is strictly contained in J for some sets I, J. An ideal I of A is said to be SFT, if there exist an integer $k \geq 1$ and a finitely generated ideal $F \subseteq I$ such that $x^k \in F$ for every $x \in I$. The ring A is called SFT if each ideal of A is SFT. In [1], Arnold showed that if A is not an SFT-ring, then $\dim(A[[X]]) = \infty$. His result motivates us to study the Krull dimension of the power series ring over an SFT-ring. In particular, it seems natural to ask if the Krull dimension of $(A/\mathrm{Nil}(A))[[X]]$ is infinite whenever A is not SFT. We give a negative answer to this question. In fact, we give a class of non-SFT rings such that $\dim((A/\mathrm{Nil}(A))[[X]])$ is finite.

In [3], Badawi defined a ring to be nonnil-Noetherian if each nonnil-ideal is finitely generated. It is obvious that Noetherian rings are both SFT-rings and nonnil-Noetherian rings but the converse is not true. Now, it is natural to investigate the relation between nonnil-Noetherian rings and SFT-rings, and it turns out that the two concepts are independent from each other. For instance, let $A = (K[X_i,\ i \geq 1])/\langle X_i^i,\ i \geq 1\rangle$, where K is a field and $X_1, X_2, \ldots$ is a

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countably family of indeterminates. Then A is nonnil-Noetherian but not SFT. Indeed, the ideal of A generated by $\{\bar{X}_i,\ i \geq 1\}$ is not SFT. Furthermore, each non-Noetherian SFT-domain (even reduced ring) is not nonnil-Noetherian.

The main purpose of this paper is to integrate the concepts of nonnil-Noetherian rings and the SFT-rings and to construct a new class of rings that contains the both classes of nonnil-Noetherian rings and SFT-rings. For this, we introduce a new class of rings called nonnil-SFT rings as follows: Let A be a ring and I an ideal of A. We say that I is a nonnil-ideal if $I \nsubseteq \mathrm{Nil}(A)$. The ideal I is said to be SFT, if there exist an integer $k \geq 1$ and a finitely generated ideal $F \subseteq I$ of A such that $x^k \in F$ for every $x \in I$. The ring A is called nonnil-SFT if each nonnil-ideal of A is an SFT-ideal. If $\mathrm{Nil}(A) = 0$, then the notion of nonnil-SFT rings coincides with that of SFT-rings. Clearly any nonnil-Noetherian ring is a nonnil-SFT ring.

This paper consists of two sections (besides the introduction). In Section 1, we investigate some basic properties of nonnil-SFT rings. In fact, we give a relation between nonnil-SFT rings and SFT-rings and study the Cohen-type theorem for nonnil-SFT rings. We also show that a nonnil-SFT ring has a Noetherian prime spectrum (i.e., satisfies the ascending chain condition for the radical ideals). We show the stability of this concept via Nagata's idealization and the flat overring of a nonnil-SFT ring. It is well known that a valuation domain is SFT if and only if each prime ideal is not idempotent. We show an analogue result for nonnil-SFT chained rings. Also, we prove that if A is a one dimensional chained ring, then A is nonnil-SFT if and only if it is nonnil-Noetherian.

Section 2 of this paper is devoted to study the stability of the nonnil-SFT property via some well known extensions (polynomial ring, power series ring and the amalgamation construction). Among our results, we show that the ring $A \bowtie^f J$ is nonnil-SFT if and only if the rings A and $f(A) + J$ are nonnil-SFT, under the assumption $A \bowtie^f J \in \mathcal{H}$, where $f : A \longrightarrow B$ is a rings homomorphism, J is a nonzero ideal of B and $\mathcal{H} = \{A$ a ring such that $\mathrm{Nil}(A)$ is a divided prime ideal of $A\}$. We also give a class of non-SFT rings such that $\dim((A/\mathrm{Nil}(A))[[X]])$ is finite. In fact, we show that if A is a finite dimensional ring such that A_M is a chained ring for every $M \in Max(A)$ and if it is nonnil-SFT which is not SFT, then $\dim((A/\mathrm{Nil}(A))[[X]]$ is finite. Recall that a ring A is called decomposable if it can be written in the form $A = A_1 \oplus A_2$ for some nonzero rings A_1 and A_2. We give a necessary and sufficient condition to a decomposable ring to be nonnil-SFT.

1. Nonnil-SFT rings

Definition 1.1. A ring A is called nonnil-SFT if each nonnil-ideal I ($I \nsubseteq \mathrm{Nil}(A)$) of A is SFT, i.e., there exist an integer $k \geq 1$ and a finitely generated ideal $F \subseteq I$ of A such that $x^k \in F$ for every $x \in I$.

Example 1.1. An SFT-ring is nonnil SFT.

The converse of Example 1.1 is false. We will give a counterexample later.

Lemma 1.2. *Let A be a ring. A maximal ideal among the nonnil-ideals of A which are not SFT is a prime ideal.*

Proof. Let $\mathcal{F}$ be the set of the nonnil-ideals which are not SFT and P be a maximal element of $\mathcal{F}$. It is clear that P is a nonnil-ideal. Assume that P is not prime. Let $a, b \in A\backslash P$ such that $ab \in P$. We have $P \subset P+aA$ and $P \subset P+bA$. It is easy to see that $P+aA$ and $P+bA$ are nonnil-ideals. By maximality of P, $P+aA$ and $P+bA$ are SFT. There exist $x_1, \ldots, x_s, y_1, \ldots, y_r \in P$ and $n, k \in \mathbb{N}^*$ such that for every $x \in P+aA$ and $y \in P+bA$, $x^n \in \langle x_1, \ldots, x_s, a\rangle$ and $y^k \in \langle y_1, \ldots, y_r, b\rangle$. Let $\alpha \in P$. Since $P \subset P+aA$ and $P \subset P+bA$, $\alpha^n = \sum_{i=1}^{s}\alpha_i x_i + a\gamma_1$ and $\alpha^k = \sum_{j=1}^{r}\beta_j y_j + b\gamma_2$, where $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_r, \gamma_1, \gamma_2 \in A$. Thus

$$\begin{aligned}\alpha^{nk} = \alpha^n\alpha^k &= \left(\sum_{i=1}^{s}\alpha_i x_i + a\gamma_1\right)\left(\sum_{j=1}^{r}\beta_j y_j + b\gamma_2\right)\\ &= \sum_{i=1}^{s}\left(\alpha_i\sum_{j=1}^{r}\beta_i y_i + \alpha_i b\gamma_2\right)x_i + \sum_{j=1}^{r}(a\gamma_1\beta_j)y_j + (\gamma_1\gamma_2)(ab)\\ &\in \langle x_1, \ldots, x_s, y_1, \ldots, y_r, ab\rangle \subseteq P\end{aligned}$$

absurd because P is not SFT. Hence P is a nonnil prime ideal. $\square$

Proposition 1.3. *A ring A is nonnil-SFT if and only if each nonnil prime ideal is SFT.*

Proof. ($\Leftarrow$) Assume that A is not nonnil-SFT. Let $\mathcal{F}$ be the set of all nonnil ideals of A which are not SFT. Thus $\mathcal{F} \neq \emptyset$. Let $(I_\lambda)_{\lambda\in\Lambda}$ be a totally ordered family of $(\mathcal{F}, \subseteq)$ and set $I = \bigcup_{\lambda\in\Lambda} I_\lambda$. It is clear that $I \nsubseteq \mathrm{Nil}(A)$. Assume that I is SFT. Then there exist a finitely generated ideal $F \subseteq I$ of A and an integer $k \geq 1$ such that $x^k \in F$ for every $x \in I$. Since $(I_\lambda)_{\lambda\in\Lambda}$ is totally ordered, there exists $\lambda \in \Lambda$ such that $F \subseteq I_\lambda$. It yields that for every $x \in I_\lambda \subseteq I$, $x^k \in F \subseteq I_\lambda$. It follows that I_λ is an SFT-ideal, which contradicts the fact that $I_\lambda \in \mathcal{F}$. Therefore, $I \in \mathcal{F}$. Hence $(\mathcal{F}, \subseteq)$ is inductive. By Zorn's lemma, $(\mathcal{F}, \subseteq)$ has a maximal element P. By Lemma 1.2, P is a nonnil prime ideal of A absurd. Thus A is nonnil-SFT. $\square$

The following example shows the difference between the concepts of SFT and nonnil-SFT.

Example 1.2. Let K be a field, $X = \{X_1, X_2, \ldots\}$ a countably set of indeterminates over K, $I = \langle X_n^n,\ n \geq 1\rangle K[X]$ and $A = K[X]/I$. The ring A is not SFT because the ideal $\langle X_1, X_2, \ldots\rangle/I$ is not SFT. On the other hand, the only prime ideal of A is $\mathrm{Nil}(A)$, then A is nonnil-SFT.

Recall that a ring A is called nonnil-Noetherian if each nonnil-ideal is finitely generated. This class of rings was introduced by Badawi in [3], and he has studied some basic properties of this rings. After, in [6] Hizem and Benhissi had generalized some results of Badawi, and they had shown that this class of rings have a principal role for the stability of the SFT property via the power series ring. It is easy to see that a nonnil-Noetherian ring is nonnil-SFT. Following [3], set $\mathcal{H} = \{A$ a ring such that $\mathrm{Nil}(A)$ is a divided prime ideal of $A\}$. In the next example, we show that the class $\mathcal{H}$ is essential in the following proposition.

Proposition 1.4. *Let $A \in \mathcal{H}$. The following statements are equivalent:*

(1) *The ring A is nonnil-SFT.*
(2) *The ring $A/\mathrm{Nil}(A)$ is SFT.*

Proof. (1) $\Rightarrow$ (2) Let Q be a prime ideal of $A/\mathrm{Nil}(A)$. There exists a prime ideal P of A such that $Q = P/\mathrm{Nil}(A)$. If $P = \mathrm{Nil}(A)$, then $Q = \bar{0}$ is SFT. Else P is an SFT-ideal. It follows that Q is SFT.

(2) $\Rightarrow$ (1) By Proposition 1.3, it suffices to show that each nonnil prime ideal is SFT. Let P be a nonnil prime ideal of A. Then $P/\mathrm{Nil}(A)$ is an SFT-ideal of $A/\mathrm{Nil}(A)$. Thus there exist $k \in \mathbb{N}^*$ and $x_1, \ldots, x_n \in P$ such that $\bar{x}^k \in \langle \bar{x_1}, \ldots, \bar{x_n} \rangle$. As $\mathrm{Nil}(A)/\mathrm{Nil}(A) = \bar{0}$, we can assume that for every i between 1 and n, $\bar{x}_i \neq \bar{0}$ (i.e., $x_i \notin \mathrm{Nil}(A)$). Therefore, $\mathrm{Nil}(A) \subset \langle x_i \rangle$ for each $1 \leq i \leq n$. It yields that $x^k \in \langle x_1, \ldots, x_n \rangle + \mathrm{Nil}(A) = \langle x_1, \ldots, x_n \rangle$. Consequently, P is an SFT-ideal. $\square$

Example 1.3. The hypothesis "$\mathrm{Nil}(A)$ is divided" can not be removed. Indeed, let B be an SFT Prüfer domain, $X, Y_1, Y_2, Y_3, \ldots$ a countably set of indeterminates over B, $I = \langle Y_i^i,\ i \geq 1 \rangle B[Y_i,\ i \geq 1]$ and $A = (B[Y_i,\ i \geq 1]/I)[X]$. We have $\mathrm{Nil}(A) = M[X]$, where $M = \langle \bar{Y}_i,\ i \geq 1 \rangle (B[Y_i,\ i \geq 1]/I)$. Since $A/\mathrm{Nil}(A) = A/M[X] \simeq ((B[Y_i,\ i \geq 1]/I)/M)[X] \simeq B[X]$ an SFT domain by [8, Proposition 10]. It shows that $\mathrm{Nil}(A)$ is a prime ideal. The ring A is not nonnil-SFT because the ideal $J = \langle X, \bar{Y}_i,\ i \geq 1 \rangle$ of A is not SFT. In fact, we have $X \notin \mathrm{Nil}(A)$, thus $J \nsubseteq \mathrm{Nil}(A)$. Assume that J is SFT. Thus there exist $k \geq 1$ and $n \geq 1$ such that $f^k \in F = \langle X, \bar{Y}_1, \ldots, \bar{Y}_n \rangle A$ for each $f \in J$. Let $m > k + n$ be an integer. It follows that $\bar{Y}_m^k \in F$. Hence $Y_m^k \in \langle Y_1, \ldots, Y_n \rangle + I$ in $B[Y_i,\ i \geq 1]$ impossible. Thus A is not nonnil-SFT.

Proposition 1.5. *The homomorphic image of a nonnil-SFT ring is nonnil-SFT.*

Proof. Let $\phi : A \longrightarrow B$ be a surjective homomorphism of rings. Assume that the ring A is nonnil-SFT. Let Q be a nonnil-ideal of B. Then $P = \phi^{-1}(Q)$ is a nonnil-ideal of A. Indeed, there exists $x \in Q \setminus \mathrm{Nil}(B)$. Since ϕ is surjective, there exists $y \in \phi^{-1}(\{x\}) \subseteq P$. If there exists $n \geq 1$ such that $y^n = 0$, then $x^n = \phi(y^n) = 0$ absurd. Thus P is a nonnil-ideal. By hypothesis, P is SFT. Consequently, there exist $n \geq 1$ and a finitely generated ideal $F \subseteq P$ of A such that $x^n \in F$ for each $x \in P$. Now let $y \in Q$. There exists $x \in P$ such that

$y = \phi(x)$. It yields that $y^n = \phi(x)^n = \phi(x^n) = \phi(F) \subseteq Q$ with $\phi(F)$ a finitely generated ideal of B. Hence Q is an SFT-ideal. It follows that the ring B is nonnil-SFT. $\square$

Example 1.4. Let A be a nonnil-SFT ring and I an ideal of A. Then the ring A/I is nonnil-SFT.

Let M be an A-module. Recall that Nagata introduced the ring extension of A, called the idealization of M in A, denoted by $A(+)M$, as the A-module $A \times M$ endowed with a multiplicative structure defined by:

$$(a, x)(a', x') = (aa', ax' + a'x) \text{ for all } a, a' \in A \text{ and } x, x' \in M.$$

For more results see [7, page 2].

Proposition 1.6. *Let A be a ring and M an A-module. Then the ring $A(+)M$ is nonnil-SFT if and only if the ring A is nonnil-SFT.*

Proof. ($\Rightarrow$) Let $\phi : A(+)M \longrightarrow A$ be the canonical projection. It is well known that ϕ is a surjective homomorphism. By Proposition 1.5, the ring A is nonnil-SFT.

($\Leftarrow$) Since every prime ideal of $A(+)M$ has the form $P(+)M$, where $P \in \operatorname{spec}(A)$, it is easy to see that $\operatorname{Nil}(A(+)M) = \operatorname{Nil}(A)(+)M$. Let

$$Q \in \operatorname{spec}(A(+)M)$$

be a nonnil-ideal. Then there exists $P \in \operatorname{spec}(A)$ such that $Q = P(+)M$. It yields that there exist $k \geq 1$ and a finitely generated ideal $F \subseteq P$ of A such that $a^k \in F$ for each $a \in P$. Let $(a, x) \in Q$. Thus $(a, x)^{k+1} = (a^{k+1}, (k + 1)a^k x) \in F(+)FM = \langle F \times \{0\}\rangle(A(+)M) \subseteq Q$. Hence Q is an SFT-ideal. By Proposition 1.3, $A(+)M$ is nonnil-SFT. $\square$

Theorem 1.7. *Let A be a nonnil-SFT ring. Then each flat overring of A is nonnil-SFT.*

Proof. Let B be a flat overring of A. Then there exists a multiplicative system of ideals S of A such that

$$B = \{x \in T \,|\, \text{there exists } I \in S \text{ such that } xI \subseteq A\},$$

where T is the total quotient ring of A. Moreover, we can choose S such that $IB = B$ for each $I \in S$ and for every $Q \in \operatorname{spec}(B)$, we have $Q = P_S$, where $P = Q \cap A$ [2, Theorem 1.3]. Let $Q \in \mathit{spec}(B)$ be a nonnil-ideal. Thus $P = Q \cap A \nsubseteq \operatorname{Nil}(A)$. Indeed, if $P \subseteq \operatorname{Nil}(A)$, then for each $x \in Q$ there exists $I \in S$ such that $xI \subseteq P$. It yields that $xIB \subseteq PB$. It follows that $x \in PB \subseteq \operatorname{Nil}(B)$ which shows that $Q \subseteq \operatorname{Nil}(B)$ absurd. As A is nonnil-SFT, there exist $k \geq 1$ and a finitely generated ideal $F \subseteq P$ of A such that $x^k \in F$ for every $x \in P$. Let $x \in Q$. There exists $I \in S$ such that $xI \subseteq P$. Then for every $a \in I$, $(xa)^k \in F \subseteq FB$. Since $IB = B$, there exist $b_1, \dots, b_r \in B$ and $c_1, \dots, c_r \in I$ such that $1 = c_1 b_1 + \cdots + c_r b_r$. Hence $1 = (c_1 b_1 + \cdots + c_r b_r)^{kr} = \sum_{\text{finite}} \alpha_i \beta_i^k$ with $\alpha_i \in B$ and $\beta_i \in I$ for each i.

Consequently, $x^k = x^k \cdot 1 = \sum_{\text{finite}} \alpha_i (x\beta_i)^k \in FB \subseteq Q$ with FB a finitely generated ideal of B. It yields that Q is an SFT-ideal. By Proposition 1.3, the ring B is nonnil-SFT. □

Remark 1.8. We note that a ring A is SFT if and only if it is nonnil-SFT and $\mathrm{Nil}(A)$ is an SFT-ideal. Indeed, let P be a prime ideal of A. If $P \nsubseteq \mathrm{Nil}(A)$ it is SFT. If $P \subseteq \mathrm{Nil}(A)$, then $P = \mathrm{Nil}(A)$ and hence it is SFT. Thus A is an SFT-ring. The other implication is clear.

Proposition 1.9. *Let A and B be two rings. Then the product ring $A \times B$ is nonnil-SFT if and only if the ring $A \times B$ is SFT if and only if the rings A and B are SFT.*

Proof. Since the ideals of $A \times B$ are of the form $I \times J$ with I (resp. J) an ideal of A (resp. B), it is easy to check that $A \times B$ is SFT if and only if A and B are SFT. Therefore, it suffices to show that if $A \times B$ is nonnil-SFT, the rings A and B are SFT. Without loosing of generality, it suffices to show that the ring A is SFT. Let I be an ideal of A. It is simple to check that $I \times B$ is a nonnil-ideal of $A \times B$, then it is SFT. It follows that there exist $k \geq 1$ and a finitely generated ideal $F \times Q \subseteq I \times B$ such that $(a,b)^k \in F \times Q$ for every $(a,b) \in I \times B$. It yields that $a^k \in F$ for each $a \in I$. Since $F \times Q$ is finitely generated in $A \times B$, F is a finitely generated ideal of A, which finishes the proof. □

Recall that a ring A is called with Noetherain spectrum (or $\mathrm{spec}(A)$ is Noetherian), if each prime ideal is the radical of a finitely generated ideal, equivalently each radical ideal is the radical of a finitely generated ideal, that is also equivalent to the ring A satisfies the ascending chain condition on the radical ideals. For more results the reader can be referred to [10].

Proposition 1.10. *Let A be a ring. If A is nonnil-SFT, then* $\mathrm{spec}(A)$ *is Noetherian. Consequently, each ideal of A has a finitely many minimal primes.*

Proof. It suffices to show that each prime ideal is the radical of a finitely generated ideal. Let P be a prime ideal of A. If $P \nsubseteq \mathrm{Nil}(A)$, then there exist $k \geq 1$ and a finitely generated ideal $F \subseteq P$ such that $x^k \in F$ for each $x \in P$. Thus P is the radical of F which is finitely generated. If $P \subseteq \mathrm{Nil}(A)$, then $P = \mathrm{Nil}(A) = \sqrt{\langle 0 \rangle}$. □

Remark 1.11. The converse of Proposition 1.10 is false. Indeed, let V be a finite dimensional non-SFT valuation domain. Since V has only finite number of prime ideals, it has a Noetherian spectrum. But V is not SFT, hence it is not nonnil-SFT because it is an integral domain.

Example 1.5. Let K be a field and $A = K[[X^{\frac{1}{\infty}}]] = \bigcup_{n=1}^{\infty} K[[X^{\frac{1}{n}}]]$. Since $K[[X]] \subset A$ is an integral extension, then $\dim(A) = \dim(K[[X]]) = 1$. The ring A is a quasi-local domain with a maximal ideal $M = \langle X^{\frac{1}{n}},\ n \geq 1 \rangle = \sqrt{\langle X \rangle}$.

Assume that M is an SFT ideal of A. Then there exist $k \geq 1$ and $n \geq 1$ such that $f^k \in \langle X, X^{\frac{1}{2}}, \ldots, X^{\frac{1}{n}} \rangle$ for every $f \in M$. It yields that

$$X^{\frac{1}{n+1}} = (X^{\frac{1}{k(n+1)}})^k \in \langle X, X^{\frac{1}{2}}, \ldots, X^{\frac{1}{n}} \rangle.$$

It follows that $X^{\frac{1}{n+1}} = \sum_{i=1}^{n} f_i X^{\frac{1}{i}}$, where $f_1, \ldots, f_n \in A$. Consequently,

$$\frac{1}{n+1} = v(X^{\frac{1}{n+1}}) \geq \min\{v(f_i) + v(X^{\frac{1}{i}}),\ 1 \leq i \leq n\} \geq \frac{1}{n},$$

where v is the natural valuation of A absurd. Hence A is not SFT. Thus A is not nonnil-SFT since it is integral. But A has Notherian spectrum because it has only two prime ideals.

Lemma 1.12. *Let A be a ring, I and J be two ideals of A such that $J \subseteq I$. If the ideals J and I/J are SFT in A and A/J, respectively, then the ideal I of A is also SFT.*

Proof. By hypothesis, there exist $k \geq 1$ and $x_1, \ldots, x_n \in I$ such that $\bar{x}^k \in \langle \bar{x}_1, \ldots, \bar{x}_n \rangle (A/J)$ for each $x \in I$. On the other hand, the ideal J is SFT. Then there exist $r \geq 1$ and a finitely generated ideal $F \subseteq J$ of A such that $y^r \in F$ for each $y \in J$. Let $x \in I$. There exist $\alpha_1, \ldots, \alpha_n \in A$ such that $x^k - \sum_{i=1}^{n} \alpha_i x_i \in J$. Thus $(x^k - \sum_{i=1}^{n} \alpha_i x_i)^r \in F$. Which implies that $x^{kr} \in F + \langle x_1, \ldots, x_n \rangle A$. Hence I is an SFT ideal of A. □

We recall that a ring A is called locally finite dimensional (LFD-ring) if each prime ideal P of A has a finite height. That is equivalent to that for every $P \in \operatorname{spec}(A)$, $\dim(A_P)$ is finite.

Theorem 1.13. *Let A be a ring.*

(1) *If* $\operatorname{Nil}(A) \notin \operatorname{spec}(A)$, *the following conditions are equivalent*:
   (i) *The ring A is nonnil-SFT.*
   (ii) *The ring A is SFT.*
   (iii) *The ring $A/\operatorname{Nil}(A)$ is SFT and each minimal prime ideal of A is SFT.*

(2) *If* $\operatorname{Nil}(A) \in \operatorname{spec}(A)$ *and A is LFD-ring, then A is nonnil-SFT if and only if the ring $A/\operatorname{Nil}(A)$ is SFT and each height one prime ideal of A is SFT.*

Proof. (1) (i)⇒(ii) Each prime ideal of A is a nonnil-ideal. Thus it is SFT. Hence the ring A is SFT.

(ii)⇒(iii) It is clear.

(iii)⇒(i) Let P be a nonnil-prime ideal of A and $P_0 \in \operatorname{Min}(A)$ such that $P_0 \subseteq P$. Since $Nil(A) \subset P_0$, we have the following isomorphism

$$(A/\operatorname{Nil}(A))/(P_0/\operatorname{Nil}(A)) \simeq A/P_0.$$

Since $A/\operatorname{Nil}(A)$ is an SFT ring, so is the ring A/P_0. It yields that the ideal P/P_0 is SFT in the ring A/P_0. Since P_0 is minimal, it is an SFT ideal of A. By Lemma 1.12, the ideal P is SFT. By Proposition 1.3, the ring A is nonnil-SFT.

(2) ($\Rightarrow$) Since $\mathrm{Nil}(A) \in \mathrm{spec}(A)$, $\mathrm{Min}(A) = \{\mathrm{Nil}(A)\}$. If $P \in \mathrm{spec}(A)$ is of height 1, then $\mathrm{Nil}(A) \subset P$. Thus P is SFT.

($\Leftarrow$) Let P be a nonnil prime ideal. Since $ht(P)$ is finite, there exists a height one prime ideal Q of A such that $\mathrm{Nil}(A) \subset Q \subseteq P$. If $P = Q$, then P is an SFT ideal. If $P \neq Q$, we have $(A/\mathrm{Nil}(A))/(Q/\mathrm{Nil}(A)) \simeq A/Q$ an SFT domain. Hence P/Q is an SFT ideal of A/Q. As Q has a height one, it is SFT by hypothesis. By Lemma 1.12, the ideal P is SFT. By Proposition 1.3, the ring A is nonnil-SFT. □

Corollary 1.14. *Let A be a chained ring with a maximal ideal M and has dimension ≤ 1.*

(1) *If $\mathrm{Nil}(A) = M$, then A is both nonnil-Noetherian and nonnil-SFT.*
(2) *If $\mathrm{Nil}(A) \neq M$, the following statements are equivalent*:
   (i) *The ring A is nonnil-SFT.*
   (ii) *The ring A is nonnil-Noetherian.*
   (iii) *M is a principal ideal.*

Proof. Since the ring A is chained, $\mathrm{Nil}(A) \in \mathrm{spec}(A)$.

(1) It is clear that $A \in \mathcal{H}$. Since $A/\mathrm{Nil}(A)$ is a field, by Proposition 1.4, the ring A is nonnil-SFT and by [3, Theorem 2.2], the ring A is nonnil-Noetherian.

(2) (i)$\Rightarrow$(ii) We have $A/\mathrm{Nil}(A)$ is a one dimensional SFT valuation domain. Then it is Noetherian. As $A \in \mathcal{H}$ by [3, Theorem 2.2], the ring A is nonnil-Noetherian.

(ii)$\Rightarrow$(i) It is easy.

(ii)$\Leftrightarrow$(iii) See [4, Corollary 2.5]. □

It is well known that a valuation domain A is SFT if and only if for each nonzero prime ideal P of A, $P \neq P^2$. Analogously, we show the following result for nonnil-SFT chained rings.

Theorem 1.15. *Let A be a chained ring. Then A is nonnil-SFT if and only if for each nonnil prime ideal P of A, $P \neq P^2$.*

Proof. It is clear that $A \in \mathcal{H}$.

($\Rightarrow$) As $A/\mathrm{Nil}(A)$ is an SFT valuation domain. Then for each nonzero prime ideal Q of $A/\mathrm{Nil}(A)$, we have $Q \neq Q^2$. Let $P \in \mathrm{spec}(A)$ be a nonnil-ideal. Thus $P/\mathrm{Nil}(A)$ is a nonzero prime ideal of $A/\mathrm{Nil}(A)$. Hence $P/\mathrm{Nil}(A) \neq (P/\mathrm{Nil}(A))^2 = P^2/\mathrm{Nil}(A)$ (since A is chained, we have $\mathrm{Nil}(A) \subseteq P^2$). It follows that $P \neq P^2$.

($\Leftarrow$) Let Q be a nonzero prime ideal of $A/\mathrm{Nil}(A)$. There exists a prime ideal P of A such that $\mathrm{Nil}(A) \subset P$ and $Q = P/\mathrm{Nil}(A)$. As $Q^2 = (P/\mathrm{Nil}(A))^2 = P^2/\mathrm{Nil}(A) \neq P/\mathrm{Nil}(A) = Q$ and $A/\mathrm{Nil}(A)$ a valuation domain, then $A/\mathrm{Nil}(A)$ is an SFT ring. By Proposition 1.4, the ring A is nonnil-SFT. □

2. Some extensions of a nonnil-SFT ring

We define the set $\mathcal{H}^*$ to be the collection of all the rings A that satisfy the following property: for each maximal ideal M of A, the ring A_M is chained. This set is not empty, because it contains the valuation domains, Prüfer domains and chained rings etc.. Recall that this section is concerned by studying the stability of the nonnil-SFT property via some extensions of a nonnil-SFT ring. We start by studying the polynomial ring and the power series ring extensions of a such ring. But first we need to generalize some results of Kang and Park in [8], which we need for Proposition 2.8. Note that the authors of [8] have studied only the case of integral domains in the set $\mathcal{H}^*$. We prove that their result is still true for every $A \in \mathcal{H}^*$. To do this we need the following lemma. First we want to recall that a Prüfer domain A is an integral domain such that A_M is a valuation domain for each maximal ideal M of A.

Lemma 2.1. *Let $A \in \mathcal{H}^*$. Then for each $P \in \operatorname{spec}(A)$, A/P is a Prüfer domain.*

Proof. Let $P \in \operatorname{spec}(A)$. If A/P is a field, there is nothing to prove. Now assume that A/P is not a field and let $Q \in \operatorname{Max}(A/P)$. There exists $M \in \operatorname{Max}(A)$ such that $P \subseteq M$ and $Q = M/P$. Consider the map $\phi : A_M \longrightarrow (A/P)_Q$ defined by $\phi(\frac{a}{s}) = \frac{\bar{a}}{\bar{s}}$ for each $\frac{a}{s} \in A_M$. It is clear that ϕ is well defined and is a ring homomorphism. Let $\frac{\bar{a}}{\bar{s}} \in (A/P)_Q$. Thus $\bar{a} \in A/P$ and $\bar{s} \in (A/P) \setminus Q$. It follows that $a \in A$, $s \in A$ and $\bar{s} \notin M/P$. Therefore, $a \in A$ and $s \in A \setminus M$. Hence ϕ is surjective. Let $\frac{a}{s} \in ker(\phi)$. We have $\frac{\bar{a}}{\bar{s}} = \phi(\frac{a}{s}) = \frac{\bar{0}}{\bar{1}}$. Consequently, there exists $\bar{t} \in (A/P) \setminus Q$ such that $\bar{t}\bar{a} = \bar{0}$ in A/P. Thus $ta \in P$. Since $\bar{t} \notin Q = M/P$, we have $t \notin M$ which follows $t \notin P$. Hence $a \in P$. We deduce that $\frac{a}{s} \in P_M$. It yields that $ker(\phi) = P_M$ and $(A/P)_Q \simeq (A_M)/(P_M)$. ☐

Theorem 2.2. *Let $A \in \mathcal{H}^*$. If A is SFT, so is $A[[X_1, \dots, X_n]]$.*

Proof. Assume that $A[[X_1, \dots, X_n]]$ is not SFT. Then there exists $Q_1 \subset Q_2 \subset \cdots$ an infinite chain of prime ideals of $A[[X_1, \dots, X_n]][[X_{n+1}]]$. Let $P = (\bigcup_{i=1}^{+\infty} Q_i) \bigcap A$. As P is the radical of a finitely generated ideal, there exists $k \geq 1$ such that $P \subseteq Q_k$. By [8, Lemma 1], $P[[X_1, \dots, X_n]][[X_{n+1}]] \subseteq Q_k$. Let $\bar{Q}_i = Q_i/P[[X_1, \dots, X_n]][[X_{n+1}]]$ and $D = A/P$. By Lemma 2.1, D is an SFT Prüfer domain. On the other hand, $\bar{Q}_k \subset \bar{Q}_{k+1} \subset \cdots$ is an infinite chain of prime ideals of $D[[X_1, \dots, X_n]][[X_{n+1}]]$ and for each $i \geq k$, $\bar{Q}_i \bigcap D = 0$. Thus $\dim(D[[X_1, \dots, X_n]][[X_{n+1}]]_{D\setminus\{0\}}) = \infty$ contradicts [8, Lemma 7 and Remark 9]. Hence $A[[X_1, \dots, X_n]]$ is an SFT-ring. ☐

Proposition 2.3. *Let $A \in \mathcal{H}^*$ be an SFT ring. If $\operatorname{Nil}(A) \in \operatorname{spec}(A)$, then*

(1) $\dim(A[[X_1, \dots, X_n]]) = n \dim(A) + 1$ *if* $\dim(A)$ *is finite.*
(2) $\dim(A[[X_1, \dots, X_n]]) = \infty$ *elsewhere.*

Proof. (1) By Lemma 2.1, $D = A/\mathrm{Nil}(A)$ is a Prüfer domain. As D is SFT, by [8, Theorem 14], $\dim(D[[X_1, \dots, X_n]]) = n\dim(D)+1 = n\dim(A)+1$ because $\dim(D) = \dim(A)$. Since

$$D[[X_1, \dots, X_n]] \simeq A[[X_1, \dots, X_n]]/\mathrm{Nil}(A)[[X_1, \dots, X_n]]$$

and by [6, Corollary 2.5], $\mathrm{Nil}(A[[X_1, \dots, X_n]]) = \mathrm{Nil}(A)[[X_1, \dots, X_n]]$, we have

$$\begin{aligned}\dim(A[[X_1, \dots, X_n]]) &= \dim(A[[X_1, \dots, X_n]]/\mathrm{Nil}(A[[X_1, \dots, X_n]])\\ &= n\dim(A)+1.\end{aligned}$$

□

Let A be a ring. Recall that the set $A[X_1]]\cdots[X_n]]$ is a ring extension of A where $[X]] = [X]$ or $[X]] = [[X]]$, is called the mixed extension of A. For more results the reader is invited to visit [8].

Proposition 2.4. *Let $A \in \mathcal{H}^*$. If A is SFT, so is the mixed extension $A[X_1]], \dots, [X_n]]$.*

Proof. The proof is similar to the case of power series ring. □

Corollary 2.5. *Let $A \in \mathcal{H}^*$. If A is SFT, so is $A[X_1, \dots, X_n]$.*

Example 2.1. Let A be an SFT Prüfer domain and I be an arbitrary ideal of A. Then the rings $(A/I)[X_1, \dots, X_n]$ and $(A/I)[[X_1, \dots, X_n]]$ are SFT. Indeed, it is clear that A/I is SFT. Now we are going to show that $A/I \in \mathcal{H}^*$. Let $Q \in \mathrm{spec}(A/I)$. There exists $P \in \mathrm{spec}(A)$ such that $I \subseteq P$ and $Q = P/I$. We consider the map $\phi : A_P \longrightarrow (A/I)_Q$ defined by $\phi(\frac{a}{s}) = \frac{\bar{a}}{\bar{s}}$. As in the proof of Lemma 2.1, we show that ϕ is well defined and it is an homomorphism. Let $\frac{\bar{a}}{\bar{s}} \in (A/I)_Q$. Then $\bar{a} \in A/I$ and $\bar{s} \in (A/I) \setminus Q$. It yields that $a \in A$ and $s \in A \setminus P$. It follows that $\frac{a}{s} \in A_P$ and $\frac{\bar{a}}{\bar{s}} = \phi(\frac{a}{s})$. Consequently, ϕ is surjective. Hence $(A/I)_Q$ is chained as the homomorphic image of a (valuation) chained ring. Which shows that $A/I \in \mathcal{H}^*$. We deduce the result by Theorem 2.2 and Corollary 2.5.

Note that the result of Example 2.1 can be deduced from [8, Proposition 10].

Remark 2.6. Let A be a ring. If A is nonnil-SFT which is not SFT, then $\mathrm{Nil}(A) \in \mathrm{spec}(A)$. Indeed, if $\mathrm{Nil}(A) \notin \mathrm{spec}(A)$, then each prime ideal is a nonnil-ideal and hence it is SFT. It follows that the ring A is SFT, contradiction.

Corollary 2.7. *Let $A \in \mathcal{H}^*$ be a finite dimensional ring. If A is nonnil-SFT which not SFT, then there exists an infinite chain of prime ideals*

$$Q_1 \subset Q_2 \subset \cdots$$

of $A[[X]]$ such that for each $k \geq 1$, $\mathrm{Nil}(A)[[X]] \nsubseteq Q_k$ and $\dim((A/\mathrm{Nil}(A))[[X]])$ is finite.

Proof. By Proposition 1.5, the ring $A/\mathrm{Nil}(A)$ is SFT. By Lemma 2.1 and Remark 2.6, $A/\mathrm{Nil}(A)$ is an SFT Prüfer domain. By Proposition 2.3,

$$\dim((A/\mathrm{Nil}(A))[[X]]) = \dim(A) + 1.$$

On the other hand, since A is not SFT, there exists an infinite chain of prime ideals $Q_1 \subset Q_2 \subset \cdots$ of $A[[X]]$. Now, assume that there exists $k \geq 1$ such that $\mathrm{Nil}(A)[[X]] \subseteq Q_k$. Then $Q_k/\mathrm{Nil}(A)[[X]] \subset Q_{k+1}/\mathrm{Nil}(A)[[X]] \subset \cdots$ is an infinite chain of prime ideals of $A[[X]]/\mathrm{Nil}(A)[[X]] \simeq (A/\mathrm{Nil}(A))[[X]]$ absurd. $\square$

Proposition 2.8. *Let $A \in \mathcal{H}^*$. The following conditions are equivalent*:

(1) *The ring A is nonnil-SFT and the ideal* $\mathrm{Nil}(A)$ *is SFT.*
(2) *The ring A is SFT.*
(3) *The ring $A[X]$ is SFT.*
(4) *The ring $A[X]$ is nonnil-SFT.*
(5) *The ring $A[[X]]$ is SFT.*
(6) *The ring $A[[X]]$ is nonnil-SFT.*

Proof. (1)$\Leftrightarrow$(2) It follows from Remark 1.8.

(2)$\Rightarrow$(3) It follow from Corollary 2.5.

(3)$\Rightarrow$(4) It is clear.

(4)$\Rightarrow$(1) Let I be an ideal of A. We consider the ideal $J = \langle I, X\rangle$ of $A[X]$. By hypothesis, J is SFT. Then there exist $k \geq 1$ and a finitely generated ideal $F \subseteq J$ of $A[X]$ such that $f^k \in F$ for each $f \in J$. Let $L = \{f(0) \mid f \in F\}$. It is clear that L is a finitely generated ideal of A and that for every $a \in I$, $a^k \in L$. Hence I is an SFT ideal of A. It follows that the ring A is SFT.

(2)$\Rightarrow$(5) It follow from Theorem 2.2.

(5)$\Rightarrow$(6) It is clear.

(6)$\Rightarrow$(1) The same proof as the implication (4)$\Rightarrow$(1). $\square$

A ring A is called decomposable if it can be written in the form $A = A_1 \oplus A_2$, where A_1 and A_2 are two nonzero rings. The decomposition of A is not unique and for each decomposition $A = A_1 \oplus A_2$, we define the two following projections, $\pi_1 : A \longrightarrow A_1$ and $\pi_2 : A \longrightarrow A_2$ by $\pi_1(x) = x_1$ and $\pi_2(x) = x_2$ for each $x = x_1 + x_2 \in A$. It is clear that $A_1 = \pi_1(A)$, $A_2 = \pi_2(A)$ and $A = \pi_1(A) \oplus \pi_2(A)$. Therefore, we can describe the set of rings of the decomposition of A by their associated projections, i.e., a family $\{\pi_i,\ i \in \Lambda\}$ of epimorphisms from A in $\pi_i(A)$ with $\pi_i(A) \neq \{0\}$ for every $i \in \Lambda$, and for each $i \in \Lambda$, there exists $j \in \Lambda$ such that $A = \pi_i(A) \oplus \pi_j(A)$.

Theorem 2.9. *Let A be a decomposable ring and $\{\pi_i,\ i \in \Lambda\}$ the set of canonical epimorphisms from A to each component of a decomposition of A. The following statements are equivalent*:

(1) *The ring A is SFT.*
(2) *The ring A is nonnil-SFT.*
(3) *For each $i \in \Lambda$, the ring $\pi_i(A)$ is SFT.*

(4) *If* $e \in A\setminus\{0,1\}$ *is an idempotent element, then each ideal of* A *contained in* $\langle e\rangle$ *is SFT.*

Proof. (1)⇒(2) It is clear.

(2)⇒(3) Let $i \in \Lambda$. Then $A = \pi_i(A) \oplus \pi_j(A)$ for some $j \in \Lambda$. Let I be an ideal of $\pi_i(A)$. We have $I \oplus \pi_j(A)$ is a nonnil-ideal of A. It follows that $I \oplus \pi_j(A)$ is SFT. There exist $k \geq 1$ and a finitely generated ideal $F \subseteq I \oplus \pi_j(A)$ of A such that $x \in I \oplus \pi_j(A)$, $x^k \in F$. Consequently, $x^k \in \pi_i(F)$ for every $x \in I$, with $\pi_i(F)$ is a finitely generated ideal of $\pi_i(A)$ contained in I. Hence $\pi_i(A)$ is an SFT ring.

(3)⇒(4) Let $e \in A \setminus \{0,1\}$ be an idempotent element and I be an ideal of A contained in $\langle e\rangle$. We have $A = \langle e\rangle \oplus \langle 1-e\rangle$. By hypothesis, the ring $\langle e\rangle$ is SFT. Thus there exist $k \geq 1$ and a finitely generated ideal $F \subseteq I$ of $\langle e\rangle$ such that $x^k \in F$ for each $x \in I$. As the ideal $F \oplus \{0\}$ of A is finitely generated and for every $x \in I$, $x^k \in F \subseteq F \oplus \{0\}$, the ideal I of A is SFT.

(4)⇒(1) Let I be an ideal of A. Since A is decomposable, $A = \langle e\rangle \oplus \langle 1-e\rangle$ for some idempotent element $e \in A \setminus \{0,1\}$. It yields that $I = I_e \oplus I_{1-e}$, where I_e and I_{1-e} are two ideals of $\langle e\rangle$ and $\langle 1-e\rangle$, respectively. There exist $i, j \in \Lambda$ such that $\langle e\rangle = \pi_i(A)$ and $\langle 1-e\rangle = \pi_j(A)$. Consequently, there exist $k, r \geq 1$ and a finitely generated ideals $E \subseteq I_e$ and $F \subseteq I_{1-e}$ of A such that for every $x \in I_e$ and $y \in I_{1-e}$, $x^k \in \pi_i(E) = E$ and $y^r \in \pi_j(F) = F$. Let $a \in I$. Set $a = x + y$ with $x \in I_e$ and $y \in I_{1-e}$. Hence

$$a^{k+r} = \sum_{i=0}^{k+r} C^i_{k+r} y^i x^{k+r-i} = \sum_{i=0}^{r} C^i_{k+r} y^i x^{k+r-i} \sum_{i=r+1}^{k+r} C^i_{k+r} x^{k+r-i} y^i \in E \oplus F,$$

where $E \oplus F$ is a finitely generated ideal of A. Therefore, A is an SFT ring. □

Corollary 2.10. *Let* $(A_i)_{i\in\Lambda}$ *be a family of rings with cardinality at least 2. We consider the product ring* $A = \prod_{i\in\Lambda} A_i$. *The following conditions are equivalent*:

(1) *The set* Λ *is finite and for each* $i \in \Lambda$, *the ring* A_i *is SFT.*
(2) *The ring* A *is SFT.*
(3) *The ring* A *is nonnil-SFT.*

Proof. (1)⇒(2) By induction using Proposition 1.9.

(2)⇒(3) It is clear.

(3)⇒(1) By Theorem 2.9, for each $i \in \Lambda$ the ring A_i is SFT. Assume that $|\Lambda| = \infty$. Consider the ideal I of A of all elements with finite support. Since I is SFT, there exist $k \geq 1$ and $n \geq 1$ such that $x^k \in \langle e_{i_1}, \ldots, e_{i_n}\rangle$ for all $x \in I$, where $e_{i_r} = (\delta_{i_r}, j)_{j\in\Lambda}$, $r = 1, \ldots, n$. Let $r \in \Lambda \setminus \{i_1, \ldots, i_n\}$. Then $e_r = e_r^k \in \langle e_{i_1}, \ldots, e_{i_n}\rangle$. Consequently, $\mathrm{supp}(e_r) \subseteq \{i_1, \ldots, i_n\}$ which is impossible. Hence Λ is finite. □

Example 2.2. (1) Let $n \geq 2$ and $A_1, \ldots, A_n$ be a finite sequence of rings. If there exists $k \in \{1, \ldots, n\}$ such that A_k is nonnil-SFT which is not SFT, then the product ring $A_1 \times \cdots \times A_n$ is never nonnil-SFT.

(2) For each ring A, the product $\prod_{i=1}^{\infty} A$ is not nonnil-SFT.

Let $f : A \longrightarrow B$ be a rings homomorphism and J an ideal of B. We recall that the set

$$A \bowtie^f J = \{(a, f(a) + j),\ a \in A,\ j \in J\}$$

is a subring of the product ring $A \times B$ called the amalgamation of A and B along J. If J is an ideal of A, we will write $A \bowtie J = A \bowtie^{id_A} J$, where $id_A : A \longrightarrow A$ defined by $id_A(x) = x$ for every $x \in A$.

Remark 2.11. Let $f : A \longrightarrow B$ be a rings homomorphism and J be a nonzero ideal of B. Set $\bar{A} = A/\text{Nil}(A)$, $\bar{B} = B/\text{Nil}(B)$ and $\bar{J} = \pi(J)$ where $\pi : B \longrightarrow \bar{B}$ the canonical epimorphism. We consider the map $\bar{f} : \bar{A} \longrightarrow \bar{B}$ defined by $\bar{f}(\bar{a}) = \overline{f(a)}$. It is clear that $\bar{f}$ is well defined and it is a rings homomorphism. By [11, Remark 2.6] $(A \bowtie^f J)/\text{Nil}(A \bowtie^f J) \simeq \bar{A} \bowtie^{\bar{f}} \bar{J}$.

If $A \bowtie^f J \in \mathcal{H}$, then $\bar{A} \bowtie^{\bar{f}} \bar{J} \simeq (A \bowtie^f J)/\text{Nil}(A \bowtie^f J)$ is an integral domain. If $\bar{J} \neq \{0\}$, by [5, Proposition 2.10], $\bar{f}^{-1}(\bar{J}) = \{0\}$. It yields that $f^{-1}(J) \subseteq \text{Nil}(A)$.

If $\bar{J} = \{0\}$, then $J \subseteq \text{Nil}(B)$. It follows that

$$\text{Nil}(A \bowtie^f J) = \text{Nil}(A) \bowtie^f J.$$

Let $x \in f^{-1}(J)$. If $\text{Nil}(A \bowtie^f J) \subseteq (x, 0)A \bowtie^f J$, then $J = \{0\}$, which is impossible. Hence $(x, 0)A \bowtie^f J \subseteq \text{Nil}(A \bowtie^f J)$. Thus $x \in \text{Nil}(A)$.

Theorem 2.12. *Let A and B be two rings, $J \neq \{0\}$ be an ideal of B and $f : A \longrightarrow B$ be a rings homomorphism. Assume that $A \bowtie^f J \in \mathcal{H}$. The following statements are equivalent*:

(1) *The ring $A \bowtie^f J$ is nonnil-SFT.*

(2) *The rings A and $f(A) + J$ are nonnil-SFT.*

Proof. Using the same notations of Remark 2.11.

(1)⇒(2) It follows from Proposition 1.5.

(2)⇒(1) Let $\Psi : f(A) + J \longrightarrow \bar{f}(\bar{A}) + \bar{J}$ be the map defined by $\Psi(f(x) + j) = \bar{f}(\bar{x}) + \bar{j}$. Ψ is well defined and is a rings homomorphism as the restriction of the canonical surjection from $B :\longrightarrow \bar{B}$. Let $\bar{x} \in \bar{f}^{-1}(\bar{J})$. We have $\overline{f(x)} = \bar{f}(\bar{x}) \in \bar{J}$. Then there exists $j \in J$ such that $f(x) - j \in Nil(B)$. Which implies that there exists $k \geq 1$ such that $(f(x) - j)^k = 0$. It follows that $f(x^k) \in J$. Thus $x^k \in \text{Nil}(A)$ and consequently, $x \in \text{Nil}(A)$. It shows that $\bar{x} = \bar{0}$ and hence $\overline{f(\bar{A})} \bigcap \bar{J} = \{\bar{0}\}$. Now, let $f(x) + j \in ker(\Psi)$. Then $\overline{f(x)} + \bar{j} = \bar{0}$. It yields that $\overline{f(x)} = \bar{0}$ and $\bar{j} = \bar{0}$. Which implies that $f(x), j \in \text{Nil}(B)$. Hence $f(x) + j \in \text{Nil}(B) \bigcap (f(A) + J) = \text{Nil}(f(A) + J)$. Consequently, $ker(\Psi) \subseteq \text{Nil}(f(A) + J)$. The other inclusion is easy. Hence $(f(A) + J)/\text{Nil}(f(A) + J) \simeq \bar{f}(\bar{A}) + \bar{J}$. As A and $f(A) + J$ are nonnil-SFT, then $\bar{A}$ and $f(A) + J/\text{Nil}(f(A) + J) \simeq \bar{f}(\bar{A}) + \bar{J}$

are nonnil-SFT rings. It follows that they are SFT since they are reduced. By [9, Theorem 3.1], the ring $\bar{A} \bowtie^{\bar{f}} \bar{J}$ is SFT. By Remark 2.11, $A \bowtie^f J/\mathrm{Nil}(A \bowtie^f J) \simeq \bar{A} \bowtie^{\bar{f}} \bar{J}$ is an SFT ring. As $A \bowtie^f J \in \mathcal{H}$, by Proposition 1.4, the ring $A \bowtie^f J$ is nonnil-SFT. □

In [11, Corollary 2.3], we have the following equivalence: $A \in \mathcal{H}$ if and only if $A \bowtie \mathrm{Nil}(A) \in \mathcal{H}$. By combining this result and Theorem 2.12, we get the following corollary.

Corollary 2.13. *Let $A \in \mathcal{H}$. Then the ring $A \bowtie \mathrm{Nil}(A)$ is nonnil-SFT if and only if the ring A is nonnil-SFT.*

Example 2.3. Let $A \in \mathcal{H}$ be a nonnil-SFT ring. For $n \geq 0$, set $A_{n+1} = A_n \bowtie \mathrm{Nil}(A_n)$, where $A_0 = A$. Then $\{A_n,\ n \geq 0\}$ is an infinite set of nonnil-SFT rings. Moreover, if A_0 is not SFT, then for each $n \geq 0$, the ring A_n is nonnil-SFT which is not SFT.

Proposition 2.14. *Let A be a ring and J be a nonnil-ideal of A. The following statements are equivalent*:

(1) *The ring $A \bowtie J$ is nonnil-SFT.*
(2) *The ring A is SFT.*
(3) *The ring $A \bowtie J$ is SFT.*

Proof. (1)⇒(2) Let I be an ideal of A. Then $I \bowtie J$ is a nonnil-ideal of $A \bowtie J$, hence it is SFT. It follows that there exist $k \geq 1$ and a finitely generated ideal $F \subseteq I \bowtie J$ of $A \bowtie J$ such that for each $(x, y) \in I \bowtie J$, $(x, y)^k \in F$. Thus $x^k \in \pi(F) \subseteq I$ for every $x \in I$, with $\pi(F)$ a finitely generated ideal of A, where $\pi : A \bowtie J \longrightarrow A$ is the first projection. Therefore, the ring A is SFT.

(2)⇒(3) It follows from [9, Theorem 3.1].

(3)⇒(1) It is clear. □

Proposition 2.15. *Let $A \subseteq B$ be a rings extension such that for each finitely generated ideal I of A $IB \bigcap A = I$. If the ring B is nonnil-SFT, so is A.*

Proof. Let I be a nonnil ideal of A. Since the ring B is nonnil-SFT and the ideal IB of B is a nonnil-ideal, there exist $k \geq 1$ and a finitely generated ideal $J \subseteq IB$ of B such that $x^k \in J$ for every $x \in IB$. Let $F \subseteq I$ be a finitely generated ideal of A such that $J \subseteq FB$. Hence $x^k \in J \bigcap A \subseteq FB \bigcap A = F$ for every $x \in I$. It is follows that I is an SFT ideal of A. Which implies that the ring A is SFT. □

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