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EIGENVALUE COMPARISON FOR THE DISCRETE $(3,3)$ CONJUGATE BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we consider a boundary value problem for a sixth order difference equation. We prove the monotone behavior of the eigenvalue of the problem as the coefficients in the difference equation change values and the existence of a positive solution for a class of problems.

1. Introduction

Sixth order boundary value problems arise from physical sciences including the study of elasticity. For example, according to Agarwal, Kovacs, and O'Regan [1], the deformation of the equilibrium state of an elastic circular ring segment with its two ends simply supported can be described by the sixth-order boundary value problem

$$u^{(6)} + 2u^{(4)} + u'' = f(t,u), \quad 0 < t < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0.$$

Sixth order boundary value problems have attracted some attention recently. For some other results on sixth order boundary value problems, we refer the readers to [3, 6–8, 18]. The reader is referred to [16] for more applications of sixth order boundary value problems in physical sciences and engineering. Motivated by these works, in this paper, we consider a boundary value problem for a discrete sixth order difference equation, which is associated with the (3,3) conjugate boundary value problem that consists of the differential equation

$$y^{(6)}(t) + f(t, y(t)) = 0, \quad 0 < t < 1, \tag{1}$$

and the boundary conditions

$$y(0) = y'(0) = y''(0) = y''(1) = y'(1) = y(1) = 0. \tag{2}$$

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The linear form of the discrete analog of the problem (1)-(2) is

$$\Delta^6 u_{k-3} + \lambda a_k u_k = 0, \quad k = 1, 2, \ldots, n; \tag{3}$$

$$u_{-2} = \Delta u_{-2} = \Delta^2 u_{-2} = \Delta^2 u_{n+1} = \Delta u_{n+2} = u_{n+3} = 0. \tag{4}$$

Here, λ is a parameter, the coefficients $a_1, a_2, \ldots, a_n$ are real numbers, and Δ is the forward difference operator, that is, $\Delta u_k = u_{k+1} - u_k$. Thus, we have

$$\Delta^2 u_k = \Delta u_{k+1} - \Delta u_k = u_{k+2} - 2u_{k+1} + u_k$$

and

$$\Delta^6 u_k = u_{k+6} - 6u_{k+5} + 15u_{k+4} - 20u_{k+3} + 15u_{k+2} - 6u_{k+1} + u_k.$$

It is easily seen that the boundary conditions (4) are equivalent to

$$u_{-2} = u_{-1} = u_0 = u_{n+1} = u_{n+2} = u_{n+3} = 0. \tag{5}$$

Denote the following $n \times n$ matrix

$$\begin{pmatrix} 20 & -15 & 6 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -15 & 20 & -15 & 6 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 6 & -15 & 20 & -15 & 6 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -1 & 6 & -15 & 20 & -15 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 6 & -15 & 20 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 20 & -15 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -15 & 20 & -15 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -15 & 20 & -15 & 6 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 6 & -15 & 20 & -15 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 6 & -15 & 20 \end{pmatrix} \tag{6}$$

by D and then, we can put the problem (3)-(4) in the matrix form

$$Du - \lambda A u = 0, \tag{7}$$

where $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ is a diagonal matrix of order n and $u = (u_1, u_2, \ldots, u_n)^T$ is a vector of order n.

In this paper we will establish the existence of a positive solution u to the system (7) for any nonzero nonnegative sequences $\{a_i\}$ and the monotone behavior of the eigenvalue λ of the system as the real sequence $\{a_i\}$ changes. Our approach directly follows the strategies employed in [9–13] for various other problems, relying on the properties of D and its inverse. Since D is a heptadiagonal symmetric Toeplitz matrix, two explicit formulas proposed in [14,15] for the inverse of general Toeplitz matrices could be directly applied to expressing the inverse of D either as a sum of products of low and upper triangular Toeplitz matrices or as a sum of products of circular matrices and upper triangular Toeplitz matrices. However, the positiveness of entries of the inverse of D seems not to be obtained easily from these formulas from [14,15]. For this reason, we take a different approach in this paper. By making a full use of special structure of D, we express the inverse D^{-1} of D as a sum of a componentwise positive rank-one matrix and several componentwise nonnegative matrices in Section 2. Finally, Section 3 will be devoted to the eigenvalue comparison and the existence of a positive solution of the system (7).

2. Preliminary results

In this section, we collect a few technical results on D which are needed in the analysis of the major results of the paper.

Theorem 2.1. *Let $n \geq 3$ and D be the matrix defined as in* (6) *and $b = (-15, 6, -1, 0)^T$, where 0 is the zero matrix in $\mathbb{R}^{1\times(n-3)}$. Then, the matrix D is nonsingular and the i-th element of $x = D^{-1}b$ is given by*

$$x_i = -\frac{(i+1)(i+2)(n+1-i)(n+2-i)(n+3-i)}{2(n+1)(n+2)(n+3)},\ i = 1, 2, \ldots, n. \tag{8}$$

Proof. Obviously, the system of linear equations $Dx = 0$ can be regarded as a difference equation

$$-x_{k-3}+6x_{k-2}-15x_{k-1}+20x_k-15x_{k+1}+6x_{k+2}-x_{k+3} = 0,\ k = 1, 2, \ldots, n \tag{9}$$

with boundary value conditions

$$x_{-2} = x_{-1} = x_0 = x_{n+1} = x_{n+2} = x_{n+3} = 0. \tag{10}$$

The characteristic polynomial of the difference equation (9) can be written as

$$p(z) = -1 + 6z - 15z^2 + 20z^3 - 15z^4 + 6z^5 - z^6 = -(z-1)^6 \tag{11}$$

which has a zero $z = 1$ of multiplicity 6. Thus, the solution to (9)-(10) is in the form of

$$x_i = \alpha_0 + \alpha_1 i + \alpha_2 i^2 + \alpha_3 i^3 + \alpha_4 i^4 + \alpha_5 i^5 \tag{12}$$

for $i = -2, -1, 0, 1, \ldots, n+3$, where the coefficients $\{\alpha_i : i = 0, 1, \ldots, 5\}$ are determined through the boundary value conditions (10). These boundary value conditions give rise to the following system of linear equations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & (-2)^2 & (-2)^3 & (-2)^4 & (-2)^5 \\ 1 & -1 & (-1)^2 & (-1)^3 & (-1)^4 & (-1)^5 \\ 1 & n+1 & (n+1)^2 & (n+1)^3 & (n+1)^4 & (n+1)^5 \\ 1 & n+2 & (n+2)^2 & (n+2)^3 & (n+2)^4 & (n+2)^5 \\ 1 & n+3 & (n+3)^2 & (n+3)^3 & (n+3)^4 & (n+3)^5 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which only has a trivial solution, i.e., $\alpha_i = 0$ for $i = 0, 1, \ldots, 5$ due to the fact that its coefficient matrix is a Vandermonde matrix whose determinant is

$$4(n+1)(n+2)^2(n+3)^3(n+4)^2(n+5) \neq 0.$$

Thus, we have $x = 0$, i.e., D is nonsingular.

Similarly, the system of linear equations $Dx = b$ can be written as the difference equation (9) with boundary value conditions

$$x_{-2} = x_{-1} = 0,\ x_0 = -1,\ x_{n+1} = x_{n+2} = x_{n+3} = 0. \tag{13}$$

Thus, the solution to $Dx = b$ is also in the form of (12) but this time the coefficients $\{\alpha_i : i = 0, 1, \ldots, 5\}$ are determined by the boundary value conditions (13). Equivalently, x_i is the polynomial of degree 5 through the following six points in $\mathbb{R}^2$

$$(-2, 0),\ (-1, 0),\ (0, -1),\ (n + 1, 0),\ (n + 2, 0),\ \text{and } (n + 3, 0).$$

The Lagrange's interpolation formula gives

$$x_i = (-1)\frac{(i - (-2))(i - (-1))(i - (n + 1))(i - (n + 2))(i - (n + 3))}{(0 - (-2))(0 - (-1))(0 - (n + 1))(0 - (n + 2))(0 - (n + 3))}$$

resulting in the expression of (8). □

In what follows, denote the submatrix consisting of the first k rows and k columns of D by D_k. Obviously, D_k is the same kind of symmetric Toeplitz matrix as D with different order and $D = D_n$.

For $k \geq 3$, define

$$\beta_k = (-15, 6, -1, 0, \ldots, 0)^T \in \mathbb{R}^k \quad \text{and} \quad \gamma_k = 20 - \beta_k^T D_k^{-1} \beta_k. \tag{14}$$

By employing the explicit expression for $D_k^{-1}\beta_k$ from Theorem 2.1, we have

$$\begin{aligned}\gamma_k &= 20 + (-15)\frac{6k(k + 1)(k + 2)}{2(k + 1)(k + 2)(k + 3)} + 6\frac{12(k - 1)k(k + 1)}{2(k + 1)(k + 2)(k + 3)} \\ &\quad - \frac{20(k - 2)(k - 1)k}{2(k + 1)(k + 2)(k + 3)} \\ &= \frac{(k + 4)(k + 5)(k + 6)}{(k + 1)(k + 2)(k + 3)}. \qquad (15)\end{aligned}$$

Theorem 2.2. *Let D be the matrix defined as in* (6) *and D_k be the submatrix of D consisting of its first k rows and k columns. Then, the determinant of D_k is*

$$\det(D_k) = \frac{(k + 1)(k + 2)^2(k + 3)^3(k + 4)^2(k + 5)}{8640} \tag{16}$$

for $k \geq 1$. Moreover, D is positive definite.

Proof. A simple calculation gives

$$\det(D_1) = 20,\ \det(D_2) = 175,\ \text{and } \det(D_3) = 980 \tag{17}$$

indicating that the expression (16) holds for $k = 1, 2$, and 3.

For any $i \geq 3$, D_i is nonsingular in view of Theorem 2.1. With the help of (14) we can write

$$D_{i+1} = \begin{pmatrix} 20 & \beta_i^T \\ \beta_i & D_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D_i \end{pmatrix} \begin{pmatrix} 1 & \beta_i^T \\ 0 & I \end{pmatrix} \begin{pmatrix} \gamma_i & 0 \\ D_i^{-1}\beta_i & I \end{pmatrix},$$

leading to

$$\det(D_{i+1}) = \gamma_i \det(D_i) \quad \text{for } i \geq 3. \tag{18}$$

For any $k \geq 4$, repeatedly using (18) along with (15) and (17), we obtain

$$\begin{aligned}\det(D_k) &= \det(D_3)\ \gamma_3\gamma_4\gamma_5 \cdots \gamma_{k-3}\gamma_{k-2}\gamma_{k-1} \\ &= 980\frac{(7)(8)(9)}{(4)(5)(6)}\frac{(8)(9)(10)}{(5)(6)(7)}\frac{(9)(10)(11)}{(6)(7)(8)} \cdots \frac{(k+3)(k+4)(k+5)}{(k)(k+1)(k+2)} \\ &= \frac{(k+1)(k+2)^2(k+3)^3(k+4)^2(k+5)}{8640},\end{aligned}$$

where the last equality comes from the fact that the numerator of γ_i can be canceled with the denominator of γ_{i+3}.

Since all the leading principal minors $\det(D_k)$ of D are positive in view of (16), D is obviously positive definite. $\square$

If only 4 decimal places are displayed, then we have

$$D_1^{-1} = (0.05),\ D_2^{-1} = \begin{pmatrix} 0.1143 & 0.0857 \\ 0.0857 & 0.1143 \end{pmatrix},$$

and

$$D_3^{-1} = \begin{pmatrix} 0.1786 & 0.2143 & 0.1071 \\ 0.2143 & 0.3714 & 0.2143 \\ 0.1071 & 0.2143 & 0.1786 \end{pmatrix}. \tag{19}$$

We observe that all entries of these matrices are positive. Next, we will show that all the entries of D^{-1} are positive as well. To this end, we define

$$p_{k+1} = \begin{pmatrix} 0, \ldots, 0, 1, -\beta_k^T D_k^{-1} \end{pmatrix}^T \in \mathbb{R}^n \quad \text{for } k \geq 3. \tag{20}$$

It is seen from Theorem 2.1 that the last $k+1$ components of p_{k+1} are positive and the other $n-k-1$ elements are all zero. In particular,

$$p_n = \begin{pmatrix} 1 \\ -D_{n-1}^{-1}\beta_{n-1} \end{pmatrix} \quad \text{for } n \geq 4$$

whose components are all positive.

Theorem 2.3. *Let D be the matrix defined as in* (6). *Then*

$$D^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & D_3^{-1} \end{pmatrix} + \frac{p_4 p_4^T}{\gamma_3} + \cdots + \frac{p_{n-1}p_{n-1}^T}{\gamma_{n-2}} + \frac{p_n p_n^T}{\gamma_{n-1}} \quad \textit{for } n \geq 4,$$

where p_i is defined as in (20) *for $i = 4, 5, \ldots, n$ and γ_k is given in* (15) *for each k. Moreover, all entries of D^{-1} are positive.*

Proof. In view of Theorem 2.1 and (15), both D_k and D_{k+1} are nonsingular and $\gamma_k > 0$ for $k \geq 3$. It is seen from

$$D_{k+1} = \begin{pmatrix} 20 & \beta_k^T \\ \beta_k & D_k \end{pmatrix}$$

that the inverse of D_{k+1} can be expressed involving D_k^{-1} and the inverse of Schur's complement as

$$D_{k+1}^{-1} = \frac{1}{\gamma_k}\begin{pmatrix} 1 & -\beta_k^T D_k^{-1} \\ -D_k^{-1}\beta_k & \gamma_k D_k^{-1} + D_k^{-1}\beta_k\beta_k^T D_k^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & D_k^{-1} \end{pmatrix} + \frac{1}{\gamma_k}\begin{pmatrix} 1 \\ -D_k^{-1}\beta_k \end{pmatrix}\begin{pmatrix} 1 \\ -D_k^{-1}\beta_k \end{pmatrix}^T. \tag{21}$$

For $k = n-1$,

$$D^{-1} = D_n^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & D_{n-1}^{-1} \end{pmatrix} + \frac{p_n p_n^T}{\gamma_{n-1}}. \tag{22}$$

With $k = n-2$ in (21), we obtain

$$D_{n-1}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & D_{n-2}^{-1} \end{pmatrix} + \frac{1}{\gamma_{n-2}}\begin{pmatrix} 1 \\ -D_{n-2}^{-1}\beta_{n-2} \end{pmatrix}\begin{pmatrix} 1 \\ -D_{n-2}^{-1}\beta_{n-2} \end{pmatrix}^T. \tag{23}$$

Substituting (23) into (22) and rewriting the first matrix on the right hand side of (22) as

$$\begin{pmatrix} 0 & 0 \\ 0 & D_{n-1}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D_{n-2}^{-1} \end{pmatrix} + \frac{1}{\gamma_{n-2}}\begin{pmatrix} 0 \\ 1 \\ -D_{n-2}^{-1}\beta_{n-2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -D_{n-2}^{-1}\beta_{n-2} \end{pmatrix}^T,$$

we obtain

$$D^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & D_{n-2}^{-1} \end{pmatrix} + \frac{p_{n-1}p_{n-1}^T}{\gamma_{n-2}} + \frac{p_n p_n^T}{\gamma_{n-1}}.$$

Repeatedly using (21), after $n-3$ iterations we eventually reach

$$D^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & D_3^{-1} \end{pmatrix} + \frac{p_4 p_4^T}{\gamma_3} + \cdots + \frac{p_{n-1}p_{n-1}^T}{\gamma_{n-2}} + \frac{p_n p_n^T}{\gamma_{n-1}} \tag{24}$$

indicating that the entry $(D^{-1})_{i,j} \geq (p_n)_i(p_n)_j/\gamma_{n-1} > 0$ since all entries of D_3^{-1} are positive, all components of p_k ($k = 4, 5, \ldots, n-1$) are nonnegative, and all components $(p_n)_i$ ($i = 1, 2, \ldots, n$) of p_n and γ_k ($k = 3, 4, \ldots, n-1$) are positive. □

3. Eigenvalue comparison and the existence of a positive solution

Lemma 3.1. *If λ is an eigenvalue of the problem* (7) *and y is a corresponding eigenvector, then* (i) *λ is real and nonzero, and $y^*Ay \neq 0$;* (ii) *if $\rho \neq \lambda$ is also an eigenvalue of the problem* (7) *and x is an eigenvector corresponding to ρ, then we have $x^*Ay = 0$.*

Proof. First we note that $\lambda y^*Ay = y^*Dy > 0$, since D is positive definite and $y \neq 0$. Hence λ and y^*Ay are both nonzero. We can write

$$\lambda y^*Ay = y^*(\lambda Ay) = y^*Dy = (Dy)^*y = (\lambda Ay)^*y = \bar{\lambda}y^*A^*y = \bar{\lambda}y^*Ay, \tag{25}$$

which implies $\lambda = \bar{\lambda}$, i.e., λ is real. Part (ii) follows from the fact that

$$(\lambda - \rho)x^* Ay = x^*(\lambda Ay) - (\rho Ax)^* y = x^* Dy - (Dx)^* y = 0.$$

The proof is complete. □

For the positive definite matrix D, there exists a unique nonsingular lower triangular matrix L such that $D = LL^T$. With the help of this Cholesky decomposition, the eigenvalue problem (7) can be converted to a regular eigenvalue problem.

Lemma 3.2. *Let $D = LL^T$ be the Cholesky decomposition of D.*

(a) *If λ is an eigenvalue of the problem (7) and y is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of $L^{-1}AL^{-T}$ and $L^T y$ is a corresponding eigenvector.*

(b) *If α is a nonzero eigenvalue of $L^{-1}AL^{-T}$ and y is a corresponding eigenvector, then $1/\alpha$ is an eigenvalue of the problem (7) and $L^{-T}y$ is a corresponding eigenvector.*

Proof. (a) If λ is an eigenvalue of the problem (7) and y is a corresponding eigenvector, then $\lambda \neq 0$ in view of Lemma 3.1. The equation $\lambda Ay = Dy$ is equivalent to the equation $\left(L^{-1}AL^{-T}\right) L^T y = \frac{1}{\lambda} L^T y$ with $L^T y \neq 0$ since L is nonsingular and $y \neq 0$. The result in (b) can be proved similarly. □

Theorem 3.3. *Let $D = LL^T$ be the Cholesky decomposition of D and let p, q be the numbers of positive and negative elements in the set $\{a_i\}_{i=1}^n$, respectively. Then there are p positive eigenvalues $\{\lambda_i^+ : i = 1, 2, \ldots, p\}$ and q negative eigenvalues $\{\lambda_i^- : i = 1, 2, \ldots, q\}$ of the problem (7). Moreover,*

$$\{1/\lambda_i^+ : i = 1, 2, \ldots, p\} \cup \{1/\lambda_i^- : i = 1, 2, \ldots, q\}$$

is the set of all nonzero eigenvalues of $L^{-1}AL^{-T}$.

Proof. The fact that $L^{-1}AL^{-T}$ is real and symmetric implies that there exists an orthogonal matrix Q such that

$$Q^T L^{-1} A L^{-T} Q = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{26}$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ are all eigenvalues of $L^{-1}AL^{-T}$. Let $x = L^{-T}Qz$. It is seen from (26) that

$$\sum_{i=1}^n \alpha_i z_i^2 = z^T \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) z = x^T Ax = \sum_{i=1}^n a_i x_i^2$$

are two representations of the real quadratic form $x^T Ax$. In view of the Law of Inertia for Quadratic Forms [4, Theorem 1, p. 297], we immediately deduce that the numbers of positive and negative elements in the set $\{\alpha_i : i = 1, 2, \ldots, n\}$ are p and q, respectively.

Thus, in view of Lemmas 3.1 and 3.2, we see that $\{1/\alpha_i : \alpha_i \neq 0\}$ gives the complete set of eigenvalues of the problem (7). Therefore, $\{\lambda_i^+ = 1/\alpha_i : i =$

$1, 2, \ldots, p\}$ and $\{\lambda_i^- = 1/\alpha_{n-i+1} : i = 1, 2, \ldots, q\}$ are the sets of all the positive and all the negative eigenvalues of the problem (7), respectively. $\square$

All the eigenvalues of the problem (7) are related to these of $L^{-1}AL^{-T}$ as specified in Theorem 3.3. We will use this relationship to study the monotone behavior of all eigenvalues as the coefficients of the problem change. To this end, we consider the following two problems

$$Du - \lambda A^{(t)} u = 0, \tag{27}$$

where $A^{(t)} = \operatorname{diag}\left(a_1^{(t)}, a_2^{(t)}, \ldots, a_{n-1}^{(t)}, a_n^{(t)}\right)$, $t = 1, 2$.

Theorem 3.4. *Let p_t and q_t be the numbers of positive and negative elements in the set $\left\{a_1^{(t)}, a_2^{(t)}, \ldots, a_n^{(t)}\right\}$, respectively, for $t = 1, 2$ and let*

$$\{\lambda_{q_t}^-(t) \le \cdots \le \lambda_2^-(t) \le \lambda_1^-(t)\} \quad \textit{and} \quad \{\lambda_1^+(t) \le \lambda_2^+(t) \le \cdots \le \lambda_{p_t}^+(t)\}$$

be the sets of all the negative and all the positive eigenvalues of problems (27)*, respectively. If $a_i^{(1)} \ge a_i^{(2)}$ for $1 \le i \le n$, then*

$$\lambda_k^+(1) \le \lambda_k^+(2), \quad 1 \le k \le p_2 \quad \textit{and} \quad \lambda_k^-(1) \le \lambda_k^-(2), \quad 1 \le k \le q_1. \tag{28}$$

If $a_i^{(1)} > a_i^{(2)}$, $1 \le i \le n$, then all the inequalities of (28) *are strict.*

Proof. Let LL^T be the Cholesky decomposition of D. Define

$$\alpha_k^+ = \frac{1}{\lambda_k^+(1)}, \quad 1 \le k \le p_1, \qquad \alpha_k^- = \frac{1}{\lambda_k^-(1)}, \quad 1 \le k \le q_1, \tag{29}$$

$$\beta_k^+ = \frac{1}{\lambda_k^+(2)}, \quad 1 \le k \le p_2, \qquad \beta_k^- = \frac{1}{\lambda_k^-(2)}, \quad 1 \le k \le q_2. \tag{30}$$

In view of Theorem 3.3, by adding $n - (p_1 + q_1)$ zeros to (29) and adding $n - (p_2 + q_2)$ zeros to (30), we deduce that

$$\alpha_1^+ \ge \alpha_2^+ \ge \cdots \ge \alpha_{p_1}^+ > 0 = \cdots = 0 > \alpha_{q_1}^- \ge \cdots \ge \alpha_2^- \ge \alpha_1^- \tag{31}$$

and

$$\beta_1^+ \ge \beta_2^+ \ge \cdots \ge \beta_{p_2}^+ > 0 = \cdots = 0 > \beta_{q_2}^- \ge \cdots \ge \beta_2^- \ge \beta_1^- \tag{32}$$

are all the eigenvalues of $L^{-1}A^{(t)}L^{-T}$ for $t = 1, 2$, respectively. If $a_i^{(1)} \ge a_i^{(2)}$ for $1 \le i \le n$, then $p_2 \le p_1$ and $q_1 \le q_2$. Furthermore, $A^{(1)} - A^{(2)}$ is positive semidefinite and so is $L^{-1}A^{(1)}L^{-T} - L^{-1}A^{(2)}L^{-T}$. If $a_i^{(1)} > a_i^{(2)}$ for $1 \le i \le n$, then $A^{(1)} - A^{(2)}$ is positive definite and so is $L^{-1}A^{(1)}L^{-T} - L^{-1}A^{(2)}L^{-T}$. It is seen from the monotonic behavior of eigenvalues of symmetric matrices [2, Theorem 3, p. 117] that $\lambda_k(L^{-1}A^{(1)}L^{-T}) \ge \lambda_k(L^{-1}A^{(2)}L^{-T})$ for each k if $a_i^{(1)} \ge a_i^{(2)}$, $1 \le i \le n$ and that $\lambda_k(L^{-1}A^{(1)}L^{-T}) > \lambda_k(L^{-1}A^{(2)}L^{-T})$ for each k if $a_i^{(1)} > a_i^{(2)}$, $1 \le i \le n$. Thus, the desired results follow immediately from (29)-(32). $\square$

Finally, we end up this paper with an existence result of positive solutions to the generalized eigenvalue problem (7).

Theorem 3.5. *Assume that* $a_i \geq 0$ *for* $i = 1, 2, \ldots, n$ *and at least one of* $\{a_1, a_2, \ldots, a_n\}$ *is positive. Then, the smallest eigenvalue of the problem* (7) *is simple and corresponds to a positive eigenvector.*

Proof. For any eigenvalue λ of (7) and any eigenvector y corresponding to λ, we have $\lambda \neq 0$ and $y^*Ay \neq 0$ in view of Lemma 3.1. Under the assumption of the theorem, we further have $y^*Ay > 0$ and thus, $\lambda = y^*Dy/y^*Ay > 0$. In view of Lemma 3.2 and Theorem 3.3, there is at least one positive eigenvalue for the problem (7). For the smallest eigenvalue λ_1 and its eigenvector y, we have

$$D^{-1}Ay = \frac{1}{\lambda_1}y.$$

Thus $1/\lambda_1$ is the maximum eigenvalue of $D^{-1}A$ and the y is an eigenvector corresponding to $1/\lambda_1$.

In the case when $a_i > 0$ for all $1 \leq i \leq n$, all elements of $D^{-1}A$ are positive due to Theorem 2.3. The result follows immediately from the Perron-Frobenius Theorem [17, p. 30] applied to $D^{-1}A$ in this case.

If some of the a_i's are zero, then, without loss of any generality, we may assume that $a_1 = a_2 = \cdots = a_t = 0$ and $a_i > 0$ for $t < i \leq n$. In such a case, we can write

$$D^{-1}A = \begin{pmatrix} O & C \\ O & B \end{pmatrix},$$

where B is an $(n-t) \times (n-t)$ matrix and C is a $t \times (n-t)$ matrix. In view of Theorem 2.3, both B and C are positive matrices. Also, $1/\lambda_1$ is the maximum eigenvalue of B. Applying the Perron-Frobenius Theorem to B, we have that $1/\lambda_1$ is simple and there exists a positive vector y_2 such that $By_2 = \frac{1}{\lambda_1}y_2$. Define $y_1 = \lambda_1 Cy_2$ and $\tilde{y} = (y_1^T, y_2^T)^T$. Obviously, all components of y_1 are positive and so are the components of $\tilde{y}$. In addition, $D^{-1}A\tilde{y} = \frac{1}{\lambda_1}\tilde{y}$ is satisfied and so is $D\tilde{y} = \lambda_1 A\tilde{y}$. Hence, the smallest eigenvalue λ_1 is simple and $\tilde{y}$ is a positive eigenvector of the problem (7) corresponding to λ_1. □

Because of the linear nature of our problem, the existence result of Theorem 3.5 for the BVPs of the difference equation (3)-(4) is established by first converting the original problem to a matrix problem and then applying the well-known Perron-Frobenius Theorem to a positive matrix. For nonlinear problems, the variational method and the fixed point theorems can be employed to obtain the existence results for solutions. For example, the reader is referred to [5,19] for some existence results for fourth order difference equations.

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