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AN EXTENSION OF SCHNEIDER'S CHARACTERIZATION THEOREM FOR ELLIPSOIDS

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Abstract. Suppose that M is a strictly convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin o in its convex side and with the outward unit normal N. For a fixed point $p \in M$ and a positive constant t, we put Φ_t the hyperplane parallel to the tangent hyperplane Φ at p and passing through the point $q = p - tN(p)$. We consider the region cut from M by the parallel hyperplane Φ_t, and denote by $I_p(t)$ the $(n+1)$-dimensional volume of the convex hull of the region and the origin o. Then Schneider's characterization theorem for ellipsoids states that among centrally symmetric, strictly convex and closed surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$, the ellipsoids are the only ones satisfying $I_p(t) = \phi(p)t$, where ϕ is a function defined on M. Recently, the characterization theorem was extended to centrally symmetric, strictly convex and closed hypersurfaces in $\mathbb{E}^{n+1}$ satisfying for a constant β, $I_p(t) = \phi(p)t^\beta$.

In this paper, we study the volume $I_p(t)$ of a strictly convex and complete hypersurface in $\mathbb{E}^{n+1}$ with the origin o in its convex side. As a result, first of all we extend the characterization theorem to strictly convex and closed (not necessarily centrally symmetric) hypersurfaces in $\mathbb{E}^{n+1}$ satisfying $I_p(t) = \phi(p)t^\beta$. After that we generalize the characterization theorem to strictly convex and complete (not necessarily closed) hypersurfaces in $\mathbb{E}^{n+1}$ satisfying $I_p(t) = \phi(p)t^\beta$.

1. Introduction

We will say that a convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ is *strictly convex* if the hypersurface is of positive Gauss-Kronecker curvature K with respect to the inward unit normal ([1]).

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Suppose that M is a strictly convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its convex side and with the outward unit normal N. For a fixed point $p \in M$, the support function $\mathfrak{h}(p) = \langle p, N(p)\rangle$ of M at p is nothing but the distance from the origin to the tangent hyperplane Φ to M at the point p. For a constant $t \in (0, \mathfrak{h}(p)]$, we consider the hyperplane Φ_t parallel to the tangent hyperplane Φ and passing through the point $q = p - tN(p)$. Then t is the distance from the point q to the tangent hyperplane Φ to M at p.

We denote by $A_p(t)$, $V_p(t)$ and $C_p(t)$ the n-dimensional area of the section in Φ_t enclosed by $\Phi_t \cap M$, the $(n+1)$-dimensional volume of the region bounded by M and the hyperplane Φ_t, and the $(n+1)$-dimensional volume of the cone with base the section in Φ_t enclosed by $\Phi_t \cap M$ and with vertex the origin o, respectively. We also denote by $I_p(t)$ the $(n+1)$-dimensional volume of the ice cream cone-shaped domain which is the convex hull of the origin o and the region of M cut off by the hyperplane Φ_t. Then, we have ([7])

$$\frac{d}{dt}V_p(t) = A_p(t), \tag{1.1}$$

$$C_p(t) = \frac{1}{n+1}A_p(t)\,(\mathfrak{h}(p) - t) \tag{1.2}$$

and

$$I_p(t) = C_p(t) + V_p(t). \tag{1.3}$$

For a constant t $(> \mathfrak{h}(p))$ such that the hyperplane Φ_t intersects M, $A_p(t)$, $V_p(t)$, $C_p(t)$ and $I_p(t)$ are also well-defined. In this case, (1.2) shows that $C_p(t)$ is (-1) times the volume of the corresponding cone with vertex the origin. Hence (1.3) implies that $I_p(t)$ is the volume of a concave domain in $\mathbb{E}^{n+1}$. See Figure 1.

Recently, in [9] the following characterization theorem was established, which is originally due to R. Schneider ([12]).

Proposition A. *Suppose that the centrally symmetric convex body B centered at the origin o in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ has smooth boundary M which is of positive Gauss-Kronecker curvature. Then, for a positive constant β and a positive function ϕ defined on M, M satisfies $I_p(t) = \phi(p)t^\beta$ if and only if $\beta = 1, n = 2$ and M is a 2-dimensional ellipsoid centered at the origin in the 3-dimensional Euclidean space $\mathbb{E}^3$. In this case, we have $I_p(t) = \alpha t/\mathfrak{h}(p)$ for some positive constant α.*

In this paper, first of all in Section 2, for a strictly convex and closed (not necessarily centrally symmetric) hypersurface M in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ we prove the following characterization theorem:

Theorem B. *Suppose that M is a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior which satisfies $I_p(t) = \phi(p)t^\beta$, where ϕ is a function on M and β is a constant.*

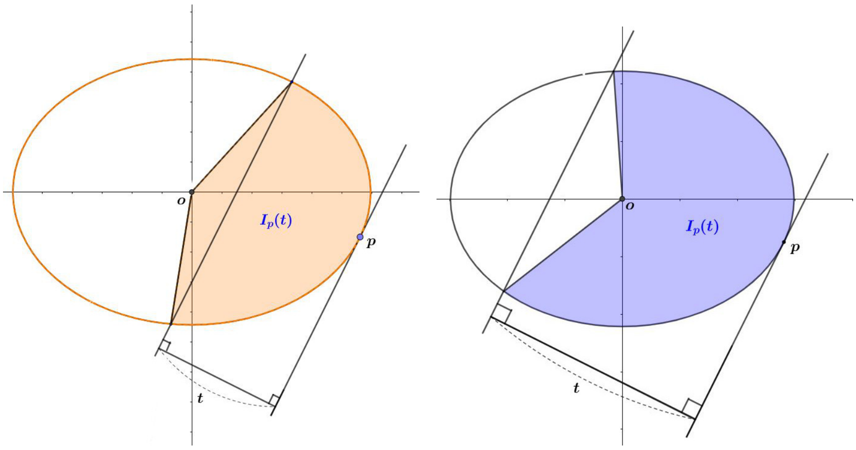

FIGURE 1. $I_p(t)$ with $t < \mathfrak{h}(p)$ and $I_p(t)$ with $t > \mathfrak{h}(p)$.

Then M is an ellipsoid centered at the origin in the 3-dimensional Euclidean space $\mathbb{E}^3$.

Next, for a strictly convex and complete (not necessarily closed) hypersurface M in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with $n = 4k+2$, $k = 0, 1, 2, \ldots$, in Section 3 we establish the following characterization theorem:

Theorem C. *Suppose that M is a strictly convex and complete hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ $(n = 4k+2, k \geq 0)$ with the origin in its convex side which satisfies $I_p(t) = \phi(p)t^\beta$, where ϕ is a function on M and β is a constant. Then $n = 2$ and M is an ellipsoid centered at the origin in the 3-dimensional Euclidean space $\mathbb{E}^3$.*

In order to prove Theorems B and C, we investigate the n-dimensional area $A_p(t)$ of the section in the hyperplane Φ_t enclosed by $\Phi_t \cap M$ and we use a characterization theorem for ellipsoids established in [9]. When $n = 4k+2$ $(k \geq 0)$, we will show that as t tends to ∞, the area $A_p(t)$ goes to $-\infty$. Hence, we have $A_p(d) = 0$ for some constant $d > 0$. This shows that the hypersurface M must be closed. Therefore Theorem B completes the proof of Theorem C.

Finally, for further study we raise a question as follows.

Question D. Are there any strictly convex and non-closed complete hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ satisfying $I_p(t) = \phi(p)t^\beta$ for a function ϕ on M and a constant β?

Some characterization theorems for hyperplanes, circular hypercylinders, hyperspheres, ellipsoids, elliptic paraboloids and elliptic hyperboloids in the Euclidean space $\mathbb{E}^{n+1}$ were established in [2–8, 11, 12]. For some characterizations of hyperbolic space in the Minkowski space $\mathbb{E}^{n+1}_1$, we refer [10].

Throughout this article, all objects are smooth (C^3) and connected, unless otherwise mentioned.

2. Preliminaries and Theorem B

In order to prove our theorems, first of all, we need the following.

Lemma 2.1. *Suppose that M is a strictly convex and closed smooth hypersurface of the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior. Then we have the following:*

$$\lim_{t\to 0} \frac{1}{(\sqrt{t})^n} A_p(t) = \frac{(\sqrt{2})^n \omega_n}{\sqrt{K(p)}}, \tag{2.1}$$

$$\lim_{t\to 0} \frac{1}{(\sqrt{t})^{n+2}} V_p(t) = \frac{(\sqrt{2})^{n+2} \omega_n}{(n+2)\sqrt{K(p)}}, \tag{2.2}$$

$$\lim_{t\to 0} \frac{1}{(\sqrt{t})^n} C_p(t) = \frac{(\sqrt{2})^n \omega_n}{(n+1)\sqrt{K(p)}} \mathfrak{h}(p), \tag{2.3}$$

$$\lim_{t\to 0} \frac{1}{(\sqrt{t})^n} I_p(t) = \frac{(\sqrt{2})^n \omega_n}{(n+1)\sqrt{K(p)}} \mathfrak{h}(p), \tag{2.4}$$

where ω_n denotes the volume of the n-dimensional unit ball in the n-dimensional Euclidean space $\mathbb{E}^n$ and $\mathfrak{h}(p)$ the support function of M at $p \in M$.

Proof. For proofs of (2.1) and (2.2), see Lemma 8 of [7]. Together with (2.1) and (2.2), it follows from (1.2) and (1.3) that (2.3) and (2.4) hold. □

Next, we need the following characterization theorem for ellipsoids.

Proposition 2.2. *Let M denote a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior. We denote by $K(p)$ and $\mathfrak{h}(p)$ the Gauss-Kronecker curvature of M at p and the support function of M at p, respectively. Then M satisfies $K(p) = \alpha\mathfrak{h}(p)^{n+2}$ for a nonzero constant α if and only if M is an ellipsoid in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ centered at the origin.*

Proof. For a proof, see Theorem A of [9]. □

Now, we prove Theorem B as follows.

Suppose that M is a strictly convex and closed hypersurface M in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior which satisfies for a constant β

$$I_p(t) = \phi(p)t^{\beta}, \tag{2.5}$$

where $\phi(p)$ denotes a function of $p \in M$. Then we get from (2.4) that $\beta = n/2$ and

$$\phi(p) = c_n \frac{\mathfrak{h}(p)}{\sqrt{K(p)}}, \quad c_n = \frac{(\sqrt{2})^n \omega_n}{(n+1)}, \tag{2.6}$$

where ω_n is the volume of the n-dimensional unit ball in the n-dimensional Euclidean space $\mathbb{E}^n$.

For a fixed point $p \in M$, we denote by $\bar{p}$ the unique point on M, where the tangent hyperplane to M is parallel to the tangent hyperplane to M at p. We put $a = \mathfrak{h}(p)$ and $b = \mathfrak{h}(\bar{p})$. Then, for the volume V of the interior of M we have

$$V = I_p(a+b) = \phi(p)(a+b)^{n/2} \tag{2.7}$$

and

$$V = I_{\bar{p}}(a+b) = \phi(\bar{p})(a+b)^{n/2}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\phi(p) = \phi(\bar{p}). \tag{2.9}$$

Furthermore, we have

$$V = I_p(a) + I_{\bar{p}}(b) = \phi(p)a^{n/2} + \phi(\bar{p})b^{n/2} = \phi(p)(a^{n/2} + b^{n/2}), \tag{2.10}$$

where the third equality follows from (2.9). Together with (2.7), this implies

$$(a+b)^{n/2} = a^{n/2} + b^{n/2}. \tag{2.11}$$

Since a and b are positive, we have

$$\begin{cases} (a+b)^{n/2} < a^{n/2} + b^{n/2}, & \text{if } n = 1, \\ (a+b)^{n/2} > a^{n/2} + b^{n/2}, & \text{if } n \geq 3. \end{cases} \tag{2.12}$$

Together with (2.11), this shows that $n = 2$ and hence $\beta = 1$. Thus we obtain from (2.5) and (2.6)

$$I_p(t) = \phi(p)t, \tag{2.13}$$

and

$$\phi(p) = c_2 \frac{\mathfrak{h}(p)}{\sqrt{K(p)}}, \quad c_2 = \frac{2}{3}\pi. \tag{2.14}$$

In this case, for later use in the proof of Theorem C we suppose that M is a strictly convex and complete (not necessarily closed) surface in the 3-dimensional Euclidean space $\mathbb{E}^3$ with the origin in its convex side which satisfies (2.13) with (2.14). Recall the definition of $I_p(t)$. Then we have

$$I_p(t) = \frac{1}{3}(\mathfrak{h}(p) - t)A_p(t) + V_p(t). \tag{2.15}$$

Together with the definition of $I_p(t)$, (2.13) implies

$$3\phi(p)t = (\mathfrak{h}(p) - t)A_p(t) + 3V_p(t). \tag{2.16}$$

Since $V_p'(t) = A_p(t)$, differentiating (2.16) with respect to t gives

$$A_p'(t) + \frac{2}{\mathfrak{h}(p) - t} A_p(t) = 3 \frac{\phi(p)}{\mathfrak{h}(p) - t}. \tag{2.17}$$

Using the integrating factor $(\mathfrak{h}(p) - t)^{-2}$ of the first order differential equation (2.17), one obtains

$$A_p(t) = c(p)(\mathfrak{h}(p) - t)^2 + \frac{3}{2}\phi(p), \tag{2.18}$$

where $c(p)$ is a constant depending only on p. Since $A_p(0) = 0$, we get

$$c(p) = -\frac{\pi}{\sqrt{K(p)}} \frac{1}{\mathfrak{h}(p)}. \tag{2.19}$$

Hence we have

$$A_p(t) = \frac{\pi}{\sqrt{K(p)}} \frac{t}{\mathfrak{h}(p)} (2\mathfrak{h}(p) - t), \tag{2.20}$$

which shows that $A_p(2\mathfrak{h}(p)) = 0$. Thus we see that the surface M is a closed surface and $I_p(2\mathfrak{h}(p))$ is the volume V of the interior of M.

It follows from (2.13) and (2.14) that

$$V = \frac{4\pi}{3} \frac{\mathfrak{h}(p)^2}{\sqrt{K(p)}}, \tag{2.21}$$

which implies

$$K(p) = \left(\frac{4\pi}{3V}\right)^2 \mathfrak{h}(p)^4. \tag{2.22}$$

Therefore Proposition 2.2 shows that M is a 2-dimensional ellipsoid centered at the origin in the 3-dimensional Euclidean space $\mathbb{E}^3$.

Conversely, it is straightforward to show that the 2-dimensional ellipsoid given by

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \tag{2.23}$$

satisfies

$$I_p(t) = \alpha \frac{t}{\mathfrak{h}(p)}, \quad \alpha = \frac{2\pi}{3abc}. \tag{2.24}$$

This completes the proof of Theorem B.

3. Proof of Theorem C

Suppose that M is a strictly convex and complete hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its convex side which satisfies for a constant β

$$I_p(t) = \phi(p)t^\beta, \tag{3.1}$$

where $\phi(p)$ denotes a function of $p \in M$. Then we get from (2.4) that $\beta = n/2$ and

$$\phi(p) = c_n \frac{\mathfrak{h}(p)}{\sqrt{K(p)}}, \quad c_n = \frac{(\sqrt{2})^n \omega_n}{(n+1)}, \tag{3.2}$$

where ω_n is the volume of the n-dimensional unit ball in the n-dimensional Euclidean space $\mathbb{E}^n$.

Hereafter we fix a point $p \in M$ and put $a = \mathfrak{h}(p)$ the support function of M at $p \in M$. It follows from the definition of $I_p(t)$ that

$$I_p(t) = \frac{1}{n+1}(a-t)A_p(t) + V_p(t). \tag{3.3}$$

Together with the definition of $I_p(t)$, (3.1) implies

$$(n+1)\phi(p)t^{n/2} = (a-t)A_p(t) + (n+1)V_p(t). \tag{3.4}$$

Using $V_p'(t) = A_p(t)$, we differentiate (3.4) with respect to t. Then we get

$$A_p'(t) + \frac{n}{a-t}A_p(t) = d_n(p)\frac{t^{(n-2)/2}}{a-t}, \tag{3.5}$$

where we put

$$d_n(p) = \frac{n(n+1)}{2}\phi(p). \tag{3.6}$$

In order to solve the first order differential equation (3.5), we use the integrating factor $(a-t)^{-n}$. Then we obtain

$$A_p(t) = d_n(p)(a-t)^n J_n(t), \tag{3.7}$$

where we let

$$J_n(t) = \int \frac{t^{(n-2)/2}}{(a-t)^{n+1}}dt. \tag{3.8}$$

In this section, we prove Theorem C as follows. Recall that when $n = 2$, the proof of Theorem C was completed in the proof of Theorem B. Hence we may assume that $n = 2k$ with $k \geq 2$. Then we have

$$\begin{aligned} J_n(t) &= \int \frac{t^{k-1}}{(a-t)^{2k+1}}dt \\ &= \sum_{j=0}^{k-1} \int \frac{A_{2k+1-j}}{(a-t)^{2k+1-j}}dt \\ &= \sum_{j=0}^{k-1} \frac{A_{2k+1-j}}{(2k-j)(a-t)^{2k-j}} + d(p), \end{aligned} \tag{3.9}$$

where $d(p)$ is an integration constant depending only on p and we put

$$A_{2k+1-j} = (-1)^j \binom{k-1}{j} a^{k-1-j}, \quad 0 \leq j \leq k-1. \tag{3.10}$$

It follows from (3.7) that

$$A_p(t) = d_n(p)\left\{\sum_{j=0}^{k-1}\frac{A_{2k+1-j}}{(2k-j)}(a-t)^j + d(p)(a-t)^{2k}\right\}. \tag{3.11}$$

Since $A_p(0) = 0$, one obtains

$$d(p) = \frac{\alpha(k)}{a^{k+1}}, \tag{3.12}$$

where $\alpha(k)$ is defined by

$$\alpha(k) = -\sum_{j=0}^{k-1}\frac{(-1)^j}{2k-j}\binom{k-1}{j}. \tag{3.13}$$

Note that $\alpha(k)$ can be written as follows:

$$\alpha(k) = -\int_0^1 x^k(x-1)^{k-1}dx. \tag{3.14}$$

Hence we see that when k is odd (even, resp.), the constant $\alpha(k)$ is negative (positive, resp.). Together with (3.11), this shows that

$$\begin{cases}\lim_{t\to\infty} A_p(t) = -\infty, & \text{if } k \text{ is odd,}\\ \lim_{t\to\infty} A_p(t) = \infty, & \text{if } k \text{ is even.}\end{cases} \tag{3.15}$$

Thus, for $n = 2k$ with an odd number k there exists a positive number d satisfying $A_p(d) = 0$. This implies that the hypersurface M is closed. Therefore Theorem B completes the proof of Theorem C.

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