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SOME RESULTS ON 2-STRONGLY GORENSTEIN PROJECTIVE MODULES AND RELATED RINGS

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Abstract. In this paper, we give some results on 2-strongly Gorenstein projective modules and related rings. We first investigate the relationship between strongly Gorenstein projective modules and periodic modules and then give the structure of modules over strongly Gorenstein semisimple rings. Furthermore, we prove that a ring R is 2-strongly Gorenstein hereditary if and only if every ideal of R is Gorenstein projective and the class of 2-strongly Gorenstein projective modules is closed under extensions. Finally, we study the relationship between 2-Gorenstein projective hereditary and 2-Gorenstein projective semisimple rings, and we also give an example to show the quotient ring of a 2-Gorenstein projective hereditary ring is not necessarily 2-Gorenstein projective semisimple.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity; in particular, R denotes such a ring, and all modules are assumed to be unitary modules.

In 1969, Auslander and Bridger [1] introduced the notion of finitely generated modules having Gorenstein dimension zero over Noetherian rings. For any module over a general ring, Enochs and Jenda [12] introduced the notion of Gorenstein projective modules, which coincides with that of modules having Gorenstein dimension zero for finitely generated modules over Noetherian rings. Recall that an R-module M is called Gorenstein projective, if there exists an acyclic complex of projective modules

$$\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

such that $M \cong \ker(P_1 \to P_0)$ and such that $\mathrm{Hom}_R(-, P)$ is acyclic complex for every projective R-module P. The complex satisfying the condition above is also called totally acyclic. In 2007, Bennis and Mahdou [6] introduced the notion of strongly Gorenstein projective modules, which sit between projective

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modules and Gorenstein projective modules. Recall that an R-module M is called strongly Gorenstein projective, if there exists an acyclic complex

$$0 \to M \to P \to M \to 0,$$

where P is projective such that $\mathrm{Hom}_R(-, Q)$ is an acyclic complex for every projective R-module Q. Then they proved that a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. As a generalization of strongly Gorenstein projective modules, the authors [3, 5] introduced the notion of n-strongly Gorenstein projective modules, in which 1-strongly Gorenstein projective modules are just strongly Gorenstein projective modules. Recall that an R-module M is called n-strongly Gorenstein projective, if there exists an acyclic complex

$$0 \to M \to P_n \to P_{n-1} \to \cdots \to P_1 \to M \to 0,$$

where each P_i is projective such that $\mathrm{Hom}_R(-, Q)$ is an acyclic complex for every projective R-module Q. Rings over which every R-module is n-strongly Gorenstein projective are called n-SG semisimple rings. These modules and rings have been studied extensively by many authors.

Recently, n-periodic modules are introduced in [2, 4]. Recall that an R-module M is said to be n-periodic if it admits a projective resolution of this form

$$0 \to M \to P_n \to \cdots \to P_1 \to M \to 0.$$

It is simply called periodic if it is 1-periodic. It is easy to see that M is n-strongly Gorenstein projective if it is Gorenstein projective and n-periodic, and M is strongly Gorenstein projective if it is Gorenstein projective and periodic. Therefore, a natural question, arises:

Question 1.1. Does there exist an n-periodic module which is not n-strongly Gorenstein projective?

Here, we find an acyclic complex of projective modules which is not totally acyclic, and thus there exists a periodic module which is not strongly Gorenstein projective.

In [4], the author raised a question of which are the rings over which every module is n-periodic? In [13], it is proved that every module is weak Gorenstein projective (cycle of acyclic complex of projective R-modules) if and only if R is Gorenstein semisimple. Hence, we get that rings over which every module is n-periodic are just the n-strongly Gorenstein semisimple rings.

In [9], it has been proved that if R is a local Gorenstein semisimple ring, then every R-module M is of the form: $M = R^{(I)} \oplus N$, where I is an index set and N is an R-module with $Ann(x) \neq 0$ for every element x of N. In [10, Theorem 1.1], it is proved that if $(R, (u))$ is a local 2-strongly Gorenstein semisimple ring, then for any R-module M, M is decomposed into a direct sum of the free $R/(u^i)$-modules:

$$M = R^{(I_k)} \oplus (R/(u^{k-1}))^{(I_{k-1})} \oplus \cdots \oplus (R/(u^2))^{(I_2)} \oplus (R/(u))^{(I_1)},$$

where (I_i) is the index set of the free module $R/(u^i)$. By [9, Theorem 3.5], every strongly Gorenstein semisimple ring R has a unique ideal $\mathfrak{m} = (u)$. Therefore we have the following result: Let $(R, (u))$ be a local strongly Gorenstein semisimple ring. Then for each R-module M, M is decomposed into a direct sum of the form: $M = R^{(I_1)} \oplus (R/(u))^{(I_2)}$, where (I_1), (I_2) are the index sets of the free R-module and free $R/(u)$-module, respectively.

The concepts of strongly Gorenstein hereditary rings and 2-strongly Gorenstein hereditary rings are introduced in [16, 20]. Recall that a ring R is called a 2-strongly Gorenstein hereditary ring (resp. strongly Gorenstein hereditary ring) if every submodule of a projective module is 2-strongly Gorenstein projective (resp. strongly Gorenstein projective), and R is called a 2-strongly Gorenstein Dedekind (resp. strongly Gorenstein Dedekind) if it is a 2-strongly Gorenstein hereditary domain (resp. strongly Gorenstein domain). In [15], it has been proved that R is a strongly Gorenstein hereditary ring if and only if every ideal of R is Gorenstein projective and the class of strongly Gorenstein projective modules is closed under extensions. So it is natural to ask if 2-strongly Gorenstein hereditary rings have the same properties.

Question 1.2. Can we conclude that a ring R is 2-strongly Gorenstein hereditary if and only if every ideal of R is Gorenstein projective and the class of 2-strongly Gorenstein projective modules is closed under extensions?

Using the same way in [16], we prove that if the class of 2-strongly Gorenstein projective modules is closed under extensions, then it is projectively resolving. Thus we prove this.

Finally, we study the relationship between 2-strongly Gorenstein hereditary rings and 2-strongly Gorenstein semisimple rings. Let R be a ring such that the class of 2-strongly Gorenstein projective modules is closed under extensions, and let x be a non-zero-divisor and non-unit element in R, denote R/xR by $\overline{R}$. Then R is 2-strongly Gorenstein hereditary if and only if $\overline{R}$ is a 2-strongly Gorenstein semisimple ring. We also give an example to show that $\overline{R}$ is not necessarily a 2-strongly Gorenstein semisimple ring.

For unexplained concepts and notations, one can refer to [3, 7, 9, 11, 12].

2. Main results

We first study the relationship between the strongly Gorenstein projective modules and periodic modules. Since every strongly Gorenstein projective module is periodic, we have to find an acyclic complex of projective modules which is not totally acyclic. In a recent paper [19], it is proved that every acyclic complex of projective modules is totally acyclic if and only if the cycles of every acyclic complex of Gorenstein projective modules are Gorenstein projective. If R is a Gorenstein ring, then every acyclic complex of projective R-modules is totally acyclic.

However, for any arbitrary ring R, an acyclic complex of projective R-modules is not always totally acyclic. Here we give an example to show it.

Example 2.1. Let $\mathbb{Q}$ be the field of rational numbers and X be an indeterminate element. Consider the ring $R = \mathbb{Q} + X^3\mathbb{Q}[X]$ and the ideal $I = (X^3, X^4, X^5)$. Let $I_i = I$. Then the following complex $\cdots \longrightarrow \bigoplus_{i=1}^{\infty} R_i \longrightarrow \bigoplus_{i=1}^{\infty} R_i \longrightarrow \bigoplus_{i=1}^{\infty} R_i \longrightarrow \cdots$ is an acyclic complex of projective R-modules, but not totally acyclic.

Proof. Consider the following acyclic complex

$$0 \longrightarrow I_1 \oplus I_2 \xrightarrow{\alpha} R \oplus R \oplus R \xrightarrow{\beta} I \longrightarrow 0. \tag{1}$$

Let α and β be homomorphisms defined by: $\alpha((i_1, i_2)) = (i_1X, -i_1, 0) + (0, i_2X, -i_2)$, $\beta((r_1, r_2, r_3)) = r_1X^3 + r_2X^4 + r_3X^5$. The image of α is obviously contained in the kernel of β. To prove that the complex is acyclic, we only need to prove that the kernel of β is contained in the image of α.

Let $r_1 = q_{10} + q_{13}X^3 + q_{14}X^4 + \cdots + q_{1k_1}X^{k_1}$, $r_2 = q_{20} + q_{23}X^3 + q_{24}X^4 + \cdots + q_{2k_2}X^{k_2}$, $r_3 = q_{30} + q_{33}X^3 + q_{34}X^4 + \cdots + q_{3k_3}X^{k_3}$. Suppose that $k_1 - 1 > k_2$ and $k_2 - 1 > k_3$. Then $\beta((r_1, r_2, r_3)) = 0$, so $r_1X^3 + r_2X^4 + r_3X^5 = 0$ and we have $q_{10} = q_{20} = q_{30} = q_{13} = 0$. It follows that

$$q_{14} + q_{23} = q_{15} + q_{24} + q_{33} = \cdots = q_{1k_1} + q_{2,k_1-1} + q_{3,k_1-2} = 0.$$

Therefore, we get that

$$(q_{14}X^4, q_{23}X^3, 0) = \alpha((-q_{23}X^3, 0)),$$
$$(q_{15}X^5, q_{24}X^4, q_{33}X^3) = \alpha((q_{15}X^4, -q_{33}X^3)), \ldots,$$
$$(q_{1,k_1}X^{k_1}, q_{2,k_1-1}X^{k_1-1}, q_{3,k_1-2}X^{k_1-2}) = \alpha((q_{1,k_1}X^{k_1-1}, -q_{3,k_1-2}X^{k_1-2})).$$

Thus, (r_1, r_2, r_3) is in the image of α. Hence, we have the acyclic complex above. Let $R_i = R$. Since $(\bigoplus_{i=1}^{\infty} I_i) \oplus (\bigoplus_{j=1}^{\infty} I_j) \cong \bigoplus_{i=1}^{\infty} I_i$ and $(\bigoplus_{i=1}^{\infty} R_i) \oplus (\bigoplus_{j=1}^{\infty} R_j) \oplus (\bigoplus_{k=1}^{\infty} R_k) \cong \bigoplus_{i=1}^{\infty} R_i$, by adding the acyclic complex (1), we obtain the following acyclic complex

$$\cdots \longrightarrow \bigoplus_{i=1}^{\infty} R_i \longrightarrow \bigoplus_{i=1}^{\infty} R_i \longrightarrow \bigoplus_{i=1}^{\infty} R_i \longrightarrow \cdots. \tag{2}$$

Next, we will show that $\mathrm{Ext}_R^1(\bigoplus_i I_i, R) \neq 0$, which implies (2) is not totally acyclic. If $\mathrm{Ext}_R^1(\bigoplus_i I_i, R) = 0$, then the acyclic complex (1) induces an acyclic complex:

$$0 \longrightarrow \mathrm{Hom}_R(I, R) \longrightarrow \mathrm{Hom}_R(R \oplus R \oplus R, R) \longrightarrow \mathrm{Hom}_R(I_1 \oplus I_2, R) \longrightarrow 0.$$

Let φ be a homomorphism:

$$\varphi : I_1 \oplus I_2 \longrightarrow R$$
$$\varphi((X^3, 0)) = X^3X^2 = X^5.$$

This is possible because $\mathrm{Hom}(I, R) = I^{-1} = \mathbb{Q}[X]$. If there exists a homomorphism θ from $R \oplus R \oplus R$ to R such that $\varphi = \theta\alpha$, there must exist $r \in R$, $r' \in R$ such that $X^5 = rX^4 - r'X^3$. However, it is contradicting the hypothesis that the right side of this equality will never have a nonzero item of the degree of X^5. Thus, consider the acyclic complex $0 \longrightarrow \mathrm{Hom}_R(I_1 \oplus I_2, R) \longrightarrow \mathrm{Ext}^1_R(I, R) \longrightarrow 0$. Since $\mathrm{Hom}_R(I_1 \oplus I_2, R) \neq 0$, it follows that $\mathrm{Ext}^1_R(\bigoplus_i I_i, R) = \prod \mathrm{Ext}^1_R(I, R) \neq 0$. □

Remark 2.2. By Example 2.1, there exists a short exact sequence $0 \to \bigoplus_{i=1}^{\infty} I_i \to \bigoplus_{i=1}^{\infty} R_i \to \bigoplus_{i=1}^{\infty} I_i \to 0$. One can see that $\bigoplus_{i=1}^{\infty} I_i$ is a periodic module but not strongly Gorenstein projective, as desired.

In [4], the author raised a question of which are the rings over which every module is n-periodic? In [13], it is proved that every module is weak Gorenstein projective (a cycle of acyclic complex of projective R-modules) if and only if R is Gorenstein semisimple. Hence, we get that rings over which every module is n-periodic are just the n-strongly Gorenstein semisimple rings. In [9], it has been proved that if R is a local Gorenstein semisimple ring, then every R-module M is of the form: $M = R^{(I)} \oplus N$, where I is an index set and N is an R-module with $Ann(x) \neq 0$ for every element x of N. In [10, Theorem 1.1], it is proved that if $(R, (u))$ is a local 2-strongly Gorenstein semisimple ring, then for any R-module M, M is decomposed into a direct sum of the free $R/(u^i)$-modules:

$$M = R^{(I_k)} \oplus (R/(u^{k-1}))^{(I_{k-1})} \oplus \cdots \oplus (R/(u^2))^{(I_2)} \oplus (R/(u))^{(I_1)},$$

where I_i is the index set of the free module $R/(u^i)$.

By [9, Theorem 3.5], every strongly Gorenstein semisimple ring R has a unique ideal $\mathfrak{m} = (u)$, and so we have the following result.

Proposition 2.3. *Let $(R, (u))$ be a local strongly Gorenstein semisimple ring. Then for each R-module M, M is decomposed into a direct sum of the form: $M = R^{(I_1)} \oplus (R/(u))^{(I_2)}$, where I_1, I_2 are the index sets of the free R-module and the free $R/(u)$-module, respectively. Hence there exists an exact sequence $0 \to M \to R^{(I_1)} \oplus R^{(I_1)} \oplus R^{(I_2)} \to M \to 0$ such that $\mathrm{Hom}_R(-, Q)$ is exact for every projective R-module Q.*

Proof. Since there exists an exact sequence $0 \to (u) \to R \to R/(u) \to 0$ and $R/(u) \cong (u)$, we have the exact sequence $0 \to (u) \to R \to (u) \to 0$. Therefore we get the exact sequence $0 \to (u)^{(I_2)} \to R^{(I_2)} \to (u)^{(I_2)} \to 0$. On the other hand, there exists an exact sequence: $0 \to R^{(I_1)} \oplus \to R^{(I_1)} \oplus R^{(I_1)} \to R^{(I_1)} \to 0$. Now adding this sequence to above one, we get the exact sequence:

$$0 \to R^{(I_1)} \oplus (u)^{(I_2)} \to R^{(I_1)} \oplus R^{(I_1)} \oplus R^{(I_2)} \to R^{(I_1)} \oplus (u)^{(I_2)} \to 0.$$

Hence, we have the exact sequence: $0 \to M \to R^{(I_1)} \oplus R^{(I_1)} \oplus R^{(I_2)} \to M \to 0$. Since R is a strongly Gorenstein semisimple ring, it follows that $\mathrm{Hom}_R(-, Q)$ is exact for every projective R-module Q. □

Example 2.4. Consider residue classes of the ring of integers $R = \mathbb{Z}/(p)$, where $p = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$, and each p_j is a prime number. One can easily see that

$$R = \mathbb{Z}/(p_1^{i_1})(p_2^{i_2}) \cdots (p_k^{i_k}) \cong \mathbb{Z}/(p_1^{i_1}) \oplus \mathbb{Z}/(p_2^{i_2}) \oplus \cdots \oplus \mathbb{Z}/(p_k^{i_k})$$

is a 2-strongly Gorenstein semisimple ring, and so each ring $\mathbb{Z}/(p_j^{i_j})(1 \le j \le k)$ is a local 2-strongly Gorenstein semisimple ring. By [10, Theorem 1.1], for each $\mathbb{Z}/(p_j^{i_j})$-module M, M is decomposed into a direct sum of the free $\mathbb{Z}/(p_j^{i_m})$-modules as follows:

$$M = \mathbb{Z}^{(I_j)} \oplus (\mathbb{Z}/(p_j^{i_{j-1}}))^{(I_{j-1})} \oplus \cdots \oplus (\mathbb{Z}/(p_j^2))^{(I_2)} \oplus (\mathbb{Z}/(p_j))^{(I_1)},$$

where I_m is the index set of the free module $\mathbb{Z}/(p_j^m)$.

Example 2.5 ([6])**.** Consider the quasi-Frobenius local ring $R = k[X]/(X^2)$ where k is a field, and denote by $\overline{X}$ the residue class in R of X. It is easy to see that R is a local strongly Gorenstein semisimple ring, and so any R-module M is of the form: $M = R^{(I_i)} \oplus (R/(X))^{(I_k)} \cong R^{(I_i)} \oplus (\overline{X})^{(I_k)}$.

Example 2.6. Consider residue classes of the ring of integers $R = \mathbb{Z}/(p^2)$, where p is a prime number. One can easily see that R is a local strongly Gorenstein semisimple ring, and so any R-module M is of the form: $M = \mathbb{Z}^{(I_j)} \oplus (\mathbb{Z}/(p))^{(I_k)} \cong \mathbb{Z}^{(I_j)} \oplus (\overline{p})^{(I_k)}$.

In [15], it has been proved that R is a strongly Gorenstein hereditary ring if and only if every ideal of R is Gorenstein projective and the class of strongly Gorenstein projective modules is closed under extensions.

Then one can ask the question: R is a 2-strongly Gorenstein hereditary ring if and only if every ideal of R is Gorenstein projective and the class of 2-strongly Gorenstein projective modules is closed under extensions. Here, we give an affirmative answer.

We begin with the following:

Lemma 2.7. *Let R be a ring. If the class of 2-strongly Gorenstein projective modules is closed under extensions, then it is projectively resolving.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence such that B, C are 2-strongly Gorenstein projective. Since B is 2-strongly Gorenstein projective, by [3, Lemma 2.3], there exists a short exact sequence: $0 \to B \to P \to G \to 0$ such that P is projective and G is 2-strongly Gorenstein projective. So we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since B and C are 2-strongly Gorenstein projective, it can be seen from the right vertical sequence that T is also 2-strongly Gorenstein projective. Now applying [3, Lemma 2.3] to the middle horizontal sequence, we get that A is 2-strongly Gorenstein projective. □

Theorem 2.8. *Let R be a ring. Then R is a 2-strongly Gorenstein hereditary ring if and only if every ideal of R is Gorenstein projective and the class of 2-strongly Gorenstein projective modules is closed under extensions.*

Proof. If R is 2-strongly Gorenstein hereditary, then every ideal, as a submodule of R, must be Gorenstein projective. Also any Gorenstein projective module, as a submodule of some projective module, must be 2-strongly Gorenstein projective. So every Gorenstein projective module is 2-strongly Gorenstein projective. Therefore, the class of 2-strongly Gorenstein projective modules is closed under extension.

Conversely, if every ideal of R is Gorenstein projective, then R is a Gorenstein hereditary ring by [17, Theorem 1.2]. If the class of 2-strongly Gorenstein projective modules is closed under extensions, by Lemma 2.7, it is projectively resolving. Also notice that the class of 2-strongly Gorenstein projective modules is closed under countable direct sums, and so direct summands by [14, Proposition 1.4]. Since any Gorenstein projective module is a direct summand of some 2-strongly Gorenstein projective module (because every strongly Gorenstein projective module is 2-strongly Gorenstein projective), it follows that every Gorenstein-projective module is 2-strongly Gorenstein projective, therefore R is strongly Gorenstein-hereditary. □

By [16], the 2-strongly Gorenstein projective dimension of M as an R-module, denoted by 2-SGpd$_R(M)$, is as follows:

2-SGpd$_R(M)$ = $\inf\{k \mid \exists$ an exact sequence $0 \to G_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ of R-modules such that P_i are projective and G_k is 2-strongly Gorenstein projective$\}$.

If there is no such an exact sequence, we say that the 2-strongly Gorenstein projective dimension of M as an R-module is infinite, i.e., 2-SGpd$_R(M) = \infty$.

Theorem 2.9. *Let R be a ring such that the class of 2-strongly Gorenstein projective modules is closed under extensions, and let x be an annihilator of M. Then* 2-SGpd$_R(M)$ = 2-SGpd$_{R/(x)}(M) + 1$.

Proof. From the above theorem, we see that if the class of 2-strongly Gorenstein projective modules is closed under extensions, then the class of G-projective modules is just the class of 2-strongly Gorenstein projective modules. By [8], we get that 2-SGpd$_R(M)$=2-SGpd$_{R/(x)}(M)$+1. □

Corollary 2.10. *Let R be a ring such that the class of 2-strongly Gorenstein projective modules is closed under extensions, and let x be a non-zero-divisor and non-unit element in R, denote R/xR by $\overline{R}$. Then R is 2-strongly Gorenstein hereditary if and only if $\overline{R}$ is a 2-SG-semisimple ring.*

The above conditions "the class of 2-strongly Gorenstein projective modules is closed under extensions" is necessary. Otherwise, $\overline{R}$ is not necessarily a 2-strongly Gorenstein semisimple ring, as shown by the following example.

Example 2.11 ([18])**.** Let $R = \mathbb{Q} + x^2\mathbb{Q}[x]$ and $\overline{R} = \frac{\mathbb{Q}+x^2\mathbb{Q}[x]}{x^4}$. Then R is 2-strongly Gorenstein hereditary, but $\overline{R}$ is not a 2-strongly Gorenstein semisimple ring.

Proof. To prove this fact, we just notice that

$$R = \mathbb{Q} + x^2\mathbb{Q}[x] \cong D/(y^3 - z^2),$$

where $D = \mathbb{Q}[y, z]$ and gl.dim$D$=2. By [17, Example 1] and[16, Theorem 3.15], R is 2-strongly Gorenstein hereditary.

On the other hand, we see that

$$\overline{R} = \{q_0 + q_1x^2 + q_2x^3 + q_3x^5 \mid q_0, q_1, q_2, q_3 \in \mathbb{Q}, x^4 = x^6 = x^7 = \cdots = x^n = \cdots = 0\}.$$

This is an Artinian local ring with the maximal ideal $M = (x^2, x^3)$ and $M^2 = (x^5)$. Since the ideal M is not principal, by [5, Corollary 2.7], $\overline{R}$ is not a 2-strongly Gorenstein semisimple ring. □

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