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ON THE p-ADIC VALUATION OF GENERALIZED HARMONIC NUMBERS

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Abstract. For any prime number p, let $J(p)$ be the set of positive integers n such that the numerator of the n^{th} harmonic number in the lowest terms is divisible by this prime number p. We consider an extension of this set to the generalized harmonic numbers, which are a natural extension of the harmonic numbers. Then, we present an upper bound for the number of elements in this set. Moreover, we state an explicit condition to show the finiteness of our set, together with relations to Bernoulli and Euler numbers.

1. Introduction

The harmonic numbers are defined as the partial sums of the harmonic series. We denote the n^{th} harmonic number by

$$h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

These numbers have been studied in many aspects and we progress toward a natural generalization of them.

In 1991, Eswarathasan and Levine [5] introduced a subset of positive integers denoted by $J(p)$ for any prime number p. The set $J(p)$ consists of positive integers n in which the numerator of the corresponding harmonic number h_n in the lowest terms is divisible by the prime number p. They conjectured that the set is finite for any prime p. In particular, they computed the elements of $J(p)$ for primes $p \leq 7$.

Later on, Boyd [3] worked on the set $J(p)$ and computed its number of elements for primes $p < 550$ except for $p \in \{83, 127, 397\}$.

Then, Sanna [19] showed for any prime p and $x \geq 1$ that

$$|J(p) \cap [1, x]| < 129 p^{\frac{2}{3}} x^{0.765}.$$

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Moreover, the upper bound was improved by Wu and Chen [22] to $3x^{\frac{2}{3}+\frac{1}{25\log p}}$. However, it is not known whether $J(p)$ is finite or not, so the conjecture is still open.

Our main purpose in this note is to generalize the set $J(p)$ to *generalized harmonic numbers*. Then, we aim to give an upper bound for the number of elements in the corresponding set. The n^{th} generalized harmonic number of order s, denoted by $H_n^{(s)}$, is defined as

$$H_n^{(s)} := \sum_{k=1}^{n} \frac{1}{k^s}.$$

Note that these numbers are the partial sums of the Riemann zeta function $\zeta(s)$ for $s > 1$. Moreover, by the well known fact

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < 2,$$

we can conclude that the generalized harmonic numbers are not integers except 1.

Given a rational number $\frac{a}{b}$, we will write $p \mid \frac{a}{b}$ to mean that p divides the numerator of the number $\frac{a}{b}$ in its lowest terms. Then, we define our generalizations of $J(p)$ as follows in Section 2. For any prime p and positive integer s, we define

$$J(p,s) := \{n \in \mathbb{N} : p \mid H_n^{(s)}\} \text{ and } J(p^s,s) := \{n \in \mathbb{N} : p^s \mid H_n^{(s)}\}.$$

Note that the sets are indeed a generalization of $J(p)$ as when $s = 1$, one has that

$$J(p,1) = J(p).$$

The relation between these sets will be given in Proposition 2.3, and our attention will be on the set $J(p,s)$.

A generalization of $J(p)$ was also given in [10] for *hyperharmonic numbers*. They were defined in [4] recursively as

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}$$

for $r \geq 2$ such that $h_n^{(1)} = h_n$. These numbers also come with numerous combinatorial properties. For instance, they tend to be non-integer, as the number of pairs (n,r) where the corresponding hyperharmonic number is non-integer have the full asymptotic in the first quadrant [1,9]. However, it has been recently shown that in fact there are infinitely many hyperharmonic integers [20].

For the corresponding generalization of $J(p)$, namely for the set

$$J_p^{(r)} = \{n \in \mathbb{N} : h_n^{(r)} \equiv 0 \pmod{p}\},$$

it was proven in some cases that the set is infinite contrary to the conjecture. On the other hand, in some cases, it was shown that the set is finite [10]. We also encourage readers to see [23] for generalizations of $J(p)$ and to see [12] for *generalized hyperharmonic numbers*, which is an extension of both generalized harmonic and hyperharmonic numbers.

Throughout the paper, p will always denote a prime number and for any rational numbers a, b, we mean by $a \equiv b \pmod{m}$ that the numerator of $a-b$ in the lowest terms is divisible by m. In particular, we will work with the equivalences by taking the modulo m, a prime p or a prime power p^k. It is known by [2] that

$$h_{p-1} \equiv 0 \pmod{p}$$

whenever $p \geq 3$. The equivalence was shown for modulo p^2 and primes $p \geq 5$ in [21]. We will give an analogous result for generalized harmonic numbers in Section 2 (see [15] for a survey on the generalizations of the congruence).

Moreover, we use the p-adic valuation, or the p-adic order, ν_p defined as follows. For any integer n and a prime p, we have

$$\nu_p(n) = \begin{cases} a & \text{if } p^a \parallel n, \\ \infty & \text{if } n = 0 \end{cases}$$

and for any rational number $\frac{a}{b}$, we set

$$\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b).$$

In Section 3, we let

$$J(p,s)(x) = J_{p,s}(x) = |\{n \leq x : n \in J(p,s)\}|$$

for any positive real number x and then, construct the proof of our first theorem.

Theorem A. *Let p be any prime number, s be any positive integer and $x \geq 1$ be any real number. Then, we have*

$$J_{p,s}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}.$$

In particular, whenever $p > se^{\frac{3}{25}}$ holds, we have

$$J_{p,s}(x) = o(x).$$

Notice here that as the integer s increases, the primes p must increase to save the condition $J_{p,s}(x) = o(x)$. For instance, when $s = 10000$, the primes p must be strictly greater than 11273. For $s = 100000$, the primes p must be strictly greater than 112741.

In Section 4, we show in some cases that $J(p,s)$ is finite by considering the position of a given integer n between the powers of the prime p. In particular, our result shows that

$$J(p,s) \subseteq \{1, \ldots, p-1\}.$$

This is our second theorem.

Theorem B. *Let p be a prime number and $s \geq 2$ be a positive integer with $p-1 \nmid s$.*

If the inequality

$$\nu_p\left(H_r^{(s)}\right) \leq s-1$$

holds for all $1 \leq r \leq p-1$, then $J(p,s)$ only consists of elements from the set $\{1,\ldots,p-1\}$. In particular, $J(p,s)$ is finite.

At this point, we can present a table for the number of elements in the set $J(p,s)$ for some values of s and p which is obtained using [18]. A more detailed table will be given in Section 5 as well.

TABLE 1. The number of elements in the set $J(p,s)$ for several p and s values.

s \ p	2	3	5	7	11	13	17	19
1	0	3	3	13	638	3	3	19
2	0	0	2	3	2	2	2	2
3	0	1	1	1	3	1	3	1
4	0	0	0	2	2	2	4	2
5	0	1	1	1	3	1	1	1
6	0	0	2	0	2	4	2	2
7	0	1	1	1	1	3	3	3
8	0	0	0	2	2	2	4	4
9	0	1	1	1	1	1	4	5
10	0	0	2	2	0	2	2	2

Finally, in Section 5, we briefly discuss the case where the condition in Theorem B fails. We present some examples which indicate some of those cases, and the following proposition is our last result.

Proposition C. *We have*

$$J(7,2)=\{3,6,26\} \text{ and } J(37,3)=\{4,13,23,32,36,1340,1360\}.$$

Next, we continue our investigation with Bernoulli and Euler numbers. The k^{th} Bernoulli number B_k is defined via the identity

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \qquad \text{for } k=0,1,2,\ldots$$

and the Euler numbers E_n are defined recursively as (see [17])

$$E_0 = 1, \quad \sum_{\substack{0 \le i \le n \\ i:\text{even}}} \binom{n}{i} E_{n-i} = 0$$

for $n \in \mathbb{Z}^{>0}$. Then, we introduce the *irregular primes* and using some congruence relations, we shed some light on the failed cases.

Short outline of the paper: We introduce the sets $J(p, s)$ and $J(p^s, s)$ in Section 2. In Section 3, we construct the counting arguments for the number of elements in $J(p, s)$ and prove Theorem A. In Section 4, we prove Theorem B and in Section 5, we prove Proposition C and continue with the concluding remarks.

2. Properties of $J(p, s)$ and $J(p^s, s)$

We begin with a crucial congruence which will be frequently used throughout the paper. The result was first shown in [6] and we omit the proof (see also [11]).

Proposition 2.1. *Let p be a prime number. Then, we have*

$$H_{p-1}^{(s)} \equiv 0 \pmod{p}$$

whenever $p - 1 \nmid s$.

Now, we show that if we have an element of $J(p, s)$, then dividing it by p yields another element of the set.

Lemma 2.2. *Let p be a prime and n be an element of $J(p, s)$. Write $n = pm+r$ for some integers $0 \le r \le p-1$ and $m > 0$. Then, m belongs to $J(p, s)$ whenever $p - 1 \nmid s$. In particular, $\nu_p(H_m^{(s)}) \ge s$ holds.*

Proof. Suppose that $n = pm + r$ in $J(p, s)$ for some m, r as above. We know by Proposition 2.1 that whenever $p - 1 \nmid s$, we have

$$H_{p-1}^{(s)} = 1 + \frac{1}{2^s} + \cdots + \frac{1}{(p-1)^s} \equiv 0 \pmod{p}.$$

Now, if we write $H_n^{(s)} = H_{pm+r}^{(s)}$ as

$$\begin{aligned} H_n^{(s)} = 1 &+ \frac{1}{2^s} + \cdots + \frac{1}{(p-1)^s} + \frac{1}{p^s} \\ &+ \frac{1}{(p+1)^2} + \frac{1}{(p+2)^2} + \cdots + \frac{1}{(2p-1)^s} + \frac{1}{(2p)^2} \\ &\vdots \\ &+ \frac{1}{(pm-p+1)^s} + \frac{1}{(pm-p+2)^s} + \cdots + \frac{1}{(pm)^s} \end{aligned}$$

$$+\frac{1}{(pm+1)^s}+\frac{1}{(pm+2)^s}+\cdots+\frac{1}{(pm+r)^s},$$

then we see that

$$H_n^{(s)} \equiv H_r^{(s)} + \frac{1}{p^s}H_m^{(s)} \equiv 0 \pmod p. \tag{1}$$

In other words, $\nu_p\left(H_r^{(s)}+\frac{1}{p^s}H_m^{(s)}\right) \geq 1$. However, as $0 \leq r \leq p-1$, we have $\nu_p(H_r^{(s)}) \geq 0$. Therefore, if $\nu_p\left(\frac{1}{p^s}H_m^{(s)}\right) < 0$ holds, then we would have $\nu_p\left(H_r^{(s)}+\frac{1}{p^s}H_m^{(s)}\right) < 0$ by the properties of the p-adic order. Thus, $\nu_p\left(\frac{1}{p^s}H_m^{(s)}\right) = \nu_p\left(H_m^{(s)}\right) - s \geq 0$ so that $\nu_p(H_m^{(s)}) \geq s$, which completes the proof. □

Proposition 2.3. *Let p be a prime. Then, $J(p^s, s)$ is finite if and only if $J(p, s)$ is finite, whenever $p-1 \nmid s$.*

Proof. If $J(p, s)$ is finite, then we can directly say that the set $J(p^s, s)$ is finite. On the contrary, suppose that $\{n : p^s \mid H_n^{(s)}\}$ is finite but $J(p, s)$ is not. Let us write $J(p^s, s) = \{n : p^s \mid H_n^{(s)}\} = \{n_1 < \cdots < n_c\}$ and let m be an element of $J(p, s)$ so that $m > p(n_c+1)$. Then, we can write $m = pn+r$ with $0 \leq r \leq p-1$. In addition, by Lemma 2.2, we have that

$$H_m^{(s)} \equiv \frac{1}{p^s}H_n^{(s)} + H_r^{(s)} \equiv 0 \pmod p$$

with $0 \leq r \leq p-1$ and $\nu_p(H_n^{(s)}) \geq s$. Thus, $n \in \{n : p^s \mid H_n^{(s)}\} = J(p^s, s)$ but $n > n_c$ since

$$n = \frac{m-r}{p} \geq \frac{m-p+1}{p} > \frac{p(n_c+1)-p+1}{p} = \frac{pn_c+1}{p} > n_c,$$

contradicting the finiteness of $J(p^s, s)$. □

3. Proof of Theorem A

Suppose for the rest of the paper that $p-1 \nmid s$. In this section, we give an upper bound for the counting function $J(p,s)(x) = J_{p,s}(x)$ that we define as

$$J(p,s)(x) = |\{n \leq x : n \in J(p,s)\}| \tag{2}$$

for any positive real number x. Then, let us define inductively the sets

$${J_{p,s}}^{(1)} := \{1 \leq n \leq p-1 : p \mid H_n^{(s)}\},$$

$${J_{p,s}}^{(k+1)} := \{pn+r \in J(p,s) : n \in J_{p,s}^{(k)},\ 0 \leq r \leq p-1\} \text{ for } k = 1, 2, \ldots.$$

Lemma 3.1. *We have $J_{p,s}^{(k)} = J(p,s) \cap [p^{k-1}, p^k - 1]$ for any positive integer k.*

Proof. The proof can be directly given by induction on k. In fact, for $k = 1$, we have

$$J_{p,s}^{(1)} = \{1 \le n \le p-1 : p \mid H_n^{(s)}\} = J(p,s) \cap [1, p-1].$$

Now, suppose that the claim is true for any $i \le k$ and let us show that

$$J_{p,s}^{(k+1)} = J(p,s) \cap [p^k, p^{k+1} - 1].$$

To begin with, if $m \in J_{p,s}^{(k+1)}$, then $m = pn + r \in J(p,s)$ for some integers $n \in J_{p,s}^{(k)}$ and $0 \le r \le p-1$. Moreover, we have $n \in J_{p,s}^{(k)} = J(p,s) \cap [p^{k-1}, p^k - 1]$, which implies that $pn \in [p^k, p^{k+1} - p]$. Hence,

$$p^k \le m = pn + r \le p^{k+1} - p + p - 1 = p^{k+1} - 1$$

so that $m \in [p^k, p^{k+1} - 1]$.

On the other hand, if $m \in J(p,s) \cap [p^k, p^{k+1} - 1]$, then we can write $m = pn + r$ for some positive integer n and an integer $0 \le r \le p-1$. Also, by Lemma 2.2, we get $n \in J(p,s)$. Now, if we show that $n \in [p^{k-1}, p^k - 1]$, then we are done. However, since

$$p^k \le m = pn + r \le p^{k+1} - 1$$

holds, we obtain that $n \in [p^{k-1}, p^k - 1]$. Hence, we have $n \in J_{p,s}^{(k)}$, and the proof is done. $\square$

In conclusion, we can partition $J(p,s)$, which will ease the upcoming counting arguments.

Corollary 3.2. $J(p,s) = \bigcup_{k=1}^{\infty} J_{p,s}^{(k)}$.

Now, we are ready to give an upper bound for the elements in a short interval belonging to the set $J(p,s)$. To do so, we will generalize the techniques from [19] to the generalized harmonic numbers.

Lemma 3.3. *Let p be a prime, $p-1 \nmid s$ and x, y be real numbers with $1 \le y < p$. Suppose that*

$$|J(p,s) \cap [x, x+y]| = \{n_1 < \cdots < n_t\}.$$

Then, for any integer $d \ge 1$ we have

$$|\{i : n_{i+1} - n_i = d\}| \le s(d-1).$$

Proof. For each $i = 1, \ldots, t-1$, let us set $d_i = n_{i+1} - n_i$. Notice that

$$\begin{aligned} H_{n_{i+1}}^{(s)} - H_{n_i}^{(s)} &= \frac{1}{(n_i+1)^s} + \cdots + \frac{1}{(n_{i+1})^s} \\ &= \frac{1}{(n_i+1)^s} + \cdots + \frac{1}{(n_i+d_i)^s} \\ &\equiv 0 \pmod{p}. \end{aligned}$$

Now, let $f_d(x) = (x+1)(x+2)\cdots(x+d)$ for any $d \ge 1$. Taking the logarithm of both sides, one has that

$$\log(f_d(x)) = \log((x+1)(x+2)\cdots(x+d)).$$

Then, if we differentiate both sides, we obtain that

$$\frac{f_d'(x)}{f_d(x)} = \frac{1}{x+1} + \cdots + \frac{1}{x+d}.$$

Next, define $g_d(x)$ to be $\frac{f_d'(x)}{f_d(x)}$. It can be shown that

$$g_d^{(k)}(x) = (-1)^k k! \sum_{i=1}^{d} (x+i)^{-(k+1)}$$

for $k \geq 0$. Therefore, for any positive integer s, we have

$$g_d^{s-1}(x) = (-1)^{s-1}(s-1)! \left(\frac{1}{(x+1)^s} + \cdots + \frac{1}{(x+d)^s} \right)$$

so that

$$\begin{aligned} g_{d_i}^{(s-1)}(n_i) &= (-1)^{s-1}(s-1)! \left(\frac{1}{(n_i+1)^s} + \cdots + \frac{1}{(n_i+d_i)^s} \right) \\ &= (-1)^{s-1}(s-1)! \big(H_{n_i+1}^{(s)} - H_{n_i}^{(s)} \big). \end{aligned}$$

Also, since $H_{n_i+1}^{(s)} - H_{n_i}^{(s)} \equiv 0 \pmod p$, we obtain that $g_{d_i}^{(s-1)}(n_i) \equiv 0 \pmod p$. Notice that the numerator of the function $g_{d_i}^{(s-1)}(x)$ is a polynomial of degree $s(d_i - 1)$. Thus, $g_{d_i}^{(s-1)}(x)$ has at most $s(d_i - 1)$ many solutions modulo p. Hence, by taking $d = d_i$, we conclude the result. □

Now, we generalize the argument given in [22] to the generalized harmonic numbers, which we will use for counting the number of elements in short intervals lying in our set.

Lemma 3.4. *Suppose that $y \geq \frac{8}{3}$ is a real number, s is a positive integer and $a_1, \ldots, a_t$ are integers satisfying $0 \leq a_k \leq s(k-1)$ for $k = 1, \ldots, t$. Suppose also that*

$$\sum_{k=1}^{t} k a_k \leq y.$$

Then, we have

$$1 + \sum_{k=1}^{t} a_k \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Proof. First, if $\frac{y}{s} \geq \frac{8}{3}$, then setting y to be $\frac{y}{s}$ and a_k's to be $\frac{a_k}{s}$ yields the result by [22]. However, our objective will be bounding y by a prime number p to count the number of elements of $J(p, s)$ in short intervals. Therefore, we will prove the general case.

To begin with, if $a_1 + \cdots + a_t \leq 1$, then we have

$$1 + \sum_{k=1}^{t} a_k \leq 2 \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}$$

as $y \geq \frac{8}{3}$ and $s \geq 1$. Therefore, we may assume that $a_1 + \cdots + a_t \geq 2$.

Now, we are going to rewrite each pair (a_i, a_{i+1}) of integers as follows. If $a_{k_0} < s(k_0 - 1)$ and $a_{k_0+1} > 0$, then replace a_{k_0} with $a'_{k_0} = a_{k_0} + 1$ and a_{k_0+1} by $a'_{k_0+1} = a_{k_0+1} - 1$ and leave the numbers as they were, otherwise. Then, for these new integers $a'_1, \ldots, a'_t$, we have

$$0 \leq a'_k \leq s(k-1)$$

with $k = 1, \ldots, t$.

Notice that for any pair (a_k, a_{k+1}), one has

$$a'_k + a'_{k+1} = a_k + 1 + a_{k+1} - 1 = a_k + a_{k+1}$$

and also

$$\begin{aligned} ka'_k + (k+1)a'_{k+1} &= ka_k + k + (k+1)a_{k+1} - (k+1) \\ &= ka_k + (k+1)a_{k+1} - 1 \\ &< ka_k + ka_{k+1}. \end{aligned}$$

Therefore, we obtain

$$\sum_{k=1}^{t} ka'_k < \sum_{k=1}^{t} ka_k \leq y \tag{3}$$

and

$$\sum_{k=1}^{t} a'_k = \sum_{k=1}^{t} a_k.$$

Continuing the process, we obtain $w \leq t$ integers $b_1, \ldots, b_w$ with

$$b_k = s(k-1) \text{ for } 1 \leq k < w, \text{ and } 1 \leq b_w \leq s(w-1)$$

such that

$$\sum_{k=1}^{w} b_k = \sum_{k=1}^{t} a_k.$$

Furthermore, we can write via (3) that

$$\sum_{k=1}^{w} kb_k \leq y.$$

Notice that this process generates an increasing sequence of non-negative integers that increases by s at each step. In the end, we only consider non-zero elements of the sequence and know the exact values of those, except for the last one, b_w.

Now, we have

$$y \geq \sum_{k=1}^{w} kb_k = wb_w + \sum_{k=1}^{w-1} kb_k = wb_w + \sum_{k=1}^{w-1} k(s(k-1)) = wb_w + s\sum_{k=1}^{w-1}(k^2 - k)$$

such that

$$y \geq s\left(\frac{w(w-1)(w-2)}{3}\right) + wb_w. \tag{4}$$

Here, (4) implies that

$$s\,(w(w-1)(w-2)) + 3wb_w \leq 3y,$$

and consequently

$$(sw(w-1)(w-2) + 3wb_w)^{2/3}\, s^{1/3} \leq (3y)^{2/3} s^{1/3} \tag{5}$$

holds. Furthermore, we have

$$\begin{aligned}\sum_{k=1}^{t} a_k = \sum_{k=1}^{w} b_k = b_w + \sum_{k=1}^{w-1} b_k = b_w + \sum_{k=1}^{w-1} s(k-1)\\ = b_w + \left(s\sum_{k=1}^{w-1} k - 1\right) = s\left(\frac{(w-1)(w-2)}{2}\right) + b_w.\end{aligned}$$

Next, since $1 \leq b_w \leq w-1$ and $\sum_{k=1}^{t} a_k \geq 2$, we have

$$\left(s\frac{(w-1)(w-2)}{2} + b_w\right) \geq 2$$

so that $w \geq 3$. Now, we consider two cases, namely, when $b_w = 1$ and $b_w \geq 2$.

Case i) $b_w = 1$.

Observe that (5) can be written as

$$(sw(w-1)(w-2) + 3w)^{2/3}\, s^{1/3} \leq (3y)^{2/3} s^{1/3},$$

and if

$$\begin{aligned}2\left(1 + \sum_{k=1}^{t} a_k\right) &= 2\left[1 + \left(s\frac{(w-1)(w-2)}{2} + 1\right)\right]\\ &= s(w-1)(w-2) + 4\\ &\leq \left(sw(w-1)(w-2) + 3w\right)^{2/3} s^{1/3}\end{aligned}$$

is satisfied, then we are done. In other words, if

$$(s(w-1)(w-2) + 4)^3 \leq s\,(sw(w-1)(w-2) + 3w)^2 \tag{6}$$

holds, then we are done. Here, when $s = 1$, we get that

$$((w-1)(w-2) + 4)^3 \leq (w(w-1)(w-2) + 3w)^2\,.$$

However, the inequality is satisfied for any $w \geq 3$ as it can be checked from the proof of [22, Lemma 2.3].

Hence, we can assume $s \geq 2$ for the rest of the proof. Next, let us set

$$v = (w-1)(w-2)$$

so that $v \geq 2$ as $w \geq 3$. Therefore, the inequality (6) is written as

$$s(swv+3w)^2-(sv+4)^3 \geq 0.$$

Consequently, our aim is now to show that

$$sw^2(sv+3)^2-(sv+4)^3 \geq 0.$$

However, for the left-hand side, we have

$$\begin{aligned}
&sw^2(sv+3)^2-(sv+4)^3\\
&= sw^2(s^2v^2+6sv+9)-(s^3v^3+12s^2v^2+48sv+64)\\
&= (s^3w^2v^2+6s^2w^2v+9sw^2)-(s^3v^3+12s^2v^2+48sv+64)\\
&= (s^3w^2v^2-s^3v^3)+(6s^2w^2v+9sw^2-12s^2v^2-48sv-64)\\
&= (3s^3v^2w-2s^3v^2)+\big(-12s^2v^2+(6s^2w^2v-48sv)+(9sw^2-64)\big)\\
&= \big(s^2v^2(3sw-2s-12)\big)+\big((6s^2w^2v-48sv)+(9sw^2-64)\big)\\
&= \left(s^2v^2\underbrace{(3sw-2s-12)}_{\geq 0}\right)+\left(6sv\underbrace{(sw^2-8)}_{\geq 0}+\underbrace{(9sw^2-64)}_{\geq 0}\right) \geq 0
\end{aligned}$$

since $s \geq 2, w \geq 3$ and $v \geq 2$.

Case ii) $b_w \geq 2$.

First, assume that $w \geq \left(\frac{3y}{s}\right)^{\frac{1}{3}}$ holds. We know by (4) that

$$y \geq s\left(\frac{w(w-1)(w-2)}{3}\right)+wb_w = B.$$

Therefore,

$$\begin{aligned}
1+\sum_{k=1}^{t} a_k &= \frac{s}{2}(w-1)(w-2)+b_w+1\\
&= \frac{3}{w}\frac{w}{3}\left(\frac{s}{2}(w-1)(w-2)\right)+b_w+1\\
&= \frac{3}{2w}\left(\frac{sw(w-1)(w-2)}{3}+\frac{2}{3}wb_w+\frac{2}{3}w\right)\\
&= \frac{3}{2w}(B-\frac{w}{3}b_w+\frac{2}{3}w)\\
&\leq \frac{3}{2w}(B-\frac{2}{3}w+\frac{2}{3}w) \quad \text{as } b_w \geq 2\\
&\leq \frac{3y}{2w}.
\end{aligned}$$

Hence, we get

$$1+\sum_{k=1}^{t} a_k \le \frac{3y}{2w} \le \frac{3}{2}\frac{y}{(\frac{3y}{s})^{\frac{1}{3}}} = \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Finally, consider the case when $w < \left(\frac{3y}{s}\right)^{\frac{1}{3}}$. Recall that $w \ge 3$ and $b_w \le s(w-1)$ holds. Thus, we get that

$$\begin{aligned} 1+\sum_{k=1}^{t} a_k &= \frac{s}{2}(w-1)(w-2) + b_w + 1 \\ &\le \frac{s}{2}(w-1)(w-2) + s(w-1) + 1 \\ &= \frac{1}{2}(sw^2) - \frac{sw}{2} + 1 \\ &\le \frac{1}{2}sw^2 \\ &< \frac{1}{2}s\left(\frac{3y}{s}\right)^{\frac{2}{3}} \\ &= \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}} \end{aligned}$$

and the proof is complete. □

Lemma 3.5. *For any real numbers x, y and for any prime number p with $\frac{8}{3} \le y < p$ and for any positive integer s, we have*

$$|J(p,s) \cap [x, x+y]| \le \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Proof. We can write

$$|J(p,s) \cap [x, x+y]| = \{n_1 < \cdots < n_t\}$$

so that if we set

$$a_k = |\{1 \le i \le t-1 : n_{i+1} - n_i = k\}|,$$

then $0 \le a_k \le s(k-1)$ holds by Lemma 3.3. Moreover, we have

$$\sum_k k a_k = \sum_{i=1}^{t-1} (n_{i+1} - n_i) \le y$$

such that the assumptions of Lemma 3.4 are satisfied. Finally, we have that

$$|J(p,s) \cap [x, x+y]| = t = 1 + \sum_k a_k \le y$$

and the proof is done by Lemma 3.4. □

Now, we are set to prove our first result, Theorem A.

Theorem A. *Let p be any prime number, s be any positive integer and $x \geq 1$ be any real number. Then, we have*

$$J_{p,s}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}.$$

In particular, whenever $p > se^{\frac{3}{25}}$ holds, we have

$$J_{p,s}(x) = o(x).$$

Proof of Theorem A. To begin with, for $s = 1$, we have $J(p,s) = J(p,1) = J(p)$ such that the result follows from [22]. Also, we can omit the cases $p = 2, 3, 5$ from Table 1, so let $p \geq 7$, $s > 1$ and set

$$A = \left(\frac{9}{8}\right)^{\frac{1}{3}} (p-1)^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Then, by the definition of $J_{p,s}^{(1)}$ and by Lemma 3.5, we can write that

$$\left| J_{p,s}^{(1)} \right| = \left| J(p,s) \cap [1, p-1] \right| \leq A.$$

In fact, by Lemma 3.1, we write

$$\left| J_{p,s}^{(k+1)} \right| = \sum_{n \in J_{p,s}^{(k)}} |J(p,s) \cap [pn, pn+p-1]| \leq \left| J_{p,s}^{(k)} \right| A.$$

As a conclusion, for any positive integer k, we get

$$\left| J_{p,s}^{(k)} \right| \leq A^k.$$

Next, we start our work on $J_{p,s}(x)$. One can find a positive integer m satisfying

$$p^{m-1} \leq x < p^m.$$

Then, we write

$$J_{p,s}(x) = J_{p,s}(p^{m-1}-1) + |J(p,s) \cap [p^{m-1}, x]| \tag{7}$$

and work with the summands separately. First, we have

$$\begin{aligned} J_{p,s}(p^{m-1}-1) &= \sum_{k=1}^{m-1} \left| J(p,s) \cap [p^{k-1}, p^k - 1] \right| \\ &= \sum_{k=1}^{m-1} \left| J_{p,s}^{(k)} \right| \\ &\leq \sum_{k=1}^{m-1} A^k < \frac{A}{A-1} A^{m-1} \qquad (8) \end{aligned}$$

by Lemma 3.1. Next, for $|J(p,s)\cap[p^{m-1},x]|$, we have

$$\begin{aligned}|J(p,s)\cap[p^{m-1},x]| &\le \sum_{\substack{n\in J_{p,s}^{(m-1)}\\ pn\le x}} |J(p,s)\cap[pn,pn+p-1]|\\ &\le A\sum_{\substack{n\in J_{p,s}^{(m-1)}\\ pn\le x}} 1\\ &= A\left|J(p,s)\cap\left[p^{m-2},\frac{x}{p}\right]\right|.\end{aligned}$$

In addition, observe that we have

$$A\left|J(p,s)\cap\left[p^{m-2},\frac{x}{p}\right]\right| \le A^2\left|J(p,s)\cap\left[p^{m-3},\frac{x}{p^2}\right]\right|$$

and if we continue in this manner, we obtain that

$$\begin{aligned}|J(p,s)\cap[p^{m-1},x]| &\le A\left|J(p,s)\cap\left[p^{m-2},\frac{x}{p}\right]\right|\\ &\le A^{m-1}\left|J(p,s)\cap\left[1,\frac{x}{p^{m-1}}\right]\right|. \qquad (9)\end{aligned}$$

For $\left|J(p,s)\cap\left[p^1,\frac{x}{p^{m-1}}\right]\right|$, if $x<3p^{m-1}$ holds, then we get

$$\left|J(p,s)\cap\left[1,\frac{x}{p^{m-1}}\right]\right| \le 1 \le \left(\frac{9}{8}\right)^{\frac13}\left(\frac{x}{p^{m-1}}\right)^{\frac23}s^{\frac13}.$$

If $x\ge 3p^{m-1}$ holds, then by Lemma 3.5, we have

$$\left|J(p,s)\cap\left[1,\frac{x}{p^{m-1}}\right]\right| \le \left(\frac{9}{8}\right)^{\frac13}\left(\frac{x}{p^{m-1}}\right)^{\frac23}s^{\frac13}.$$

Therefore, (9) can be written as

$$\begin{aligned}|J(p,s)\cap[p^{m-1},x]| &\le A^{m-1}\left|J(p,s)\cap\left[1,\frac{x}{p^{m-1}}\right]\right|\\ &\le A^{m-1}\left(\frac{9}{8}\right)^{\frac13}\left(\frac{x}{p^{m-1}}\right)^{\frac23}s^{\frac13}. \qquad (10)\end{aligned}$$

Now, by (8) and (10), we write for (7) that

$$\begin{aligned}J_{p,s}(x) &= J_{p,s}(p^{m-1}-1)+|J(p,s)\cap[p^{m-1},x]|\\ &\le \frac{A}{A-1}A^{m-1}+A^{m-1}\left(\frac{9}{8}\right)^{\frac13}\left(\frac{x}{p^{m-1}}\right)^{\frac23}s^{\frac13}\\ &\le 3A^{m-1}s^{\frac13}\left(\frac{x}{p^{m-1}}\right)^{\frac23}\end{aligned}$$

$$= 3\left(\left(\frac{9}{8}\right)^{\frac{1}{3}}(p-1)^{\frac{2}{3}}s^{\frac{1}{3}}\right)^{m-1} s^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}$$
$$= 3\left(\frac{9}{8}\right)^{\frac{m-1}{3}} s^{\frac{m-1}{3}} s^{\frac{1}{3}} x^{\frac{2}{3}}$$
$$= 3\left(\frac{9}{8}\right)^{\frac{m-1}{3}} s^{\frac{m}{3}} x^{\frac{2}{3}} \leq 3\left(e^{\frac{1}{25}}\right)^{m-1} s^{\frac{m}{3}} x^{\frac{2}{3}}.$$

Notice that the inequality $p^{m-1} \leq x < p^m$ yields that

$$m-1 \leq \frac{\log x}{\log p}. \tag{11}$$

Hence,

$$J_{p,s}(x) \leq 3\left(e^{\frac{1}{25}}\right)^{\frac{\log x}{\log p}} s^{\frac{m}{3}} x^{\frac{2}{3}} = 3\left(x^{\frac{1}{25\log p}}\right) s^{\frac{m}{3}} x^{\frac{2}{3}} = 3x^{\frac{2}{3}+\frac{1}{25\log p}} s^{\frac{m}{3}}. \tag{12}$$

Next, we work on $s^{\frac{m}{3}}$ to complete the proof. First, we write

$$s^{\frac{m-1}{3}} \leq s^{\frac{\log x}{3\log p}}$$

such that

$$s^{\log x} = e^{\log\left(s^{\log x}\right)} = e^{\log x \log s} = x^{\log s}.$$

Thus, we have

$$s^{\frac{\log x}{3\log p}} = x^{\frac{\log s}{3\log p}}$$

which yields that

$$s^{\frac{m-1}{3}} \leq x^{\frac{\log s}{3\log p}}. \tag{13}$$

On the other hand, for $s^{\frac{1}{3}}$ we have

$$s^{\frac{1}{3}} = x^{\frac{\log s}{3\log x}}. \tag{14}$$

Now, if we combine (13) and (14), we can write

$$s^{\frac{m}{3}} \leq x^{\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}.$$

Then, feeding our result into (12) we obtain the upper bound for our set as

$$J_{p,s}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}} x^{\frac{\log s}{3\log p}+\frac{\log s}{3\log x}} = 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}.$$

Finally, for the last part of the proof, the inequality

$$\frac{1}{25\log p} + \frac{\log s}{3\log p} + \frac{\log s}{3\log x} < \frac{1}{3}$$

must hold to have $J_{p,s}(x) = o(x)$. That is,

$$3(3\log x + 25\log s\log x + 25\log s\log p) < 75\log p\log x \text{ or}$$
$$9\log x + 75\log x\log s + 25\log p\log s < 75\log x\log p$$

must be satisfied. Now, we can say that when x is large enough, then $25\log p\log s$ is relatively small. Therefore, if

$$75\log x\log p > 9\log x + 75\log x\log s$$

holds, then we are done. That is, $\log p - \log s > \frac{9}{75}$, so that we obtain

$$p > se^{\frac{3}{25}}.$$

The proof is complete. □

4. Proof of Theorem B

In this section, we will prove Theorem B. Let us check Table 1 given in the Introduction for the number of elements in the set $J(p,s)$ before we begin the proof.

s \ p	2	3	5	7	11	13	17	19
1	0	3	3	13	638	3	3	19
2	0	0	2	3	2	2	2	2
3	0	1	1	1	3	1	3	1
4	0	0	0	2	2	2	4	2
5	0	1	1	1	3	1	1	1
6	0	0	2	0	2	4	2	2
7	0	1	1	1	1	3	3	3
8	0	0	0	2	2	2	4	4
9	0	1	1	1	1	1	4	5
10	0	0	2	2	0	2	2	2

As it can be seen from Table 1, for some levels of s, the set $J(p,s)$ is non-trivial. That is, there are additional elements inside $J(p,s)$ except for $p-1$, provided that $p-1 \nmid s$.

For instance, let us show for $s=2$ that

$$\left\{\frac{p-1}{2}, p-1\right\} \subseteq J(p,s) \text{ for } p>3.$$

Suppose that $p>3$ is a prime number. Then, by Proposition 2.1, we have

$$H_{p-1}^{(s)} \equiv 0 \pmod{p}$$

and observe that

$$\frac{1}{k^2} \equiv \frac{1}{(p-k)^2} \pmod{p} \tag{15}$$

holds for any $k = 1, 2, \dots, \frac{p-1}{2}$. Consequently,

$$H_{p-1}^{(2)} = \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 2 \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^2} = 2H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$$

such that $H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$ as $p > 3$. In fact, this result can be extended to all even integers s and $p > 3$ with $p - 1 \nmid s$ as the congruence (15) holds for all those numbers.

Now, we prove Theorem B. Let n be a positive integer, p be a prime and s be a positive integer with $p - 1 \nmid s$. We can write

$$p^{m-1} \le n < p^m$$

for some $m \in \mathbb{Z}^{>0}$. The p-adic order of $H_n^{(s)}$ is then determined by the multiples of p^{m-1} up to n. For instance, we may have $p^{m-1} < 2p^{m-1} \le n < 3p^{m-1} < p^m$ and that

$$H_n^{(s)} = 1 + \frac{1}{2^s} + \cdots + \frac{1}{(p^{m-1})^s} + \cdots + \frac{1}{(2p^{m-1})^s} + \cdots + \frac{1}{n^s}.$$

In particular, by checking the p-adic order of

$$\frac{1}{(p^{m-1})^s} H_r^{(s)}$$

for $1 \le r \le p - 1$, we may be able to find the exact p-adic order of $H_n^{(s)}$. Moreover, if the inequality

$$\nu_p \left(\frac{1}{(p^{m-1})^s} H_r^{(s)} \right) < -s(m-2) \tag{16}$$

holds for any $r \in \{1, 2, \dots, p-1\}$, then we obtain the finiteness of $J(p, s)$. That is because when $m - 1 \ge 1$, the terms with the highest exponents of p in the denominators cannot add up and increase the p-adic order $\nu_p\big(H_n^{(s)}\big)$ as they cannot reach to the level p^{m-2}. Therefore, the only integers n, which may lie in $J(p, s)$ can be the ones with $m = 1$, so we have

$$1 = p^0 \le n < p.$$

However, for the inequality (16), we have

$$-s(m-1) + \nu_p(H_r^{(s)}) < -s(m-2) \text{ so that } \nu_p(H_r^{(s)}) < s.$$

Consequently, $J(p, s)$ is a subset of the set $\{1, \dots, p-1\}$ and the proof is complete.

5. Concluding remarks

In this section, we present some explicit results on $J(p,s)$ using SageMath. The code to calculate the number of elements of $J(p,s)$ is given in Appendix A. Also, for the exact p-adic orders of some generalized harmonic numbers, we encourage interested readers to [13]. Moreover, we briefly discuss the case where the condition in Theorem B fails.

Now, for some primes p and a positive integer s with $p-1 \nmid s$, we will present the values of the numbers $H_n^{(s)}$, together with the p-adic orders $\nu_p\left(H_n^{(s)}\right)$ for $1 \le n \le p-1$. We can take $s=2$ and for the prime numbers p, we begin with $p=5$ and then, prove Proposition C.

TABLE 2. The values of $H_n^{(s)}$ together with the 5-adic orders for $s=2$.

n	$H_n^{(2)}$	$\nu_5\left(H_n^{(2)}\right)$
1	1	0
2	5/4	1
3	49/36	0
4	205/144	1

One may see from Table 2 that the condition

$$\nu_5\left(H_n^{(s)}\right) \le 1 = s-1$$

holds for all $1 \le n \le 4$. Therefore, we conclude that

$$J(5,2) \subseteq \{1,2,3,4\}$$

by Theorem B. In fact, we have $J(5,2) = \{2,4\}$.

Next, we continue with the case $p=7$ and $s=2$.

TABLE 3. The values of $H_n^{(s)}$ together with the 7-adic orders for $s=2$.

n	$H_n^{(2)}$	$\nu_7\left(H_n^{(2)}\right)$
1	1	0
2	5/4	0
3	49/36	**2**
4	205/144	0
5	5269/3600	0
6	5369/3600	1

For $p = 7$ and $s = 2$, the condition in Theorem B fails, namely,

$$\nu_7\left(H_r^{(2)}\right) \leq 2 - 1 = 1$$

is not satisfied for all $r \in \{1, \ldots, 6\}$. We have $H_3^{(2)} = \frac{49}{36}$ so that

$$\nu_7\left(H_3^{(2)}\right) = 2 \not\leq 1.$$

Therefore, we may be able to find some element $n \in J(7, 2)$ with $n > 7$. It turns out that we have $26 \in J(7, 2)$, as

$$\nu_7\left(H_{26}^{(2)}\right) = \nu_7(235076082542347816492/14626411683380640000) = 1.$$

On the other hand, there is not any positive integer greater than 26 in $J(7, 2)$. Recall by Corollary 3.2 that

$$J(p, s) = \bigcup_{k=1}^{\infty} J_{p,s}^{(k)},$$

where $J_{p,s}^{(k)} = J(p, s) \cap [p^{k-1}, p^k - 1]$ for $k \in \mathbb{Z}^{>0}$. Moreover, using [18] with our work given in Appendix A, we have

$$J_{7,2}^{(1)} = \{3, 6\}, J_{7,2}^{(2)} = \{26\} \text{ and } J_{7,2}^{(3)} = \emptyset.$$

In conclusion, we have

$$J(7, 2) = J_{7,2}^{(1)} \cup J_{7,2}^{(2)} = \{3, 6, 26\}.$$

For $p = 37$ and $s = 3$, we have

$$\nu_{37}\left(H_{36}^{(3)}\right) = 3 \not\leq 3 - 1 = 2.$$

Therefore, we may find some element in $J(37, 3)$ that is greater than 37. Now, again using [18], we have the following levels for $J(37, 3)$:

$$J_{37,3}^{(1)} = \{4, 13, 23, 32, 36\}, J_{37,3}^{(2)} = \{1340, 1360\} \text{ and } J_{7,2}^{(3)} = \emptyset.$$

Hence, we find the elements $\{1340, 1360\}$ that are greater than 37. However, there is not any other elements of $J(37, 3)$, so that we have $J(37, 3) = \{4, 13, 23, 32, 36, 1340, 1360\}$, which proves Proposition C.

One may infer that the natural candidate, for which the condition $\nu_p\left(H_r^{(s)}\right) \leq s - 1$ fails is when $r = p - 1$. Hence, the exact p-adic valuation of $H_{p-1}^{(s)}$ or congruence relations modulo powers of p is needed for that purpose. For instance, we have the following congruence relations, provided that $p \geq s + 3$ (see [7] and [8]) as

$$\sum_{k=1}^{p-1} \frac{1}{k^s} \equiv \begin{cases} \frac{s}{s+1} p B_{p-1-s} & \pmod{p^2} \text{ if } s \text{ is even,} \\ -\frac{s(s+1)}{2(s+2)} p^2 B_{p-2-s} & \pmod{p^3} \text{ if } s \text{ is odd,} \end{cases} \tag{17}$$

where B_k is the k^{th} Bernoulli number defined via the identity

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \qquad \text{for } k = 0, 1, 2, \ldots .$$

The first few Bernoulli numbers are $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = \frac{1}{30}$ and in fact, one has $B_{2k+1} = 0$ for any $k \geq 1$.

Now, for $p = 37$ and $s = 3$, using [18], we see that

$$\nu_p(B_{p-2-s}) = \nu_{37}(B_{32}) = 1$$

and the congruence (17) yields

$$H_{36}^{(3)} \equiv 0 \pmod{p^3}.$$

In fact, 37 is the smallest *irregular prime*. That is, a prime $p > 3$ is called irregular if p divides at least one of the numbers $B_2, B_4, \ldots, B_{p-3}$. Also, we note that if we have an irregular prime $p > 3$, then

$$\nu_p(H_{p-1}^{(s)}) > 1$$

is satisfied for more than $\frac{1}{2}$ of the whole positive integers s by [13], so that working on these primes may be one approach to understand our set more.

Furthermore, we briefly discuss the congruence relations with Euler numbers. They are defined recursively as

$$E_0 = 1, \sum_{\substack{0 \leq i \leq n \\ i:\text{even}}} \binom{n}{i} E_{n-i} = 0$$

for $n \in \mathbb{Z}^{>0}$. For instance, let $s = 2$ and $p > 3$ be a prime. Then, we have by [14] that

$$\sum_{k=1}^{\lfloor \frac{p}{4} \rfloor} \frac{1}{k^2} \equiv (-1)^{\left(\frac{p-1}{2}\right)} 4E_{p-3} \pmod{p}. \tag{18}$$

Thus, in this case, the p-adic valuation of these numbers are related to the Euler numbers E_{p-3} so that if the right-hand side is 0, we have $\nu_p(E_{p-3}) \geq 1$. Therefore, if we conclude that the p-adic order is greater than 1, our condition fails and we get

$$p < p\left\lfloor \frac{p}{4} \right\rfloor + r \in J(p, s)$$

for some $r \in \{0, 1, \ldots, p-1\}$. Furthermore, the prime p is called *E-irregular* in this case (see also [16]), which is similar to the definition of irregular primes defined for Bernoulli numbers.

In conclusion, the irregular primes may have significant importance on the finiteness of the set $J(p, s)$.

Finally, we close our section with the sets $J(p,s)$ for primes p up to 100 and $s = 2,3,4,5$. The table is obtained using [18] with the code provided in Appendix A as follows. We have by Corollary 3.2 that

$$J(p,s) = \bigcup_{k=1}^{\infty} J_{p,s}^{(k)}.$$

The code runs until finding a level k with $J_{p,s}^{(k)} = \emptyset$, so that we are able to get the whole set $J(p,s)$.

TABLE 4. The sets $J(p,s)$ for several p and s values.

p	$J(p,2)$	$J(p,3)$	$J(p,4)$	$J(p,5)$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	$\emptyset$	$\{2\}$	$\emptyset$	$\{2\}$
5	$\{2,4\}$	$\{4\}$	$\emptyset$	$\{4\}$
7	$\{3,6,26\}$	$\{6\}$	$\{3,6\}$	$\{6\}$
11	$\{5,10\}$	$\{4,6,10\}$	$\{5,10\}$	$\{2,8,10\}$
13	$\{6,12\}$	$\{12\}$	$\{6,12\}$	$\{12\}$
17	$\{8,16\}$	$\{7,9,16\}$	$\{2,8,14,16\}$	$\{16\}$
19	$\{9,18\}$	$\{18\}$	$\{9,18\}$	$\{18\}$
23	$\{11,22\}$	$\{22\}$	$\{11,22\}$	$\{22\}$
29	$\{14,28\}$	$\{28\}$	$\{14,28\}$	$\{6,22,28\}$
31	$\{15,30\}$	$\{8,22,30\}$	$\{15,30\}$	$\{30\}$
37	$\{15,18,21,36\}$	$\{4,13,23,32,36,1340,1360\}$	$\{18,36\}$	$\{6,9,12,18,24,27,30,36\}$
41	$\{4,20,36,40\}$	$\{40\}$	$\{18,20,22,40\}$	$\{40\}$
43	$\{11,21,31,42\}$	$\{42\}$	$\{21,42\}$	$\{42\}$
47	$\{23,46\}$	$\{5,41,46\}$	$\{23,46\}$	$\{14,32,46\}$
53	$\{26,52\}$	$\{6,46,52\}$	$\{26,52\}$	$\{52\}$
59	$\{6,24,29,34,52,58\}$	$\{58\}$	$\{15,29,43,58\}$	$\{58\}$
61	$\{30,60\}$	$\{60\}$	$\{30,60\}$	$\{60\}$
67	$\{33,66\}$	$\{28,30,36,38,66\}$	$\{24,33,42,66\}$	$\{66\}$
71	$\{35,70\}$	$\{70\}$	$\{28,35,42,70\}$	$\{9,61,70\}$
73	$\{36,72\}$	$\{72\}$	$\{36,72\}$	$\{72\}$
79	$\{39,78\}$	$\{78\}$	$\{6,39,72,78\}$	$\{78\}$
83	$\{41,82\}$	$\{82\}$	$\{41,82\}$	$\{3,15,21,61,67,79,82\}$
89	$\{44,88\}$	$\{10,43,45,78,88\}$	$\{44,88\}$	$\{88\}$
97	$\{15,48,81,96\}$	$\{96\}$	$\{38,48,58,96\}$	$\{3,22,74,93,96\}$

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Appendix A. The code to compute $J(p,s)$

In Section 4, the SageMath code we used to find the elements of $J(p,s)$ is as below.

```
def dp(p,k):
finalArray = []
if k==1:
finalArray =[p^(k-1)..p^(k)-1]
else:
for a in dp(p,k-1):
for j in [0..p-1]:
finalArray.append(p*a+j)
return finalArray

## Set p and s
p = 7
s = 2

num = 1
k = 1
mainArray = []

while True:
if num == 0:
print "Finished"
break
print "Level k =",k,":"
subInterval = [n for n in dp(p,k) if valuation(harmonic_number(n,s),p)>0]
print subInterval
mainArray = mainArray + subInterval
num = len(subInterval)
k = k+1
print "J(p,s) = ","J(",p,",",s,")"
print mainArray
print "number of elements: ", len(mainArray)
```

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