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RESIDUAL SUPERSINGULAR IWASAWA THEORY OVER QUADRATIC IMAGINARY FIELDS

Parham Hamidi

Abstract. Let p be an odd prime. Let E be an elliptic curve defined over a quadratic imaginary field, where p splits completely. Suppose E has supersingular reduction at primes above p. Under appropriate hypotheses, we extend the results of [17] to $\mathbb{Z}_p^2$-extensions. We define and study the fine double-signed residual Selmer groups in these settings. We prove that for two residually isomorphic elliptic curves, the vanishing of the signed μ-invariants of one elliptic curve implies the vanishing of the signed μ-invariants of the other. Finally, we show that the Pontryagin dual of the Selmer group and the double-signed Selmer groups have no non-trivial pseudo-null submodules for these extensions.

1. Introduction

Consider two elliptic curves defined over $\mathbb{Q}$ with good ordinary reduction at prime p whose residual Galois representations are isomorphic. Greenberg and Vatsal in [7] showed that the vanishing of the μ-invariant attached to the Pontryagin dual of the Selmer group over the cyclotomic $\mathbb{Z}_p$-extension of one curve implies the vanishing of the μ-invariant for the other. Their work uses an auxiliary Selmer group called *nonprimitive dual Selmer group* which has the same μ-invariant as the Pontryagin dual of the Selmer group. For an elliptic curve $E/\mathbb{Q}$ with good supersingular reduction at p, the Selmer group over the cyclotomic extension is no longer cotorsion. Kobayashi in [12] defined the signed Selmer group over the cyclotomic $\mathbb{Z}_p$-extension using moderately stronger local conditions at p and showed that they are cotorsion over Iwasawa modules. A similar approach has been taken in the study of elliptic curves with supersingular reduction for the signed Selmer groups. The μ-invariants of the Pontryagin dual of the signed Selmer groups are referred to as the *signed μ-invariants*. An analogue of Greenberg–Vatsal was proved by B. D. Kim in [11] for the signed μ-invariants over cyclotomic $\mathbb{Z}_p$-extension using nonprimitive dual Selmer groups. Later, in [17] the authors used a new technique to work with residual representations of elliptic curves and they improved upon

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the results of Kim in [11]. They defined a new Selmer group called *the fine residual signed Selmer group* and studied its structure as an Iwasawa module, in particular, the vanishing of the signed μ-invariants (*cf.* [17, Theorem 4.12]).

In this paper, we follow the strategy of [17] to prove new results for the signed μ-invariants of $\mathbb{Z}_p^2$-extensions of quadratic imaginary fields. However, the Iwasawa theory of $\mathbb{Z}_p^2$-extensions has additional technicalities compared to the cyclotomic $\mathbb{Z}_p$-extensions. We define the fine signed residual Selmer groups in these settings (*cf.* Definition 2.2). In Proposition 3.3 we show that these residual singed Selmer groups depend only on the isomorphism class of the residual Galois representation of elliptic curves. Therefore, these groups provide a natural method to study congruences of elliptic curves. We then relate the structure of the fine signed residual Selmer groups to that of the signed Selmer groups as Iwasawa modules (*cf.* Proposition 4.4). Theorem 4.5 relates the module structure of the fine signed residual Selmer groups to the vanishing of the signed μ-invariants. We use this theorem to show that if two elliptic curves have isomorphic residual representations, then vanishing of the signed μ-invariants for one implies the same for the other. Furthermore, we relate these results to analogous results over cyclotomic extensions proved in [17] (*cf.* Corollary 4.12).

Moreover, we prove in that the Pontryagin dual of the signed Selmer groups and the classical Selmer group have no non-zero pseudo-null submodules (*cf.* Theorem 5.7). This is an important property for an Iwasawa module, since the structure of finitely generated modules over commutative Iwasawa algebras is known up to pseudo-isomorphism. Hence, whenever such modules have no non-zero pseudo-null submodules we have a better understanding of their structure. To this, we invoke Auslander–Buchsbaum–Serre formula and compute the depth of the signed Selmer groups by studying Galois cohomology of these groups.

This paper consists of five sections including this introductory section and it is organized as follows. In Section 2, we introduce preliminary definitions, notations, and assumptions. Our goal in Section 3 is to prove some of the essential components of the proof of our main results. In Section 4, we prove Theorem 4.5 and record some important consequences of it. Finally, in Section 5 we use purely algebraic tools to compute the depth of signed Selmer groups and prove Theorem 5.7 which states that the Pontryagin dual of the Selmer group and the double-signed Selmer groups have no non-trivial pseudo-null submodules.

2. Preliminaries

Let E/L be an elliptic curve defined over a quadratic imaginary number field $L/\mathbb{Q}$. Suppose p is an odd prime and let S_p denote the set of all primes of L over p. Denote the finite set of primes above p, where E/L has supersingular reduction by $S^{\mathrm{ss}} \subseteq S_p$ and let S^{bad} denote the finite set of primes in L, where

E has bad reduction. Let S be the disjoint union of the sets S_p and S^{bad}. For any field extension $\mathcal{L}/L$, let $S_p(\mathcal{L})$ be the set primes in $\mathcal{L}$ above the set S_p. Similarly, let $S^*(\mathcal{L})$ for $* \in \{\emptyset, \text{ bad}, \text{ ss}\}$ denote the set of primes in $\mathcal{L}$ above the corresponding finite set S^*. Throughout this article we assume the following hypotheses, which we refer to as Hyp 1.

Hyp 1(i): The prime p is odd.

Hyp 1(ii): The number field $L/\mathbb{Q}$ is a quadratic imaginary field extension. Furthermore, assume that p splits completely in $L/\mathbb{Q}$. We denote the primes of L over p by $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. In particular, $\mathfrak{p} \neq \bar{\mathfrak{p}}$ and $L_{\mathfrak{p}} = L_{\bar{\mathfrak{p}}} = \mathbb{Q}_p$.

Hyp 1(iii): The elliptic curve E/L has good supersingular reduction at both primes above prime p. Therefore, the sets $S^{\text{ss}} = S_p = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$.

Hyp 1(iv): We have $a_{\mathfrak{p}} = 1 + N_{L/\mathbb{Q}}(\mathfrak{p}) - \#\widetilde{E}_{\mathfrak{p}}(\mathcal{O}_L/\mathfrak{p}) = a_{\bar{\mathfrak{p}}} = 0$, where $N_{L/\mathbb{Q}} : L \to \mathbb{Q}$ is the usual norm map and $\widetilde{E}_{\mathfrak{p}}$ is the reduced curve modulo $\mathfrak{p}$.

Let L_{cyc}/L denote the cyclotomic $\mathbb{Z}_p$-extension of L and let L_∞/L denote the compositum of all $\mathbb{Z}_p$-extensions of L. Leopoldt's conjecture implies that $\text{Gal}(L_\infty/L) \cong \mathbb{Z}_p^2$. Note that Leopoldt's conjecture is known for abelian extensions of rational numbers, and therefore it is not a hypothesis in this case (*cf.* Corollary 5.32 and Theorem 13.4 of [22]). Let L^S be the maximal algebraic extension of L unramified outside of the primes inside S and G_L^S denote the Galois group of the extension L^S/L. Note that $L_\infty \subset L^S$. Let

$$G_\infty^S := \text{Gal}(L^S/L_\infty),\ G := \text{Gal}(L_\infty/L) \cong \mathbb{Z}_p^2, \text{ and } \Gamma := \text{Gal}(L_{\text{cyc}}/L) \cong \mathbb{Z}_p.$$

Given a prime w in $S_p(L_\infty)$, by abuse of notation, let the prime below w in L_n be again denoted by w. It should be clear from the context whether the prime w is in L_n for some $n \geq 0$ or is in L_∞. Finally, for an abelian group M and an integer $t \geq 1$ we write M_t for the subgroup of elements of M that are annihilated by t. Moreover, for a prime p, we define the p-primary torsion subgroup of M, which we denote by M_{p^∞}, as

$$M_{p^\infty} := \cup_{i \geq 1} M_{p^i}.$$

2.1. Plus and minus norm groups

Suppose $K/\mathbb{Q}_p$ is a finite unramified extension of $\mathbb{Q}_p$. Denote the cyclotomic (*resp.* the unramified) $\mathbb{Z}_p$-extension of K by K_{cyc} (*resp.* K^{ur}) which is totally ramified. For any integer $r \geq 0$, let K_r be the unique intermediate field extension of K_{cyc}/K such that $\text{Gal}(K_r/K) = \mathbb{Z}/p^r\mathbb{Z}$ and for any integer $l \geq 0$, let $K^{(l)}$ be the unique intermediate field extension of K^{ur}/K such that $\text{Gal}(K^l/K)$ is isomorphic to $\mathbb{Z}/p^l\mathbb{Z}$. Further, let $K_r^{(l)}$ be $K^l K_r$, so $\text{Gal}(K_r^{(l)}/K)$ is $\mathbb{Z}/p^l\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z}$ and

$$K_{\text{cyc}}^{\text{ur}} = K^{\text{ur}} K_{\text{cyc}} = \cup_{r,l \geq 0} K^l K_r,$$

with $\mathrm{Gal}(K_{\mathrm{cyc}}^{\mathrm{ur}}/K)$ equal to $\mathbb{Z}_p^2$. Let μ_{p^r} denote the p^r-th roots of unity and let $\mathbb{K}_r^{(l)}$ to be $K^{(l)}(\mu_{p^{r+1}})$ for $l \geq 0$ and $r \geq -1$. For all $l, r \geq 0$, denote

$$\Delta := \mathrm{Gal}(K(\mu_p)/K) = \mathrm{Gal}(\mathbb{K}_0^{(0)}/K_0^{(0)}) \cong \mathrm{Gal}(\mathbb{K}_r^l/K_r^l). \tag{1}$$

In this article, with abuse of notation, we write $(\mathbb{K}_r^{(l)})^\Delta$ is equal to $K_r^{(l)}$.

In what follows, let $\widehat{E}(\mathfrak{m}_\mathcal{L})$ by $\widehat{E}(\mathcal{L})$, where $\mathcal{L}/\mathbb{Q}_p$ is any local field over $\mathbb{Q}_p$ and $\mathfrak{m}_\mathcal{L}$ is the maximal ideal of the valuation ring of $\mathcal{L}$. Denote the ring of integers of $\mathcal{L}$ by $\mathcal{O}_\mathcal{L}$. Suppose $E/\mathbb{Q}_p$ is an elliptic curve and let $\widehat{E}$ denote the formal group over $\mathbb{Z}_p$ associated to the minimal model of E over $\mathbb{Q}_p$.

Definition 2.1. Following [12, Definition 8.16], define the *plus and minus norm groups* as

$$\widehat{E}^\pm(\mathbb{K}_r^{(l)}) := \Big\{P \in \widehat{E}(\mathbb{K}_r^{(l)}) \mid \mathrm{Tr}_{m+1}^r(P) \in \widehat{E}(\mathbb{K}_m^{(l)}), -1 \leq m \leq r-1 \text{ s.t. } (-1)^m = \pm 1\Big\},$$

$$E^\pm(\mathbb{K}_r^{(l)}) := \Big\{P \in E(\mathbb{K}_r^{(l)}) \mid \mathrm{Tr}_{m+1}^r(P) \in E(\mathbb{K}_m^{(l)}), -1 \leq m \leq r-1 \text{ s.t. } (-1)^m = \pm 1\Big\},$$

$$\widehat{E}^\pm(K_r^{(l)}) := \Big(\widehat{E}^\pm(\mathbb{K}_r^{(l)})\Big)^\Delta, \text{ and } E^\pm(K_r^{(l)}) := \Big(E^\pm(\mathbb{K}_r^{(l)})\Big)^\Delta,$$

$$\widehat{E}^\pm(\mathbb{K}_{\mathrm{cyc}}^{(\mathrm{ur})}) := \bigcup_{r,l\geq 0} \widehat{E}^\pm(\mathbb{K}_r^{(l)}), \text{ and } E^\pm(\mathbb{K}_{\mathrm{cyc}}^{\mathrm{ur}}) := \bigcup_{r,l\geq 0} \widehat{E}^\pm(\mathbb{K}_r^{(l)}).$$

Suppose $E/\mathbb{Q}_p$ is an elliptic curve with good reduction at p and such that $a_p = 0$. Then, by Proposition 8.7 of [12], for all $l, r \geq 0$, $\widehat{E}(\mathbb{K}_r^{(l)})$ has no non-trivial p-torsion points. This implies that

$$E(\mathbb{K}_r^{(l)})_p = E(K_r^{(l)})_p = \{0\}. \tag{2}$$

2.2. Signed Kummer maps

Suppose L and S are as Hyp 1. For any $n \geq 0$, let L_n be the unique sub-extension of L_∞/L such that

$$\mathrm{Gal}(L_n/L) \cong (\mathbb{Z}/p^n\mathbb{Z})^2, \quad L = L_0, \quad L_\infty := \cup_{n\geq 0} L_n. \tag{3}$$

For simplicity, let G_n^S denote $\mathrm{Gal}(L^S/L_n)$. Note that if $w|v$ is a prime of L_∞ not above p, then $L_{\infty,w}$ is the unique unramified $\mathbb{Z}_p$-extension (*cf.* for example [22, Proposition 13.2]) of the local field L_v. Let $\mathcal{L} \in \{K_r^{(l)}, \mathbb{K}_r^{(l)}, L_n\}$ and $\mathcal{G} \in \{G_{K_r^{(l)}}, G_{\mathbb{K}_r^{(l)}}, G_n^S\}$. For any integer $t \geq 0$, there is a short exact sequence of $\mathcal{G}$-modules

$$0 \longrightarrow E(\mathcal{L})/p^t E(\mathcal{L}) \xrightarrow{\kappa_\mathcal{L}^{p^t}} H^1(\mathcal{G}, E_{p^t}) \longrightarrow H^1(\mathcal{G}, E)_{p^t} \longrightarrow 0,$$

where the map $\kappa_\mathcal{L}^{p^t}$ is the Kummer map for E_{p^t} over $\mathcal{L}$. Taking the direct limit of the above sequence we obtain the following

$$0 \longrightarrow E(\mathcal{L}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\kappa_\mathcal{L}^{p^\infty}} H^1(\mathcal{G}, E_{p^\infty}) \longrightarrow H^1(\mathcal{G}, E)_{p^\infty} \longrightarrow 0.$$

For $v \in S$, we define the local condition

$$J_v(E_{p^\infty}/L_n) := \bigoplus_{w|v} H^1(L_{n,w}, E)_{p^\infty} \cong \bigoplus_{w|v} \frac{H^1(L_{n,w}, E_{p^\infty})}{E(L_{n,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}, \tag{4}$$

where the isomorphism is due to the Kummer map (*cf.* [4, Section 1.6]). The *classical* p^∞*-Selmer group* of the elliptic curve E over L_n is defined by the following sequence

$$0 \longrightarrow \mathrm{Sel}(E_{p^\infty}/L_n) \to H^1(L^S/L_n, E_{p^\infty}) \xrightarrow{\lambda_n} \bigoplus_{v \in S} J_v(E_{p^\infty}/L_n), \tag{5}$$

where λ_n consists of restriction maps coming from Galois cohomology. By definition of the plus/minus norm groups, there are inclusions

$$E^\pm(L_{n,w}) \hookrightarrow E(L_{n,w}), \text{ and } \widehat{E}^\pm(L_{n,w}) \hookrightarrow \widehat{E}(L_{n,w}),$$

where w is a prime in L_n above p, $\mathfrak{m}_n$ is the maximal ideal of the ring of integers of $L_{n,w}(\mu_p)$. By Lemma 3.4 of [17], the above maps remain injective after applying the functor $- \otimes \ \mathbb{Z}/p^t$ for any $t \geq 1$

$$E^\pm(L_{n,w})/p^t E^\pm(L_{n,w}) \hookrightarrow E(L_{n,w})/p^t E(L_{n,w}),$$

where w is a prime in L_n above p. The Kummer map $\kappa_{\mathcal{L}}^{p^t}$ induces the following injective map

$$\kappa^{\pm,p^t}_{L_{n,w}} \ : \ E^\pm(L_{n,w})/p^t E^\pm(L_{n,w}) \hookrightarrow H^1(L_{n,w}, E_{p^t}). \tag{6}$$

Similarly, for any $n, t \geq 1$, there is an injection

$$\kappa^{\pm,p^t}_{L_{n,w}} \ : \ \widehat{E}^\pm(L_{n,w})/p^t \widehat{E}^\pm(L_{n,w}) \hookrightarrow H^1(L_{n,w}, \widehat{E}_{p^t}).$$

Refer to the maps $\kappa^{\pm,p^t}_{L_{n,w}}$ as the *signed Kummer maps* for E_{p^t} over $L_{n,w}$.

2.3. Selmer group, signed Selmer groups, and fine signed residual Selmer groups

Let E/L be an elliptic curve satisfying Hyp 1 and let L_n/L be as in (3). If w belongs to $S^{\mathrm{ss}}(L_n) = S_p(L_n)$, then $w|\mathfrak{p}$ or $w|\bar{\mathfrak{p}}$, by Hyp 1(iii). Denote a prime in L_n which lies over $\mathfrak{p}$ (*resp.* $\bar{\mathfrak{p}}$) by $\mathfrak{q}$ (*resp.* $\bar{\mathfrak{q}}$). Similarly, if $w \in S_p(L_\infty)$ and $w|\mathfrak{p}$ (*resp.* $w|\bar{\mathfrak{p}}$), denote w by $\mathfrak{q}$ (*resp.* $\bar{\mathfrak{q}}$). Finally, if $\mathfrak{q}$ (*resp.* $\bar{\mathfrak{q}}$) is in L_∞, denote its restriction to L_n again by $\mathfrak{q}$ (*resp.* $\bar{\mathfrak{q}}$). It should be clear from the context whether $\mathfrak{q}$ and $\bar{\mathfrak{p}}$ represent primes in L_n or in L_∞. Following [17], for

$v \in S$ define the local cohomological groups

$$(7)\qquad {}^{\pm}\widetilde{K}_v(E_p/L_n) := \begin{cases} \bigoplus_{w|\mathfrak{l}} H^1(L_{n,w}, E_p) & \text{if } v = \mathfrak{l} \in S^{\text{bad}}, \\ \bigoplus_{\mathfrak{q}|\mathfrak{p}} H^1(L_{n,\mathfrak{q}}, E_p)/\text{Im } \kappa^{\pm,p}_{L_{n,\mathfrak{q}}} & \text{if } v = \mathfrak{p} \in S^{\text{ss}}, \\ \bigoplus_{\bar{\mathfrak{q}}|\bar{\mathfrak{p}}} H^1(L_{n,\bar{\mathfrak{q}}}, E_p)/\text{Im } \kappa^{\pm,p}_{L_{n,\bar{\mathfrak{q}}}} & \text{if } v = \bar{\mathfrak{p}} \in S^{\text{ss}}. \end{cases}$$

Similarly

$$(8)\qquad J_v^{\pm}(E_{p^\infty}/L_n) := \begin{cases} \bigoplus_{w|\mathfrak{l}} H^1(L_{n,w}, E_{p^\infty}) & \text{if } v = \mathfrak{l} \in S^{\text{bad}}, \\ \bigoplus_{\mathfrak{q}|\mathfrak{p}} H^1(L_{n,\mathfrak{q}}, E_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,\mathfrak{q}}} & \text{if } v = \mathfrak{p} \in S^{\text{ss}}, \\ \bigoplus_{\bar{\mathfrak{q}}|\bar{\mathfrak{p}}} H^1(L_{n,\bar{\mathfrak{q}}}, E_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,\bar{\mathfrak{q}}}} & \text{if } v = \bar{\mathfrak{p}} \in S^{\text{ss}}. \end{cases}$$

Here, when $v \in S^{\text{ss}}$, the sign of ${}^{\pm}\widetilde{K}_v(E_p/L_n)$ (*resp.* $J_v^{\pm}(E_{p^\infty}/L_n)$) agrees with the choice of the sign of the Kummer map $\kappa^{\pm,p}$ (*resp.* $\kappa^{\pm,p^\infty}$) in the direction of $\mathfrak{q}|\mathfrak{p}$ or in the direction of $\bar{\mathfrak{q}}|\bar{\mathfrak{p}}$. When $v \in S^{\text{bad}}$, the sign choice of the sign does not matter. Similarly, when $v \in S^{\text{bad}}$ then $J_v^{\pm}(E_{p^\infty}/L_n)$ coincides with $J_v(E_{p^\infty}/L_n)$ for any choice of the sign. The following definition is the analogue of [17, Definition 3.6] over $\mathbb{Z}_p^2$-extensions.

Definition 2.2. Let $n \geq 0$. For every intermediate field L_n in the tower L_∞/L, let ${}^{\pm}\widetilde{K}_v(E_p/L_n)$ be as (7) for any $v \in S$. Define the *fine signed residual Selmer group* $\mathcal{R}^{\pm/\pm}(E_p/L_n)$ of E_p over L_n by:

$$\mathcal{R}^{\pm/\pm}(E_p/L_n) := \ker\left(H^1(G_n^S, E_p) \to \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_n)\right).$$

The choice of the first sign of $\mathcal{R}^{\pm/\pm}(E_p/L_n)$ is in accordance with the choice of ${}^{+}\widetilde{K}_{\mathfrak{p}}(E_p/L_n)$ or ${}^{-}\widetilde{K}_{\mathfrak{p}}(E_p/L_n)$ and the second sign is in accordance with the choice of ${}^{+}\widetilde{K}_{\bar{\mathfrak{p}}}(E_p/L_n)$ or ${}^{-}\widetilde{K}_{\bar{\mathfrak{p}}}(E_p/L_n)$.

Here, we use the notation $\mathcal{R}^{\pm/\pm}(E_p/L_n)$ for convenience. It unifies notation for any of the four possibilities: "$\mathcal{R}^{+/+}(E_p/L_n)$, or $\mathcal{R}^{+/-}(E_p/L_n)$, or $\mathcal{R}^{-/+}(E_p/L_n)$, or $\mathcal{R}^{-/-}(E_p/L_n)$". Similar notation is used throughout the article for simplicity.

Definition 2.3. For every $n \geq 0$ let $J_v^{\pm}(E_{p^\infty}/L_n)$ be as (8) for any $v \in S$. Define the *signed Selmer groups* $\text{Sel}^{\pm/\pm}(E_{p^\infty}/L_n)$ of E_{p^∞} over the intermediate field L_n in the tower L_∞/L as follows:

$$\mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_n) := \ker\left(H^1(G_n^S, E_{p^\infty}) \to \bigoplus_{v\in S} J_v^{\pm}(E_{p^\infty}/L_n)\right).$$

The first sign of $\mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_n)$ corresponds to the choice of $J_{\mathfrak{p}}^{+}(E_{p^\infty}/L_n)$ or $J_{\mathfrak{p}}^{-}(E_{p^\infty}/L_n)$ and the second sign corresponds to the choice of $J_{\bar{\mathfrak{p}}}^{+}(E_{p^\infty}/L_n)$ or $J_{\bar{\mathfrak{p}}}^{-}(E_{p^\infty}/L_n)$.

Remark 2.4. Suppose S' is a finite set of primes containing set S. Then for any prime $v \in S'\backslash S$, define the local condition $J_v^{\pm}(E_{p^\infty}/L_n)$ or ${}^{\pm}\widetilde{K}_v(E_p/L_n)$ the same way we did for the bad primes in S^{bad}. We note that adding a finite set of primes to the set S does not change these Selmer groups (*cf.* Chapter 1, Section 1.7 of [2]).

Let $G_n := \mathrm{Gal}(L_n/L)$. Then, there is an exact sequence of $\Lambda(G_n)$-modules (*resp.* $\Omega(G_n)$-modules):

$$0 \longrightarrow \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_n) \longrightarrow H^1(G_n^S, E_{p^\infty}) \longrightarrow \bigoplus_{v\in S} J_v^{\pm}(E_{p^\infty}/L_n) \tag{9}$$

resp.

$$0 \longrightarrow \mathcal{R}^{\pm/\pm}(E_p/L_n) \longrightarrow H^1(G_n^S, E_p) \longrightarrow \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_n). \tag{10}$$

By taking the direct limit over the intermediate field extensions L_n of the exact sequences (5), (9), and (10) we obtain

$$0 \longrightarrow \mathrm{Sel}(E_{p^\infty}/L_\infty) \longrightarrow H^1(G_\infty^S, E_{p^\infty}) \xrightarrow{\lambda_\infty} \bigoplus_{v\in S} J_v(E_{p^\infty}/L_\infty), \tag{11}$$

$$0 \longrightarrow \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty) \longrightarrow H^1(G_\infty^S, E_{p^\infty}) \xrightarrow{\xi^{\pm/\pm}} \bigoplus_{v\in S} J_v^{\pm}(E_{p^\infty}/L_\infty), \tag{12}$$

$$0 \longrightarrow \mathcal{R}^{\pm/\pm}(E_p/L_\infty) \longrightarrow H^1(G_\infty^S, E_p) \xrightarrow{\xi_p^{\pm/\pm}} \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_\infty). \tag{13}$$

The above exact sequences (11), (12), and (13) have a natural G-action. This action gives these Selmer groups a module structure over Iwasawa algebras, which we now introduce.

Definition 2.5. The *Iwasawa algebra* of $\mathcal{G}$, denoted by $\Lambda(\mathcal{G})$ (*resp.* $\Omega(\mathcal{G})$), to be the completed group algebra of $\mathcal{G}$ over $\mathbb{Z}_p$ (*resp.* $\mathbb{F}_p$). That is,

$$\Lambda(\mathcal{G}) := \mathbb{Z}_p[[\mathcal{G}]] = \varprojlim_{\mathcal{N}\subset\mathcal{G}} \mathbb{Z}_p[\mathcal{G}/\mathcal{N}],\ (\textit{resp. } \Omega(\mathcal{G}) := \mathbb{F}_p[[\mathcal{G}]] = \Lambda(\mathcal{G})/p\Lambda(\mathcal{G})),$$

where $\mathcal{N}$ runs over open normal sub-groups of $\mathcal{G}$.

In this article, we deal with $\mathcal{G} \cong \mathbb{Z}_p^n$ from some $n \in \{1, 2\}$. When the group $\mathcal{G} = G \cong \mathbb{Z}_p^2$ then $n = 2$ and when we have $\mathcal{G} = \Gamma \cong \mathbb{Z}_p$ then $n = 1$. The Iwasawa algebra $\Lambda(\mathcal{G})$ (*resp.* $\Omega(\mathcal{G})$) is isomorphic to the ring of formal power series $\mathbb{Z}_p[[T_1, \ldots, T_n]]$ (*resp.* $\mathbb{F}_p[[T_1, \ldots, T_n]]$) with indeterminate variables $T_1, \ldots, T_n$. Therefore $\Lambda(\mathcal{G})$ is a commutative, regular local ring of Krull dimension $n + 1$.

Note that (11) and (12) are exact sequences of $\Lambda(G)$-modules and (13) is an exact sequence of $\Omega(G)$-modules. Let $\mathfrak{X}(E_{p^\infty}/L_\infty)$ denote the Pontryagin dual of $\mathrm{Sel}(E_{p^\infty}/L_\infty)$. It is important to note here that $\mathfrak{X}(E_{p^\infty}/L_\infty)$ is conjectured to have positive rank, and therefore it is not a torsion $\Lambda(G)$-module. This is shown over the cyclotomic $\mathbb{Z}_p$-extensions (*cf.* [4, Theorem 2.5]). For a Galois extension with a pro-p Galois group without p-torsion which contains the cyclotomic $\mathbb{Z}_p$-extension, this is conjectured to hold (*cf.* [18]). However, the Pontryagin dual of the signed Selmer group $\mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty)$, which we denote by $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$, is conjectured to be torsion as a $\Lambda(G)$-module. For a more generalized version of this see Conjecture 4.11 of [13]. We denote

$$\mu_G^{\pm/\pm}(E_{p^\infty}/L_\infty) := \mu_G(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)),$$

and we call them the *signed μ-invariants*. Suppose that M is a discrete cofinitely generated $\Lambda(\mathcal{G})$-module and let $M^\wedge$ denote its Pontryagin dual. Then, there is an isomorphism $(M_p)^\wedge \cong M^\wedge/pM^\wedge$ and the inequality

$$\mathrm{corank}_{\Lambda(\mathcal{G})}(M) \leq \mathrm{corank}_{\Omega(\mathcal{G})}(M_p). \tag{14}$$

The inequality (14) becomes an equality exactly when the μ-invariant of $M^\wedge$ is zero.

3. Local and global cohomology calculations

Throughout this section, we assume that E/L is an elliptic curve satisfying Hyp 1 and L_n/L is as defined in equation (3) for any $n \geq 0$. Let $\mathcal{Y}^{\pm/\pm}(E_p/L_\infty)$ denote the Pontryagin dual of the fine signed residual Selmer group $\mathcal{R}^{\pm/\pm}(E_p/L_\infty)$ defined in Definition 2.2. Our goal in this article is to show that if E_1/L and E_2/L are two elliptic curves satisfying Hyp 1 and they are such that their residual Galois representations $(E_1)_p$ and $(E_2)_p$ are isomorphic, then

$$\mu_G^{\pm/\pm}((E_1)_{p^\infty}/L_\infty) = 0 \iff \mu_G^{\pm/\pm}((E_2)_{p^\infty}/L_\infty) = 0$$

given that $\mathcal{Y}^{\pm/\pm}((E_j)_p/L_\infty)$ is a torsion $\Omega(G)$-module for $j \in \{1, 2\}$.

3.1. Comparison of the local and global cohomology groups

Let $n \geq 0$ and let L_n/L be as in (3). For convenience, given a local field $\mathcal{K}$, let $G_{\mathcal{K}}$ denote the Galois group of the extension $\overline{\mathcal{K}}/\mathcal{K}$, where $\overline{\mathcal{K}}$ denotes the separable closure of $\mathcal{K}$. Suppose $\mathcal{G} \in \{G_n^S, G_{L_{n,w}}\}$, where $w \in S(L_n)$. There is a short exact sequence

$$0 \longrightarrow E(\mathcal{L})_{p^\infty}/pE(\mathcal{L})_{p^\infty} \longrightarrow H^1(\mathcal{G}, E_p) \xrightarrow{\psi_{\mathcal{G},n}} H^1(\mathcal{G}, E_{p^\infty})_p \longrightarrow 0. \tag{15}$$

Recall that $L_{n,w} = K_r^{(l)}$ for some $l, r \geq 0$. Equation (2) tells us that $E(L_{n,w})_p = E(L_n)_p = \{0\}$. This means that the first term in the short exact sequence (15) vanishes. Therefore for any $n \geq 0$, the map $\psi_{\mathcal{G},n}$ is an isomorphism. It remains to investigate $\psi_{\mathcal{G},n}$ when $\mathcal{G} = G_{L_{n,w}}$ for some $w \in S^{\mathrm{bad}}(L_n)$. Note that $\ker(\psi_{\mathcal{G},n}) = E(L_{n,w})_{p^\infty}/pE(L_{n,w})_{p^\infty}$ has a finite $\mathbb{F}_p$-dimension and by [23, Theorem 3.2]

$$\dim_{\mathbb{F}_p}(\ker(\psi_{\mathcal{G},n})) = \dim_{\mathbb{F}_p}(E(L_{n,w})_p) \leq \dim_{\mathbb{F}_p}(E(\overline{L_w})_p) = 2.$$

Passing the exact sequence (15) to direct limit, define the surjective map

$$\psi_{\mathcal{G},\infty} : H^1(\mathcal{G}, E_p) \twoheadrightarrow H^1(\mathcal{G}, E_{p^\infty})_p, \tag{16}$$

where $\mathcal{G} \in \{G_\infty^S, G_{L_{\infty,w}}\}$ for $w \in S(L_\infty)$ and $G_\infty^S := \mathrm{Gal}(L_\infty^S/L_\infty)$.

Proposition 3.1. *Let $\mathcal{G} \in \{G_\infty^S, G_{L_{\infty,w}}\}$ for $w \in S(L_\infty)$ and let $\psi_{\mathcal{G},\infty}$ be the map described in* (16)*.*

(1) *If $\mathcal{G} \in \{G_\infty^S, G_{L_{\infty,w}}\}$ for $w \in S_p(L_\infty)$, then $\psi_{\mathcal{G},\infty}$ is an isomorphism.*
(2) *If $\mathcal{G} = G_{L_{\infty,w}}$ for $w \in S^{\mathrm{bad}}(L_\infty)$, then $\dim_{\mathbb{F}_p}(\ker(\psi_{\mathcal{G},\infty})) \leq 2$.*

Proof. The proof is similar to Proposition 4.1 in [17]. □

Suppose $w \in S(L_\infty)$ and assume $w|\mathfrak{p}$ (*resp.* $w|\bar{\mathfrak{p}}$). Denote w by $\mathfrak{q}$ (*resp.* by $\bar{\mathfrak{q}}$) and the isomorphism $\psi_{\mathcal{G},\infty}$, where $\mathcal{G} = \mathrm{Gal}(\overline{L_{\infty,\mathfrak{q}}}/L_{\infty,\mathfrak{q}})$ by $\psi_{\mathfrak{q},\infty}$ (*resp.* $\psi_{\bar{\mathfrak{q}},\infty}$). Finally, when $w \in S^{\mathrm{bad}}(L_\infty)$ then $L_{\infty,w} = L_{\mathrm{cyc},w}$ and $\mathcal{G} = \mathrm{Gal}(\overline{L_{\infty,w}}/L_{\infty,w})$. Denote the surjective map $\psi_{\mathcal{G},\infty}$ by $\psi_{w,\infty}$.

Corollary 3.2. *Let $\mathfrak{q}$ (resp. $\bar{\mathfrak{q}}$) be a prime in $S_p(L_\infty)$ above $\mathfrak{p}$ (resp. $\bar{\mathfrak{p}}$). Then, the isomorphism $\psi_{\mathfrak{q},\infty}$ (resp. $\psi_{\bar{\mathfrak{q}},\infty}$) induces an isomorphism*

$$\psi_{\mathfrak{q},\infty}^{\pm} : H^1(L_{\infty,\mathfrak{q}}, E_p)/\mathrm{Im}\left(\kappa_{L_{\infty,\mathfrak{q}}}^{\pm,p}\right) \xrightarrow{\cong} \left(H^1(L_{\infty,\mathfrak{q}}, E_{p^\infty})/\mathrm{Im}\left(\kappa_{L_{\infty,\mathfrak{q}}}^{\pm,p^\infty}\right)\right)_p$$

resp.

$$\psi_{\bar{\mathfrak{q}},\infty}^{\pm} : H^1(L_{\infty,\bar{\mathfrak{q}}}, E_p)/\mathrm{Im}\left(\kappa_{L_{\infty,\bar{\mathfrak{q}}}}^{\pm,p}\right) \xrightarrow{\cong} \left(H^1(L_{\infty,\bar{\mathfrak{q}}}, E_{p^\infty})/\mathrm{Im}\left(\kappa_{L_{\infty,\bar{\mathfrak{q}}}}^{\pm,p^\infty}\right)\right)_p.$$

Proof. The proof is similar to [17, Proposition 4.1, part d)]. □

3.2. Fine signed residual Selmer group and residual representations

This section explains why the fine signed residual Selmer group only depends on the residual Galois representation E_p. To do this, we show that the local conditions ${}^{\pm}\widetilde{K}_v(E_p/L_\infty)$, which define the fine signed residual Selmer group (*cf.* Definition 2.2), only depend on the residual Galois representation (*cf.* Proposition 3.3). Suppose $\mathcal{L}$ is an algebraic extension of $\mathbb{Q}_p$. By Lemma 3.3 of [17], there exists an exact sequence

$$0 \longrightarrow \widehat{E}(\mathcal{L}) \longrightarrow E(\mathcal{L}) \longrightarrow D \longrightarrow 0,$$

where D is a finite group of order prime-to-p. In particular, for any $n \geq 0$ and any prime $w \in S^{\rm ss}(L_n) = S_p(L_n)$,

$$\widehat{E}^{\pm}(L_{n,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong E^{\pm}(L_{n,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

If w is a prime in $S^{\rm ss}(L_\infty)$, then $\widehat{E}^{\pm}(L_{\infty,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong E^{\pm}(L_{\infty,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Putting this together with equation (6), for all $n \geq 0$

$$\text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}}(E^{\pm}_{p^\infty}(L_{n,w})) \cong \text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}}(\widehat{E}^{\pm}_{p^\infty}(L_{n,w})). \tag{17}$$

Furthermore, the exact sequence (3.2) induces an isomorphism for any $n \geq 0$ and any prime $w \in S^{\rm ss}(L_n)$, $H^1(L_{n,w}, E_{p^\infty}) \cong H^1(L_{n,w}\widehat{E}_{p^\infty})$. The isomorphisms in the above equation and equation (17) are compatible for all $n \geq 0$ and they induce the isomorphism

$$H^1(L_{n,w}, E_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}} \cong H^1(L_{n,w}, \widehat{E}_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}}. \tag{18}$$

Note that for $n \geq 0$ and $w \in S^{\rm ss}(L_n)$, then $L_{n,w} = K_r^{(l)}$ for some $l, r \geq 0$. Let $\mathcal{O}$ denote the ring of integers of $K_0^{(l)}$. By Honda theory (*cf.* [9]), there exists a formal group $\mathcal{F}_{\rm ss}$ in $\mathcal{O}[[X]]$, known as the supersingular formal group of Honda type $t^2 + p$ which is an Eisenstein polynomial. Further, part (ii) and (iv) of Hyp 1 implies that the formal group $\widehat{E}$ is also of Honda type t^2+p (*cf.* Section 8.2 of [12]). Therefore, there exists an $\mathcal{O}$-isomorphism

$$\exp_{\widehat{E}} \circ \log_{\mathcal{F}_{\rm ss}} \ : \ \mathcal{F}_{\rm ss}(\mathfrak{m}) \to \widehat{E}(\mathfrak{m}), \tag{19}$$

where $\mathfrak{m}$ is the maximal ideal of the valuation ring $\mathcal{O}$ of the local field $L_{n,w}$. Since the Honda type t^2+p is independent of the choice of the elliptic curve, the supersingular formal group $\mathcal{F}_{\rm ss}$, and hence the formal group $\widehat{E}$ by the above isomorphism, are independent of the choice of elliptic curve E/L, assuming E/L satisfies Hyp 1.

For simplicity we denote $\mathcal{F}_{\rm ss}(\mathfrak{m})$ by $\mathcal{F}_{\rm ss}(L_{n,w})$ and define the plus minus norm groups $\mathcal{F}^{\pm}_{\rm ss}(L_{n,w}) \subseteq \mathcal{F}_{\rm ss}(L_{n,w})$ using the isomorphism (19)

$$\mathcal{F}^{\pm}_{\rm ss}(L_{n,w}) := \log_{\mathcal{F}_{\rm ss}} \circ \exp_{\widehat{E}}(\widehat{E}^{\pm}(L_{n,w})).$$

Moreover, the signed Kummer maps, defined for the formal groups in equation (6), can be defined for the supersingular formal group $\mathcal{F}_{\rm ss}(L_{n,w})$

$$0 \to \mathcal{F}^{\pm}_{\rm ss}(L_{n,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\kappa^{\pm,p^\infty}_{L_{n,w}}} H^1(L_{n,w}, (\mathcal{F}_{\rm ss})_{p^\infty}) \to H^1(L_{n,w}, (\mathcal{F}_{\rm ss})_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}} \to 0.$$

The isomorphism (19) is $\mathcal{O}_{K_0^{(l)}}$-linear and thus it commutes with $G_{L_{n,w}}$-action. Hence, there is an isomorphism

$$H^1(L_{n,w}, (\mathcal{F}_{\rm ss})_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}} \cong H^1(L_{n,w}, \widehat{E}_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}}. \tag{20}$$

Combining (18) and (20), for $w \in S^{\rm ss}(L_n)$

$$H^1(L_{n,w}, (\mathcal{F}_{\rm ss})_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}} \cong H^1(L_{n,w}, E_{p^\infty})/\text{Im } \kappa^{\pm,p^\infty}_{L_{n,w}}. \tag{21}$$

Since the left-hand side is independent of E, so is the right-hand side. This implies that $J^{\pm}_{\mathfrak{p}}(E_{p^\infty}/L_n)$ and $J^{\pm}_{\bar{\mathfrak{p}}}(E_{p^\infty}/L_n)$ are independent of the choice of E/L for any $n \geq 0$. Passing to the direct limit we have the following isomorphism

$$H^1(L_{\infty,w}, (\mathcal{F}_{\mathrm{ss}})_{p^\infty})/\mathrm{Im}\ \kappa^{\pm,p^\infty}_{L_{\infty,w}} \cong H^1(L_{\infty,w}, E_{p^\infty})/\mathrm{Im}\ \kappa^{\pm,p^\infty}_{L_{\infty,w}}, \tag{22}$$

where $w \in S^{\mathrm{ss}}(L_\infty) = S_p(L_\infty)$ which implies that the same is true for that the local conditions $J^{\pm}_{\mathfrak{p}}(E_{p^\infty}/L_\infty)$ and $J^{\pm}_{\bar{\mathfrak{p}}}(E_{p^\infty}/L_\infty)$. Suppose E_1/L and E_2/L are two elliptic curves satisfying Hyp 1. Let $j \in \{1,2\}$ and let $S_j = \{\mathfrak{p}, \bar{\mathfrak{p}}\} \cup S_j^{\mathrm{bad}}$, where S_j^{bad} is the set of primes where E_j has bad reduction. From now on, we shall now enlarge the set S by declaring

$$S := S_1 \cup S_2 = \{\mathfrak{p}, \bar{\mathfrak{p}}\} \cup S_1^{\mathrm{bad}} \cup S_2^{\mathrm{bad}}.$$

Recall from Remark 2.4 that adding a finite set of primes to the set S_j does not change $\mathcal{R}^{\pm/\pm}((E_j)_p/L_\infty)$ and $\mathrm{Sel}^{\pm/\pm}((E_j)_{p^\infty}/L_\infty)$.

Proposition 3.3. *Let E_1/L and E_2/L be two elliptic curves that satisfy Hyp 1. Moreover, suppose $(E_1)_p \cong (E_2)_p$ as $\mathrm{Gal}(\overline{L}/L)$-modules. Then, there is an isomorphism*

$$\mathcal{R}^{\pm/\pm}((E_1)_p/L_\infty) \cong \mathcal{R}^{\pm/\pm}((E_2)_p/L_\infty).$$

Proof. Let $j \in \{1,2\}$ and S be as above. Suppose $v = \mathfrak{p}$. Then, for any $\mathfrak{q}|\mathfrak{p}$ in L_∞ by Corollary 3.2 and isomorphism (22)

$$\begin{aligned} H^1(L_{\infty,\mathfrak{q}}, (E_j)_p)/\mathrm{Im}\left(\kappa^{\pm,p}_{L_{\infty,\mathfrak{q}}}\right) &\cong \left(H^1(L_{\infty,\mathfrak{q}}, (E_j)_{p^\infty})/\mathrm{Im}\left(\kappa^{\pm,p^\infty}_{L_{\infty,\mathfrak{q}}}\right)\right) \\ &\cong \left(H^1(L_{\infty,w}, (\mathcal{F}_{\mathrm{ss}})_{p^\infty})/\mathrm{Im}\ \kappa^{\pm,p^\infty}_{L_{\infty,w}}\right)_p. \end{aligned}$$

The right-hand side is independent of the choice $j \in \{1,2\}$. Therefore,

$$H^1(L_{\infty,\mathfrak{q}}, (E_1)_p)/\mathrm{Im}\left(\kappa^{\pm,p}_{L_{\infty,\mathfrak{q}}}\right) \cong H^1(L_{\infty,\mathfrak{q}}, (E_2)_p)/\mathrm{Im}\left(\kappa^{\pm,p}_{L_{\infty,\mathfrak{q}}}\right)$$

which implies that ${}^{\pm}\widetilde{K}_{\mathfrak{p}}((E_1)_p/L_\infty) \cong {}^{\pm}\widetilde{K}_{\mathfrak{p}}((E_2)_p/L_\infty)$. Similar argument works for when $v = \bar{\mathfrak{p}}$. Now let $v \in S \backslash \{\mathfrak{p}, \bar{\mathfrak{p}}\}$. Since $(E_1)_p \cong (E_2)_p$, it follows that for any prime $w|v$

$$H^1(L_{n,w}, (E_1)_p) \cong H^1(L_{n,w}, (E_2)_p)$$

and thus ${}^{\pm}\widetilde{K}_v((E_1)_p/L_n) \cong^{\pm} \widetilde{K}_v((E_2)_p/L_n)$. This means that for any $v \in S$ the local terms match, and therefore we have the result. □

Proposition 3.4. *Let E/L be an elliptic curve satisfying Hyp 1. Then, there exists an injective map*

$$\varphi^{\pm/\pm} : \mathcal{R}^{\pm/\pm}(E_p/L_\infty) \hookrightarrow \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty)_p$$

such that the $\mathrm{coker}(\varphi^{\pm/\pm})$ *is a cotorsion* $\Omega(G)$*-module. Therefore, the following equality holds*

$$\mathrm{corank}_{\Omega(G)}(\mathcal{R}^{\pm/\pm}(E_p/L_\infty)) = \mathrm{corank}_{\Omega(G)}(\mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty)_p).$$

Proof. Let $v \in S$ and suppose $w|v$ is a prime in L_∞. Set

$$\varphi_w^{\pm} := \begin{cases} \psi_{w,\infty} & \text{if } w|\mathfrak{l} \in S^{\mathrm{bad}}, \\ \psi_{\mathfrak{q},\infty}^{\pm} & \text{if } w = \mathfrak{q}|\mathfrak{p}, \\ \psi_{\bar{\mathfrak{q}},\infty}^{\pm} & \text{if } w = \bar{\mathfrak{q}}|\bar{\mathfrak{p}}. \end{cases}$$

The maps $\psi_{\mathfrak{q},\infty}^{\pm}$ and $\psi_{\bar{\mathfrak{q}},\infty}^{\pm}$ are isomorphisms by Proposition 3.1(1) and the $\mathbb{F}_p$-dimension of $\ker(\psi_{w,\infty})$ is less than or equal to 2 by Proposition 3.1(2). Define the map

$$\varphi_v^{\pm} : {}^{\pm}\widetilde{K}_v(E_p/L_\infty) \to \left(J_v^{\pm}(E_{p^\infty}/L_\infty)\right)_p \tag{23}$$
$$\varphi_v^{\pm} := \bigoplus_{w|v} \varphi_w^{\pm}.$$

Therefore, $\varphi_v^{\pm}$ is an isomorphism when $v \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$. Define the map $\varphi^{\pm/\pm}$ using the following commutative diagram (*cf.* [17, Corollary 4.5])

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{R}^{\pm/\pm}(E_p/L_\infty) & \longrightarrow & H^1(G_\infty^S, E_p) & \xrightarrow{\xi_p^{\pm/\pm}} & \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_\infty) \\ & & \downarrow{\scriptstyle \varphi^{\pm/\pm}} & & \downarrow{\scriptstyle \psi_{G_\infty^S,\infty}}\ \cong & & \downarrow{\scriptstyle \bigoplus_{v\in S}\varphi_v^{\pm}} \\ 0 & \longrightarrow & \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty)_p & \longrightarrow & H^1(G_\infty^S, E_{p^\infty})_p & \xrightarrow[\xi^{\pm/\pm}]{} & \bigoplus_{v\in S} \left(J_v^{\pm}(E_{p^\infty}/L_\infty)\right)_p. \end{array} \tag{24}$$

The middle vertical map is an isomorphism by Proposition 3.1(1). This implies that $\varphi^{\pm/\pm}$ is injective and

$$\mathrm{coker}(\varphi^{\pm/\pm}) \subseteq \ker\Big(\bigoplus_{v\in S} \varphi_v^{\pm}\Big) = \bigoplus_{v\in S^{\mathrm{bad}}} \ker(\varphi_v^{\pm}).$$

Suppose $v \in S^{\mathrm{bad}}$ and fix a prime $w|v$ in L_∞. Then we have

$$\ker(\varphi_v^{\pm}) = \mathrm{Ind}_{G_w}^{G} \ker(\psi_{w,\infty}) \cong \ker(\psi_{w,\infty}) \widehat{\otimes}_{\Omega(G_w)} \Omega(G),$$

where $G_w = \mathrm{Gal}(L_{\mathrm{cyc},w}, L_v) \cong \mathbb{Z}_p$ is the decomposition group of G at the prime w and $-\widehat{\otimes}_{\Omega(G_w)}-$ denotes the completed tensor product over $\Omega(G_w)$. Recall that the map $\psi_{w,\infty}$ is surjective with finite kernel by Proposition 3.1(2). Thus $\ker(\psi_{w,\infty})$ is a cotorsion $\Omega(G_w)$-module. As $\Omega(G)$ is a flat $\Omega(G_w)$-module (*cf.* [19, Lemma 3.3]), we have $\bigoplus_{v\in S^{\mathrm{bad}}} \ker(\varphi_v^{\pm})$ is cotorsion as a $\Omega(G)$-module. □

4. Signed Selmer and fine residual Selmer groups as Iwasawa modules

Here, we prove our main theorem. Throughout this section, we assume that E/L is an elliptic curve satisfying the assumptions of Hyp 1.

Conjecture 4.1 ([13, Conjecture 4.11])**.** *For any choice of the signs,*

$$\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$$

is a torsion $\Lambda(G)$*-module.*

If E is defined over $\mathbb{Q}$, then Lei and Sprung showed in [14] that the above conjecture holds (*cf.* proof of [14, Theorem 4.4]).

Hyp 2 $^{\pm/\pm}$: $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is a torsion $\Lambda(G)$-module.

Note that Conjecture 4.1 implies Hyp $2^{\pm/\pm}$ for all choices of the signs. In [17], to prove their main results, the authors assume (*cf.* [17, Hyp 2]):

Hyp 2 $^{\pm/\pm}$*(cyc)* : $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}})$ is a torsion $\Lambda(\Gamma)$-module.

Here, the $\Lambda(\Gamma)$-module $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}})$ denotes the Pontryagin dual of the Selmer groups $\mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}})$. We show later that Hyp $2^{\pm/\pm}(\mathrm{cyc})$ implies Hyp $2^{\pm/\pm}$ (*cf.* Proposition 5.1). If $\mathrm{Sel}(E_{p^\infty}/L) = \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L)$ is finite, then Hyp $2^{\pm/\pm}(\mathrm{cyc})$ is automatically satisfied (*cf.* Remark 4.5 of [15]).

Proposition 4.2. *Hyp* $2^{\pm/\pm}$*(cyc) (resp. Hyp* $2^{\pm/\pm}$*) holds if and only if*

(1) $H^2(G^S_{\mathrm{cyc}}, E_{p^\infty})$ *(resp.* $H^2(G^S_\infty, E_{p^\infty})$*) vanishes; and*

(2) *The map* $\xi^{\pm/\pm}_{\mathrm{cyc}}$ *in the exact sequence (resp. the map* $\xi^{\pm/\pm}$ *defined in* (12))

$$0 \to \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}}) \to H^1(G^S_{\mathrm{cyc}}, E_{p^\infty}) \xrightarrow{\xi^{\pm/\pm}_{\mathrm{cyc}}} \bigoplus_{v\in S} J^{\pm}_v(E_{p^\infty}/L_{\mathrm{cyc}}) \tag{25}$$

is surjective.

Proof. See Proposition 4.4 of [15] (*resp.* Proposition 4.12 of [13]). □

4.1. Cassels–Poitou–Tate exact sequence for fine signed residual Selmer group

Here, we aim to produce an analogous statement to Remark 4.2 for signed fine residual Selmer groups. More specifically, we would like to show that the Pontryagin dual of the signed fine residual Selmer group, $\mathcal{Y}^{\pm/\pm}(E_p/L_\infty)$, is a torsion $\Omega(G)$-module exactly when the exact sequence (13) is short exact. In other words, exactly when the map $\xi^{\pm/\pm}_p$ is surjective. Our strategy is the same as [17, Section 4.2]. We do this by studying the exact sequence (13) using Cassels–Poitou–Tate exact sequence (*cf.* [4, Theorem 1.5] for details). We begin by defining some new modules involved in the sequence. Let $n \geq 0$ and $w \in S(L_n)$. Define

$$\mathcal{W}^{\pm}_w(E_p/L_n) := \begin{cases} 0 & \text{if } w \in S^{\mathrm{bad}}(L_n), \\ \mathrm{Im}\ \kappa^{\pm,p}_{L_{n,\mathfrak{q}}} & \text{if } w = \mathfrak{q}|\mathfrak{p}, \\ \mathrm{Im}\ \kappa^{\pm,p}_{L_{n,\bar{\mathfrak{q}}}} & \text{if } w = \bar{\mathfrak{q}}|\bar{\mathfrak{p}}, \end{cases} \tag{26}$$

where the choice of the sign, as usual, depends on the sign of the Kummer map $\kappa^{\pm,p}$ in the direction of primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. For $v \in S$ (*cf.* equation (7))

$$^{\pm}\widetilde{K}_v(E_p/L_n) = \bigoplus_{w|v} H^1(L_{n,w}, E_p)/\mathcal{W}_w^{\pm}(E_p/L_n).$$

Let $\mathcal{W}_w^{\pm}(E_p/L_n)^{\perp} \subset H^1(L_{n,w}, E_p)$ denote the orthogonal complement of the group $\mathcal{W}_w^{\pm}(E_p/L_n)$ with respect to the Tate local duality (*cf.* [4, Section 1.1]). Define $\mathfrak{R}^{\pm/\pm}(E_p/L_n)$, a $\Omega(G_n)$-module, using the exact sequence

$$0 \to \mathfrak{R}^{\pm/\pm}(E_p/L_n) \to H^1(G_n^S, E_p) \to \bigoplus_{w\in S(L_n)} H^1(L_{n,w}, E_p)/\mathcal{W}_w^{\pm}(E_p/L_n)^{\perp}.$$

For any $n \geq 0$, the Cassels–Poitou–Tate exact sequence gives

$$\begin{aligned} 0 &\to \mathcal{R}^{\pm/\pm}(E_p/L_n) \to H^1(G_n^S, E_p) \to \bigoplus_{w\in S(L_n)} H^1(L_{n,w}, E_p)/\mathcal{W}_w^{\pm}(E_p/L_n) \\ &\to \mathfrak{R}^{\pm/\pm}(E_p/L_n)^{\wedge} \to H^2(G_n^S, E_p) \to \bigoplus_{w\in S(L_n)} H^2(L_{n,w}, E_p) \to 0. \end{aligned} \tag{27}$$

The final zero is due to equation (2) (*cf.* [4, Theorem 1.5]). Define

$$\mathscr{R}^{\pm/\pm}(E_p/L_\infty) := \varprojlim_{\text{cores}} \mathfrak{R}^{\pm/\pm}(E_p/L_n) \subset H^1_{\text{Iw}}(L, E_p), \tag{28}$$

where the Iwasawa cohomology module $H^1_{\text{Iw}}(L, E_p) := \varprojlim_{\text{cores}} H^1(G_n^S, E_p)$. Since $H^0(G_n^S, E_p)$ vanishes by equation (2), using Jannsen's spectral sequence (*cf.* [10, Corollary 13]), there is an isomorphism

$$H^1_{\text{Iw}}(L, E_p) \cong \text{Hom}_{\Omega(G)}\left(\text{Hom}_{\mathbb{F}_p}(H^1(G_\infty^S), \mathbb{F}_p), \Omega(G)\right).$$

The left-hand side is a torsion-free $\Omega(G)$-module and therefore so is the right-hand side. In particular, $\mathscr{R}^{\pm/\pm}(E_p/L_\infty)$ is also torsion-free being a $\Omega(G)$-submodule of $H^1_{\text{Iw}}(L, E_p)$. Moreover (*cf.* [17, Lemma 4.6.])

$$\mathscr{R}^{\pm/\pm}(E_p/L_\infty)^{\wedge} = \text{Hom}(\varprojlim_{\text{cores}} \mathfrak{R}^{\pm/\pm}(E_p/L_n), \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim_{\text{cores}^{\wedge}} \mathfrak{R}^{\pm/\pm}(E_p/L_n)^{\wedge}.$$

Taking the direct limit of the exact sequence (27) yields

$$\begin{aligned} 0 &\to \mathcal{R}^{\pm/\pm}(E_p/L_\infty) \to H^1(G_\infty^S, E_p) \xrightarrow{\xi_p^{\pm/\pm}} \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_\infty) \\ &\to \mathscr{R}^{\pm/\pm}(E_p/L_\infty)^{\wedge} \to H^2(G_\infty^S, E_p) \to \bigoplus_{w\in S(L_\infty)} H^2(L_{\infty,w}, E_p) \to 0. \end{aligned} \tag{29}$$

Our plan now is to analyze this exact sequence. In [3], the authors introduce the $\mu = 0$-conjecture for the fine Selmer group (*cf.* Conjecture A of [2]). Its original formulation asserts that for an elliptic curve E over any number filed F, the Pontryagin dual of the classical fine Selmer group over the cyclotomic extension F_{cyc} is a finitely generated $\mathbb{Z}_p$-module. Here, we use an equivalent cohomological version of Conjecture A (*cf.* [17, Proposition 4.7]).

Conjecture A: Let $\Gamma = \mathrm{Gal}(L_{\mathrm{cyc}}/L)$ and $G_{\mathrm{cyc}}^S := \mathrm{Gal}(L^S/L_{\mathrm{cyc}})$. Then the $\Omega(\Gamma)$-module $H^2(G_{\mathrm{cyc}}^S, E_p)$ is trivial.

4.2. Main theorem

We prove the main theorem (*cf.* Theorem 4.5) of this article. This is an analogue of [17, Proof of Theorem 4.12].

Lemma 4.3. *Let $v \in S$ and $w|v$ be a prime in L_∞.*

(1) *If $w \nmid p$, then the Pontryagin dual of $H^1(L_{\infty,w}, E_{p^\infty})$ has μ-invariant equal to zero. This implies that*

$$\mathrm{corank}_{\Lambda(G)} H^1(L_{\infty,w}, E_{p^\infty}) = \mathrm{corank}_{\Omega(G)} H^1(L_{\infty,w}, E_p).$$

(2) *Suppose Conjecture A holds. Then, the Pontryagin dual of $H^1(G_\infty^S, E_{p^\infty})$ has μ-invariant equal to zero. This implies that*

$$\mathrm{corank}_{\Lambda(G)} H^1(G_\infty^S, E_{p^\infty}) = \mathrm{corank}_{\Omega(G)} H^1(G_\infty^S, E_p).$$

Proof. When $w \nmid p$, then $L_{\infty,w}$ is equal to $L_{\mathrm{cyc},w}$ and the statement that $H^1(L_{\infty,w}, E_{p^\infty})^\wedge$ has μ-invariant equal to zero is proven in [17, Lemma 4.9, part i]. By Proposition 3.1, there is an isomorphism $(H^1(G_\infty^S, E_{p^\infty}))_p \cong H^1(G_\infty^S, E_p)$ and equation (14) implies

$$\mathrm{corank}_{\Lambda(G)} H^1(L_{\infty,w}, E_{p^\infty}) = \mathrm{corank}_{\Omega(G)} H^1(L_{\infty,w}, E_p).$$

For part (2), Lemma 5.6 in [15] tells us $H^1(G_\infty^S, E_{p^\infty})^H \cong H^1(G_{\mathrm{cyc}}^S, E_{p^\infty})$, where $H = \mathrm{Gal}(L_\infty, L_{\mathrm{cyc}}) \cong \mathbb{Z}_p$. Then $\mu_\Gamma(H^1(G_{\mathrm{cyc}}^S, E_{p^\infty})^\wedge) = 0$ by [17, Lemma 4.9, part ii]. Let $M = H^1(G_\infty^S, E_{p^\infty})^\wedge$. Note that $\mu_\Gamma(M_H)$ vanishes, where M_H denotes the H-coinvariant of $\Lambda(G)$-module M. Then, a version of the topological Nakayama's lemma (*cf.* [1, Theorem 2, page 5]) implies that $\mu_G(M) = 0$. The equality about the coranks is implied by Proposition 3.1 and equation (14). □

Thus, given Conjecture A, the last two modules in the exact sequence (29) vanish. Therefore, to show equality (30), it is enough to show that

$$\mathrm{corank}_{\Omega(G)}(H^1(G_\infty^S, E_p)) = \mathrm{corank}_{\Omega(G)}(\bigoplus_{v \in S} {}^{\pm}\widetilde{K}_v(E_p/L_\infty)).$$

Proposition 4.4. *Let E/L be an elliptic curve satisfying Hyp 1 and Hyp $2^{\pm/\pm}$. Moreover, suppose Conjecture A holds. Then*

$$\mathrm{corank}_{\Omega(G)}(\mathcal{R}^{\pm/\pm}(E_p/L_\infty)) = \mathrm{corank}_{\Omega(G)}(\mathscr{R}^{\pm/\pm}(E_p/L_\infty)^\wedge). \tag{30}$$

Proof. By Theorems 3.2 and 4.1 of [18], $\mathrm{corank}_{\Lambda(G)}(H^1(G_\infty^S, E_{p^\infty})) = [L : \mathbb{Q}] = 2$. Putting this together with Lemma 4.3(2) we get

$$\mathrm{corank}_{\Lambda(G)}(H^1(G_\infty^S, E_{p^\infty})) = \mathrm{corank}_{\Omega(G)}(H^1(G_\infty^S, E_p)) = 2. \tag{31}$$

For a prime $w \nmid p$ in L_∞ Proposition 2 in [6] implies that $\Lambda(G)$-corank of $H^1(L_{\infty,w}, E_{p^\infty})$ is zero. This means that for $v \nmid p$, the $\Lambda(G)$-corank of $J_v^\pm(E_{p^\infty}/L_\infty)$ is zero. Hence, we have

$$\operatorname{corank}_{\Lambda(G)}(\bigoplus_{v\in S} J_v^\pm(E_{p^\infty}/L_\infty)) = \operatorname{corank}_{\Lambda(G)}(\bigoplus_{v\in S_p} J_v^\pm(E_{p^\infty}/L_\infty)). \tag{32}$$

Furthermore, by Lemma 4.3(2) we know

$$\begin{aligned} \operatorname{corank}_{\Lambda(G)}(H^1(L_{\infty,w}, E_{p^\infty})) &= \operatorname{corank}_{\Omega(G)} H^1(L_{\infty,w}, E_p) \\ &= \operatorname{corank}_{\Omega(G)}({}^\pm\widetilde{K}_v(E_p/L_\infty)) = 0 \\ \Longrightarrow \operatorname{corank}_{\Omega(G)}(\bigoplus_{v\in S} {}^\pm\widetilde{K}_v(E_p/L_\infty)) &= \operatorname{corank}_{\Omega(G)}(\bigoplus_{v\in S_p} {}^\pm\widetilde{K}_v(E_p/L_\infty)). \end{aligned}$$

For $v|p$, recall that the map $\varphi_v^\pm$ (*cf.* (23)) gives an isomorphism between $(J_v^\pm(E_{p^\infty}/L_\infty))_p$ and ${}^\pm\widetilde{K}_v(E_p/L_\infty)$. Therefore, $\Omega(G)$-corank of $(J_v^\pm(E_{p^\infty}/L_\infty))_p$ is equal to $\operatorname{corank}_{\Omega(G)}({}^\pm\widetilde{K}_v(E_p/L_\infty))$. Moreover, for $v|p$ the local condition $J_v^\pm(E_{p^\infty}/L_\infty)$ is a free $\Lambda(G)$-module (*cf.* [13, Corollary 3.9]) and

$$\operatorname{corank}_{\Lambda(G)}(\bigoplus_{v\in S_p} J_v^\pm(E_{p^\infty}/L_\infty)) = \operatorname{corank}_{\Lambda(G)}(H^1(G_\infty^S, E_{p^\infty}) = 2.$$

This means that $J_v^\pm(E_{p^\infty}/L_\infty)$ has μ-invariant zero. By equation (14),

$$\begin{aligned} \operatorname{corank}_{\Lambda(G)}(J_v^\pm(E_{p^\infty}/L_\infty)) &= \operatorname{corank}_{\Omega(G)}((J_v^\pm(E_{p^\infty}/L_\infty))_p) \\ &= \operatorname{corank}_{\Omega(G)}({}^\pm\widetilde{K}_v(E_p/L_\infty)). \end{aligned}$$

This, together with equation (32) gives us that

$$\operatorname{corank}_{\Omega(G)}(\bigoplus_{v\in S} {}^\pm\widetilde{K}_v(E_p/L_\infty)) = \operatorname{corank}_{\Omega(G)}(\bigoplus_{v\in S_p} {}^\pm\widetilde{K}_v(E_p/L_\infty)) = 2.$$

Comparing the above equation and equation (31) yields

$$\operatorname{corank}_{\Omega(G)}(H^1(G_\infty^S, E_p)) = \operatorname{corank}_{\Omega(G)}(\bigoplus_{v\in S} {}^\pm\widetilde{K}_v(E_p/L_\infty)) = 2. \tag{33}$$

The exact sequence (29) implies that

$$\operatorname{corank}_{\Omega(G)}(\mathcal{R}^{\pm/\pm}(E_p/L_\infty)) = \operatorname{corank}_{\Omega(G)}(\mathscr{R}^{\pm/\pm}(E_p/L_\infty)^\wedge). \qquad \square$$

We are now ready to give our main theorem which is the analogue of [17, Theorem 4.12] in our setting. It describes a criterion for the vanishing of the signed μ-invariants based completely on the structure of the fine singed residual Selmer groups as Iwasawa modules.

Theorem 4.5. *Let E/L be an elliptic curve satisfying Hyp 1 and Hyp $2^{\pm/\pm}$. Furthermore, suppose Conjecture A holds. Then the following statements are equivalent*:

(1) $\mathcal{Y}^{\pm/\pm}(E_p/L_\infty) = \mathcal{R}^{\pm/\pm}(E_p/L_\infty)^\wedge$ *is $\Omega(G)$-torsion.*

(2) *The μ-invariant* $\mu_G^{\pm/\pm}(E_{p^\infty}/L_\infty) := \mu_G(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty))$ *vanishes.*

(3) *The map* $\xi_p^{\pm/\pm}$*, described in diagram* (29)*, is surjective.*

Proof. To see (1) and (2) are equivalent, recall from Proposition 3.4

$$\mathrm{rank}_{\Omega(G)}(\mathcal{Y}^{\pm/\pm}(E_p/L_\infty)) = \mathrm{rank}_{\Omega(G)}(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)/p\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty));$$

$$0 = \mathrm{rank}_{\Lambda(G)}(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)) \leq \mathrm{rank}_{\Omega(G)}(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)/p\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty))$$

with equality exactly when $\mu_G^{\pm/\pm}(E_{p^\infty}/L_\infty)$ vanishes. Therefore, given (1) we have

$$\mathrm{rank}_{\Lambda(G)}(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)) = \mathrm{rank}_{\Omega(G)}(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)/p\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)) = 0$$

which implies $\mu_G^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is zero. On the other hand, if (2) holds, then

$$0 = \mathrm{rank}_{\Lambda(G)}(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)) = \mathrm{rank}_{\Omega(G)}(\mathcal{Y}^{\pm/\pm}(E_p/L_\infty))$$

and so (1) $\iff$ (2). Now let us show (1) $\implies$ (3). The discrete $\Omega(G)$-module $H^2(G_\infty^S, E_p)$ vanishes by Proposition 4.7 [17]. By Proposition 4.4

$$0 = \mathrm{corank}_{\Omega(G)}(\mathcal{R}^{\pm/\pm}(E_p/L_\infty)) = \mathrm{corank}_{\Omega(G)}(\mathscr{R}^{\pm/\pm}(E_p/L_\infty)^\wedge).$$

This means that $\mathscr{R}^{\pm/\pm}(E_p/L_\infty)^\wedge$ is a cotorsion $\Omega(G)$-module. However, being a $\Omega(G)$-submodule of a cotorsion-free module $H^1_{\mathrm{Iw}}(L, E_p)$, the module $\mathscr{R}^{\pm/\pm}(E_p/L_\infty)$ is also torsion-free (*cf.* equation (28)). As a result,

$$\mathscr{R}^{\pm/\pm}(E_p/L_\infty)^\wedge$$

vanishes. Therefore, the map $\xi_p^{\pm/\pm}$ (*cf.* the exact sequence (29)) is surjective. Finally to see (3) $\implies$ (1), suppose the map $\xi_p^{\pm/\pm}$ is surjective and hence we have

$$0 \to \mathcal{R}^{\pm/\pm}(E_p/L_\infty) \to H^1(G_\infty^S, E_p) \xrightarrow{\xi_p^{\pm/\pm}} \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_\infty) \to 0. \tag{34}$$

Taking Pontryagin duals of the above sequence gives

$$0 \to \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_\infty)^\wedge \xrightarrow{\xi_p^{\pm/\pm\wedge}} H^1(G_\infty^S, E_p)^\wedge \to \mathcal{Y}^{\pm/\pm}(E_p/L_\infty) \to 0.$$

In the proof of Proposition 4.4, we proved that (*cf.* equation (33)) the first two terms in the above short exact sequence have the same $\Omega(G)$-rank which is equal to 2. So, $\mathcal{Y}^{\pm/\pm}(E_p/L_\infty)$ is a torsion $\Omega(G)$-module. □

We record the following important corollary of Theorem 4.5.

Corollary 4.6. *Let E_1/L and E_2/L be two elliptic curves that satisfy Hyp 1 and Hyp $2^{\pm/\pm}$. Suppose they have isomorphic residual Galois representations and Conjecture A is satisfied for either E_1/L or E_2/L (and hence both). Then,* $\mu_G^{\pm/\pm}((E_1)_{p^\infty}/L_\infty) = 0$ *if and only if* $\mu_G^{\pm/\pm}((E_2)_{p^\infty}/L_\infty) = 0$.

Proof. Without loss of generality assume $\mu_G^{\pm/\pm}((E_1)_{p^\infty}/L_\infty)$ vanishes.

Note that Conjecture A for an elliptic curve E/L only depends on the isomorphism class of residual Galois representation E_p. By Theorem 4.5, the module $\mathcal{Y}^{\pm/\pm}((E_1)_p/L_\infty)$ is $\Omega(G)$-torsion. Proposition 3.3 gives

$$\mathcal{R}^{\pm/\pm}((E_1)_p/L_\infty) \cong \mathcal{R}^{\pm/\pm}((E_2)_p/L_\infty).$$

Thus, the module $\mathcal{Y}^{\pm/\pm}((E_2)_p/L_\infty)$ is also $\Omega(G)$-torsion. Again by Theorem 4.5, this implies that $\mu_G^{\pm/\pm}((E_2)_{p^\infty}/L_\infty)$ is equal to zero. □

4.3. Comparison with the cyclotomic level

Let H denote the Galois group $\mathrm{Gal}(L_\infty/L_{\mathrm{cyc}})$ and consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{R}^{\pm/\pm}(E_p/L_{\mathrm{cyc}}) & \longrightarrow & H^1(G^S_{\mathrm{cyc}}, E_p) & \xrightarrow{\xi^{\pm/\pm}_{p,\mathrm{cyc}}} & \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_{\mathrm{cyc}}) \\ & & \downarrow \alpha_p^{\pm/\pm} & & \downarrow \beta_p & & \downarrow \widetilde{\gamma}_p^{\pm/\pm} := \bigoplus_{v\in S} \widetilde{\gamma}_v^{\pm} \\ 0 & \longrightarrow & \mathcal{R}^{\pm/\pm}(E_p/L_\infty)^H & \longrightarrow & H^1(G^S_\infty, E_p)^H & \xrightarrow[\xi_p^{\pm/\pm,H}]{} & \bigoplus_{v\in S} {}^{\pm}\widetilde{K}_v(E_p/L_\infty)^H \end{array} \tag{35}$$

Proposition 4.7. *All the vertical maps in the diagram* (35) *are isomorphisms. In particular,* $\mathcal{R}^{\pm/\pm}(E_p/L_\infty)^H \cong \mathcal{R}^{\pm/\pm}(E_p/L_{\mathrm{cyc}})$.

Proof. The map β_p is an isomorphism by [15, Lemma 5.6]. Let us show that the map $\widetilde{\gamma}_p^{\pm/\pm} := \bigoplus_{v\in S} \widetilde{\gamma}_v^{\pm}$ is an isomorphism. For any $v \in S\backslash S_p$, the map $\widetilde{\gamma}_v^{\pm}$ is the identity map (*cf.* proof of Lemma 5.10 in [15]). Let us assume $v = \mathfrak{p}$ and let $A := J_{\mathfrak{p}}^{\pm}(E_{p^\infty}/L_\infty)$. Note that, by Corollary 3.2, $A_p \cong {}^{\pm}\widetilde{K}_{\mathfrak{p}}(E_p/L_\infty)$ and by Lemma 5.10 of [15], $A^H \cong J_{\mathfrak{p}}^{\pm}(E_{p^\infty}/L_{\mathrm{cyc}})$. Furthermore, by [17, Proposition 4.1-d], there is the following isomorphism

$$\psi^{\pm}_{\mathfrak{q},\mathrm{cyc}} : {}^{\pm}\widetilde{K}_{\mathfrak{p}}(E_p/L_{\mathrm{cyc}}) \xrightarrow{\cong} \left(J_{\mathfrak{p}}^{\pm}(E_{p^\infty}/L_{\mathrm{cyc}})\right)_p.$$

Using the isomorphism $(A^H)_p \cong (A_p)^H$ along with the above map, we see

$$\begin{aligned} \widetilde{\gamma}_{\mathfrak{p}}^{\pm} : {}^{\pm}\widetilde{K}_{\mathfrak{p}}(E_p/L_{\mathrm{cyc}}) &\xrightarrow[\psi^{\pm}_{\mathfrak{q},\mathrm{cyc}}]{\cong} \left(J_{\mathfrak{p}}^{\pm}(E_{p^\infty}/L_{\mathrm{cyc}})\right)_p \overset{\cong}{\Longrightarrow} (A^H)_p \cong (A_p)^H \\ &\overset{\cong}{\Longrightarrow} {}^{\pm}\widetilde{K}_{\mathfrak{p}}(E_p/L_\infty)^H \end{aligned}$$

is an isomorphism. The case where $v = \bar{\mathfrak{p}}$ is similar. Finally, the snake lemma implies that $\alpha_p^{\pm/\pm}$ is also an isomorphism. □

We record the analogue of Theorem 4.5 in the cyclotomic setting.

Theorem 4.8 ([17, Theorem 4.12])**.** *Let E/L be an elliptic curve satisfying Hyp 1 and Hyp 2$^{\pm/\pm}$(cyc). Then the following are equivalent*:

(1) $\mathcal{Y}^{\pm/\pm}(E_p/L_{\mathrm{cyc}}) = \mathcal{R}^{\pm/\pm}(E_p/L_{\mathrm{cyc}})^\wedge$ *is* $\Omega(\Gamma)$*-torsion.*

(2) *The signed cyclic μ-invariant* $\mu_\Gamma^{\pm/\pm}(E_{p^\infty}/L_{\rm cyc}) := \mu_\Gamma(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_{\rm cyc}))$ *vanishes.*

(3) *The map* $\xi_{p,{\rm cyc}}^{\pm/\pm}$ *in diagram* (35) *is surjective and Conjecture A holds.*

Corollary 4.9. *Suppose E/L satisfies Hyp 1 and Hyp $2^{\pm/\pm}$(cyc). Assume one of the equivalent statements in Theorem 4.8 is satisfied. Then, the top and bottom exact sequences in diagram* (35) *are short exact.*

Proof. By Theorem 4.8, the map $\xi_{p,{\rm cyc}}^{\pm/\pm}$ is surjective and hence the top row in diagram (35) is short exact. Since the map $\widetilde{\gamma}_p^{\pm/\pm}\circ\xi_{p,{\rm cyc}}^{\pm/\pm}$ is surjective, the map $\xi_p^{\pm/\pm,H}$ is also surjective. □

We now combine Theorem 4.8 and Proposition 4.7.

Theorem 4.10. *Suppose E/L satisfies Hyp 1 and Hyp $2^{\pm/\pm}$(cyc). Furthermore, assume one of the equivalent statements in Theorem 4.8 is satisfied. Then, the equivalent statements in Theorem 4.5 are satisfied.*

Proof. Conjecture A is satisfied by part 3 of Theorem 4.8. Proposition 5.1 implies that Hyp $2^{\pm/\pm}$ holds. Using Proposition 4.7

$$\mathcal{Y}^{\pm/\pm}(E_p/L_{\rm cyc}) = \mathcal{R}^{\pm/\pm}(E_p/L_{\rm cyc})^\wedge \cong (\mathcal{R}^{\pm/\pm}(E_p/L_\infty)^H)^\wedge \cong \mathcal{Y}^{\pm/\pm}(E_p/L_\infty)_H.$$

Suppose $\mathcal{Y}^{\pm/\pm}(E_p/L_{\rm cyc})$ which is isomorphic to $\mathcal{Y}^{\pm/\pm}(E_p/L_\infty)_H$ is torsion as a $\Omega(\Gamma)$-module. Then, we can apply [8, Lemma 2.6] to get that the module $\mathcal{Y}^{\pm/\pm}(E_p/L_\infty)$ is torsion as a $\Omega(G)$-module. □

Corollary 4.11. *With the same assumption as Theorem 4.10,*

$$H^1(H,\mathcal{R}^{\pm/\pm}(E_p/L_\infty)) = 0.$$

Proof. By Theorem 4.10, we have the short exact sequence (34). Taking the long exact H-Galois cohomology of this sequence yields Coker($\xi_p^{\pm/\pm,H}$) is equal to $H^1(H,\mathcal{R}^{\pm/\pm}(E_p/L_\infty))$. By Corollary 4.9, the map $\xi_p^{\pm/\pm,H}$ is surjective which proves the claim. □

Corollary 4.12. *Suppose E/L is an elliptic curve satisfying Hyp 1 and Hyp $2^{\pm/\pm}$(cyc). Furthermore, assume the $\Omega(\Gamma)$-module $H^1(H,\mathcal{R}^{\pm/\pm}(E_p/L_\infty))$ vanishes. Then, the converse of Theorem 4.10 holds too. That is, all the statements in Theorem 4.8 and Theorem 4.5 are equivalent.*

Proof. Suppose the map $\xi_p^{\pm/\pm}$ is surjective and Conjecture A holds. Since $H^1(H,\mathcal{R}^{\pm/\pm}(E_p/L_\infty))$ vanishes, a similar argument as in Corollary 4.11 shows that the map $\xi_p^{\pm/\pm,H}$ is surjective. Using diagram (35), the map $\xi_p^{\pm/\pm,H}\circ\beta_p$ is surjective, and hence that the map $\xi_{p,{\rm cyc}}^{\pm/\pm}$ is also surjective. □

Theorem 4.10 implies that if E/L satisfies Hyp $2^{\pm/\pm}$(cyc), then

$$\mu_\Gamma^{\pm/\pm}(E_{p^\infty}/L_{\rm cyc}) = 0 \implies \mu_G^{\pm/\pm}(E_{p^\infty}/L_\infty) = 0. \tag{36}$$

Moreover, if $H^1(H, \mathcal{R}^{\pm/\pm}(E_p/L_\infty))$ vanishes and Conjecture A holds, then the two sides of the statement (36) become equivalent by Corollary 4.12.

5. Pseudo-null submodules

Let E/L be an elliptic curve satisfying Hyp 1. We show that under some mild assumptions, the $\Lambda(G)$-modules the $\Lambda(G)$-modules $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ and $\mathfrak{X}(E_{p^\infty}/L_\infty)$ have no non-trivial pseudo-null submodules (*cf.* Definition 5.1.4 of [16] for the definition of pseudo-null submodules).

The Iwasawa algebra $\Lambda(G)$ is a Noetherian regular local commutative ring and hence a Cohen–Macaulay ring. In particular, the depth of $\Lambda(G)$ coincides with its Krull dimension (*cf.* Chapter 17 of [5]). By the Auslander–Buchsbaum–Serre theorem, the global dimension of $\Lambda(G)$ is finite and it coincides with its Krull dimension. Let H be $\mathrm{Gal}(L_\infty, L_{\mathrm{cyc}})$ and consider the following diagram of $\Lambda(\Gamma)$-modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}}) & \longrightarrow & H^1(G^S_{\mathrm{cyc}}, E_{p^\infty}) & \xrightarrow{\xi^{\pm/\pm}_{\mathrm{cyc}}} & \bigoplus_{v\in S} J^{\pm}_v(E_{p^\infty}/L_{\mathrm{cyc}}) \\ & & \alpha^{\pm/\pm}\Big\downarrow\cong & & \beta\Big\downarrow\cong & & \cong\Big\downarrow\gamma^{\pm/\pm} := \bigoplus_{v\in S}\gamma^{\pm}_v \\ 0 & \longrightarrow & \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty)^H & \longrightarrow & H^1(G^S_\infty, E_{p^\infty})^H & \xrightarrow{\xi^{\pm/\pm,H}} & \bigoplus_{v\in S}\left(J^{\pm}_v(E_{p^\infty}/L_\infty)\right)^H \end{array} \tag{37}$$

The maps β and $\gamma^{\pm/\pm}$ are isomorphisms by Lemmas 5.6, 5.8, and 5.10 of [15]. By snake lemma, the map $\alpha^{\pm/\pm}$ is also an isomorphism and hence, all the vertical maps in the diagram (37) are isomorphisms.

Proposition 5.1. *Suppose E/L is an elliptic curve over quadratic number field L satisfying Hyp 1. If Hyp $2^{\pm/\pm}$(cyc) holds, then*

$$\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}}) \cong \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)_H, \tag{38}$$

where $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)_H$ denotes the of H-coinvariants of $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$. Moreover, Hyp $2^{\pm/\pm}$ holds and the group $H^1(H, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty))$ vanishes.

Proof. By Proposition 4.2, the top row of the diagram (37) becomes a short exact sequence. A diagram chase gives that, since the map $\gamma^{\pm/\pm} \circ \xi^{\pm/\pm}_{\mathrm{cyc}}$ is surjective, the map $\xi^{\pm/\pm,H}$ is also surjective and hence, the diagram (37) becomes an isomorphism of two short exact sequences. Taking the Pontryagin dual of the isomorphism $\alpha^{\pm/\pm}$ gives the first claim.

By Hyp $2^{\pm/\pm}$(cyc), $\Lambda(\Gamma)$-module $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)_H$ is finitely generated and torsion. This implies that $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is a finitely generated torsion $\Lambda(G)$-module (cf. [8, Lemma 2.6]) which is Hyp $2^{\pm/\pm}$. By Proposition 4.2, this implies that the map $\xi^{\pm/\pm}$ in (12) is surjective. Taking long exact H-cohomology of the short exact sequence (12), we get the bottom short exact sequence in the diagram (37). In particular, the map $\xi^{\pm/\pm,H}$ is surjective which implies $\mathrm{Coker}(\xi^{\pm/\pm,H}) = H^1(H, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty)) = 0$. □

Lemma 5.2. *Assuming Hyp $2^{\pm/\pm}$(cyc) holds, the cohomological group* $H^1(\Gamma, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}}))$ *vanishes if and only if* $\xi_{\mathrm{cyc}}^{\pm/\pm,\Gamma}$ *is surjective.*

Proof. By Proportion 5.1, $H^1(H, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty))$ vanishes. The inflation-restriction exact sequence implies that

$$H^1(\Gamma, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}})) = H^1(G, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty)).$$

Let $G^S := \mathrm{Gal}(L^S/L)$ and consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}(E_{p^\infty}/L) & \longrightarrow & H^1(G^S, E_{p^\infty}) & \xrightarrow{\lambda_0} & \bigoplus_{v\in S} J_v(E_{p^\infty}/L) \\ & & \Big\downarrow{\scriptstyle \alpha_{\mathrm{cyc}}^{\pm/\pm}} & & \Big\downarrow{\scriptstyle \beta_{\mathrm{cyc}}} & & \Big\downarrow{\scriptstyle \gamma_{\mathrm{cyc}}^{\pm/\pm} := \bigoplus_{v\in S} \gamma_{\mathrm{cyc},v}^{\pm}} \\ 0 & \longrightarrow & \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}})^\Gamma & \longrightarrow & H^1(G^S_{\mathrm{cyc}}, E_{p^\infty})^\Gamma & \xrightarrow{\xi_{\mathrm{cyc}}^{\pm/\pm,\Gamma}} & \bigoplus_{v\in S} \left(J_v^{\pm}(E_{p^\infty}/L_{\mathrm{cyc}})\right)^\Gamma \end{array} \tag{39}$$

The map λ_0 is as in the exact sequence (5). Note that at $L_0 = L$, the classical p^∞-Selmer and the signed Selmer groups coincide. Recall that Hyp $2^{\pm/\pm}$(cyc) implies the map $\xi^{\pm/\pm}$ is surjective by Proposition 4.2. Taking long exact Γ-cohomology of the short exact sequence (25) yields that $\mathrm{Coker}(\xi_{\mathrm{cyc}}^{\pm/\pm,\Gamma}) = H^1(\Gamma, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}}))$. □

Lemma 5.3. *The map* β_{cyc} *in diagram* (39) *is an isomorphism.*

Proof. The Hochschild–Serre spectral sequence gives

$$0 \to H^1(\Gamma, E_{p^\infty}(L_{\mathrm{cyc}})) \to H^1(G^S, E_{p^\infty}) \xrightarrow{\beta_{\mathrm{cyc}}} H^1(G^S_{\mathrm{cyc}}, E_{p^\infty})^\Gamma \to 0.$$

The last term is zero as Γ has p-cohomological dimension. Moreover, equation (2) gives that $E_{p^\infty}(L_{\mathrm{cyc}})$ is zero and so $H^1(\Gamma, E_{p^\infty}(L_{\mathrm{cyc}}))$ vanishes. □

Remark 5.4 ([15, Corollary 5.13]). If $\mathrm{Sel}(E_{p^\infty}/L)$ is finite, the map $\xi_{\mathrm{cyc}}^{\pm/\pm,\Gamma}$ is surjective and hence

$$H^1(\Gamma, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}})) = H^1(G, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty)) = 0.$$

Let γ and h be topological generators of the groups Γ and H, respectively. Since $G \cong H \times \Gamma$, there is an isomorphism

$$\Lambda(G) = \mathbb{Z}_p[[G]] \cong \mathbb{Z}_p[[H \times \Gamma]] \xrightarrow{\cong} \mathbb{Z}_p[[T_1, T_2]],$$

where $h - 1$ (*resp.* $\gamma - 1$) is mapped to the indeterminate variable T_1 (*resp.* T_2) and we extend $\mathbb{Z}_p$-linearly. In what follows we use $\Lambda(H)$ (*resp.* $\Lambda(\Gamma)$) and $\mathbb{Z}_p[[T_1]]$ (*resp.* $\mathbb{Z}_p[[T_2]]$) interchangeably.

Definition 5.5. Suppose R is a ring and let M be an R-module. A sequence of elements $f_1, \ldots, f_k$ of R is called an *M-regular sequence* if the following conditions hold:

(1) The element f_i is a non-zero divisor on $M/(f_1, \ldots, f_{i-1})M$ for each $i = 1, \ldots, k$;

(2) the module $M/(f_1, \dots, f_k)M$ is not zero.

If I is an ideal of the ring R and $f_1, \dots, f_k \in I$, then we call $f_1, \dots, f_k$ an *M-regular sequence in I*. Moreover, if $M = R$, we call $f_1, \dots, f_k$ a *regular sequence in I*.

Using the above definition we can define the depth of a finitely generated module over an Iwasawa algebra Λ.

Definition 5.6. Let I be an ideal of Λ and suppose M is a finitely generated Λ-module such that $IM \neq M$. Then the *I-depth* of M, denoted by $\mathrm{depth}_I(M)$, is the maximal length of a M-regular sequence in I. The *depth* of M as an Λ-module, denoted by $\mathrm{depth}(M)$, is defined to be

$$\mathrm{depth}(M) := \mathrm{depth}_{\mathfrak{m}}(M),$$

where $\mathfrak{m}$ is the maximal ideal of Λ.

Theorem 5.7. *Suppose E/L is an elliptic curve satisfying Hyp 1 and $\mathrm{Sel}(E_{p^\infty}/L)$ is finite. Then the following assertions hold.*

(1) *$\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ has no non-trivial pseudo-null $\Lambda(G)$-submodule.*
(2) *$\mathfrak{X}(E_{p^\infty}/L_\infty)$ has no non-trivial pseudo-null $\Lambda(G)$-submodule.*

Proof. For part (1), note that for a finitely generated $\Lambda(G)$-module M

$$\mathrm{depth}(M) \leq \mathrm{Krulldim}(M) \leq \mathrm{Krulldim}(\Lambda(G)) = 3.$$

Since the module $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is $\Lambda(G)$-torsion then $\mathrm{depth}(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)) \leq 2$. If the depth of $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is two, then the Auslander–Buchsbaum formula (*cf.* Chapter 19.3 of [5]) will imply that the projective dimension of $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is equal to one. By Proposition 3.10 of [21], any module of projective dimension at most one has no non-trivial non-trivial pseudo-null submodule. By Proposition 5.1, $H^1(H, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty))^\wedge$ vanishes, and

$$\begin{aligned} 0 &= H^1(H, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty))^\wedge \cong H_1(H, \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)) \\ &= \mathrm{Tor}_1^{\Lambda(H)}(\mathbb{Z}_p, \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)) = \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)[T_1], \end{aligned} \tag{40}$$

where $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)[T_1]$ is the set of elements in $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ that are annihilated by T_1. Therefore, the element T_1 is a non-zero divisor on $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$. Proposition 5.1 also tells us that $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)_H \cong \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}})$ as $\Lambda(\Gamma)$-modules. Note that $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)_H$ is non-zero, otherwise, by Nakayama's lemma $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is trivial as $T_1 \in \mathfrak{m}_G$. Arguing as above, we see

$$\begin{aligned} H^1(\Gamma, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}}))^\wedge &\cong H_1(\Gamma, \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)_H) \\ &= (\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)_H)[T_2]. \end{aligned}$$

By Remark 5.4, the group $H^1(\Gamma, \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_{\mathrm{cyc}}))$ vanishes. Hence, T_2 is a non-zero divisor on $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)_H$. Note that using Nakayama's lemma we can see that the $\Lambda(G)$-module $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)/(T_1,T_2)\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is non-trivial. Therefor, the set $\{T_1,T_2\}$ is an $\mathfrak{m}_G$-sequence on $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$. Since $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is $\Lambda(G)$-torsion, then $\mathrm{depth}(\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty))$ is two.

Similar to the first part, to show part (2), it suffices to show that $\mathfrak{X}(E_{p^\infty}/L_\infty)$ has projective dimension at most one. Consider the short exact sequence:

$$0 \longrightarrow \mathrm{Sel}^{\pm/\pm}(E_{p^\infty}/L_\infty) \longrightarrow \mathrm{Sel}(E_{p^\infty}/L_\infty) \longrightarrow \bigoplus_{v\in\{\mathfrak{p},\bar{\mathfrak{p}}\}} J_v^{\pm}(E_{p^\infty}/L_\infty) \longrightarrow 0.$$

The last map is exact by Proposition 4.2. Let $U := \bigoplus_{v\in\{\mathfrak{p},\bar{\mathfrak{p}}\}} J_v^{\pm}(E_{p^\infty}/L_\infty)^\wedge$, then taking the Pontryagin dual of the above sequence yields

$$0 \longrightarrow U \longrightarrow \mathfrak{X}(E_{p^\infty}/L_\infty) \longrightarrow \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty) \longrightarrow 0. \tag{41}$$

By Corollary 3.9 of [13], the $\Lambda(G)$-module U is free of rank 2 and hence it has projective dimension zero. By part (1), the depth of $\mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)$ is equal to two, and so by the Auslander–Buchsbaum–Serre formula it has projective dimension one. Now, applying Lemma 10.109.9 of [20] to the short exact sequence (41) shows that the projective dimension of $\mathfrak{X}(E_{p^\infty}/L_\infty)$ is at most one. □

Let us end this section by relating the signed μ-invariant $\mu_G^{\pm/\pm}(E_{p^\infty}/L_\infty)$ to the μ-invariant of the torsion $\Lambda(G)$-submodule of the Pontryagin dual of the Selmer group. Suppose M is a finitely generated $\Lambda(G)$-module and let $T_{\Lambda(G)}(M)$ denote the $\Lambda(G)$-torsion submodule of M. Note that

$$T_{\Lambda(G)}(M) = \ker\left(M \to M \otimes_{\Lambda(G)} K\right),$$

where $K = \mathrm{Frac}(\Lambda(G))$ is the fraction field of $\Lambda(G)$. With abuse of terminology, by the μ-invariant of M, we mean the μ-invariant of the $\Lambda(G)$-torsion submodule of M.

Proposition 5.8. *Suppose E/L is an elliptic curve satisfying Hyp 1 and Hyp $2^{\pm/\pm}$. Then, the μ-invariant of $\mathfrak{X}(E_{p^\infty}/L_\infty)$ is bounded by the signed μ-invariant $\mu_G^{\pm/\pm}(E_{p^\infty}/L_\infty)$.*

Proof. Noting that K is a flat $\Lambda(G)$-module Thus, when we apply the functor $-\otimes_{\Lambda(G)} K$ to the short exact sequence (41), we obtain the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & \mathfrak{X}(E_{p^\infty}/L_\infty) & \longrightarrow & \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U\otimes K & \longrightarrow & \mathfrak{X}(E_{p^\infty}/L_\infty)\otimes K & \longrightarrow & \mathfrak{X}^{\pm/\pm}(E_{p^\infty}/L_\infty)\otimes K & \longrightarrow & 0. \end{array}$$

The snake lemma implies the following exact sequence of $\Lambda(G)$-modules:

$$\begin{aligned} 0 &\to T_{\Lambda(G)}(\mathfrak{X}(\mathsf{E}_{p^\infty}/L_\infty)) \to \mathfrak{X}^{\pm/\pm}(\mathsf{E}_{p^\infty}/L_\infty) \to U\otimes_{\Lambda(G)} K/\Lambda(G) \\ &\to \mathfrak{X}(\mathsf{E}_{p^\infty}/L_\infty)\otimes_{\Lambda(G)} K/\Lambda(G) \to 0. \end{aligned} \tag{42}$$

The exact sequence (42) implies that the $\Lambda(G)$-module $T_{\Lambda(G)}(\mathfrak{X}(\mathsf{E}_{p^\infty}/L_\infty))$ embeds into $\mathfrak{X}^{\pm/\pm}(\mathsf{E}_{p^\infty}/L_\infty)$. This means that

$$\mu_G\left(T_{\Lambda(G)}(\mathfrak{X}(\mathsf{E}_{p^\infty}/L_\infty))\right) \leq \mu_G^{\pm/\pm}(E_{p^\infty}/L_\infty). \qquad \square$$

Remark 5.9. A similar argument shows this result for the cyclotomic $\mathbb{Z}_p$-extensions. Also, note that Proposition 5.8 does not assume Conjecture A.

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Parham Hamidi
Department of Mathematics
The University of British Columbia
Room 121, 1984 Mathematics Road
Vancouver, BC
V6T 1Z2, Canada
Email address: phamidi@math.ubc.ca