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FIXED-WIDTH PARTITIONS ACCORDING TO THE PARITY OF THE EVEN PARTS

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Abstract. A celebrated result in the study of integer partitions is the identity due to Lehmer whereby the number of partitions of n with an even number of even parts minus the number of partitions of n with an odd number of even parts equals the number of partitions of n into distinct odd parts. Inspired by Lehmer's identity, we prove explicit formulas for evaluating generating functions for sequences that enumerate integer partitions of fixed width with an even/odd number of even parts. We introduce a technique for decomposing the even entries of a partition in such a way so as to evaluate, using a finite sum over q-binomial coefficients, the generating function for the sequence of partitions with an even number of even parts of fixed, odd width, and similarly for the other families of fixed-width partitions that we introduce.

1. Introduction

An integer partition, which may be simply referred to as a partition, may be defined as an ordered tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$. If $\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)} = n$, then λ may be referred to as a partition of n, and this may be denoted as $\lambda \vdash n$. The length of λ is defined as the number $\ell(\lambda)$ of entries in the tuple λ, and the order of λ may be defined as $\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)}$. The entries or parts of a partition refer to expressions of the following forms: λ_1, λ_2, $\ldots$, $\lambda_{\ell(\lambda)}$. The width of a partition λ may be defined as the value of the initial entry λ_1. In this article, we introduce results on partitions related to the work of Lehmer [1, §8] and Gupta [5].

Let $p_{e,e}(n)$, $p_{o,e}(n)$, and $p_{d,o}(n)$, respectively, denote the number of partitions of n with an even number of even parts, the number of partitions of n with an odd number of even parts, and the cardinality of the set of expressions $\lambda \vdash n$ into distinct odd parts. Derrick Lehmer, at the 1974 International Congress of Mathematicians, proved the identity

$$p_{e,e}(n) - p_{o,e}(n) = p_{d,o}(n)$$

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using generating functions [1, §8], and Hansraj Gupta offered a combinatorial proof of this result [5]. In this article, we consider the natural combinatorial problem of enumerating fixed-width partitions with an odd or even number of even parts, using generating functions. Our work on fixed-width integer partitions is partly inspired by [4], in which monotonicity properties for generating functions for partitions with part sizes in a given set are studied; see also [2,3].

Partitions with even/odd numbers of even parts are interesting in part because there are many useful group-theoretic applications of partitions of this form. To illustrate this idea, we remark that the number of partitions of n with an even number of even parts is equal to the number of cycle types of conjugacy classes of permutations in the alternating group A_n. Also, for $n \neq 6$, there is a canonical bijection between the orbits of A_n under the action of $\mathrm{Aut}(A_n)$ and the set of all partitions of n with an even number of even parts [6].

Let $p^k_{e,e}(n)$ (resp. $p^k_{o,e}(n)$) denote the number of width-k partitions of n with an even (resp. odd) number of even parts. We prove explicit formulas for generating functions for integer sequences of the form

$$(p^k_{e,e}(n) : n \in \mathbb{N}_0) \tag{1}$$

and sequences of the form

$$(p^k_{o,e}(n) : n \in \mathbb{N}_0) \tag{2}$$

for arbitrary k. These formulas provide an efficient way to enumerate the combinatorial objects with which we had defined $p^k_{e,e}(n)$ and $p^k_{o,e}(n)$. We also use these formulas to determine identities relating such combinatorial objects to other families of integer partitions, as in Corollary 3.4 below.

The integer sequences of the form $\left(p^k_{e,e}(n) : n \in \mathbb{N}\right)$ for $k \in \{1, 2, 3, 4\}$ agree, up to offsets, with the following OEIS entries: `A000012`, `A002265`, `A025767`, and `A029002`. The sequences for $p^1_{o,e}(n)$, $p^2_{o,e}(n)$, and $p^3_{o,e}(n)$, respectively, agree, up to offsets, with the all-zeroes sequence, `A002265`, and the OEIS entry `A025767`. The sequences $\left(p^k_{e,e}(n) : k \in \mathbb{N}\right)$ and $\left(p^k_{o,e}(n) : k \in \mathbb{N}\right)$ for a parameter n seem to alternate between the rows of the number triangles given as `A026920` and `A026921` in the OEIS, but these number triangles are defined in a way that is not equivalent to our definitions for $p^k_{e,e}(n)$ and $p^k_{o,e}(n)$, and it is unclear as to how the given generating functions for `A026920` and `A026921` could be used to obtain our results introduced in Section 3 below. The data provided in the OEIS entries that we have cited suggest that the partitions used to define $p^k_{e,e}(n)$ and $p^k_{o,e}(n)$ have not been studied previously.

2. A decomposition of the even entries of a partition

We recall that the q-shifted factorial is such that

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

for $n \in \mathbb{N}$. In this section, we prove the following result.

Lemma 2.1. *The generating function* $\sum_{n\geq 0} p_{e,e}^{2m+1}(n)x^n$ *is equal to*

$$\frac{\left(1+\sum_{\lambda<\mu<\cdots<\eta} x^{|\lambda|+|\mu|+\cdots+|\eta|}\right)x^{2m+1}}{(x;x^2)_{m+1}(x^4;x^4)_m}, \tag{3}$$

where the above sum is over chains of partitions $\lambda < \mu < \cdots < \eta$*, where the order relation* $<$ *is such that given partitions* α *and* β*,* $\alpha < \beta$ *if and only if each entry of* β *is strictly greater than each entry of* α*, and where a chain in the above sum consists of length-*2 *partitions* p *with distinct even entries such that the width of* p *is less than or equal to* $2m$*.*

Proof. To evaluate the generating function $\sum_{n\geq 0} p_{e,e}^{2m+1}(n)x^n$, we decompose, in the following manner, the collection of even entries of odd-width partitions with an even number of even entries. Let λ be such a partition of width $2m+1$. For $i \in \mathbb{N}$, let $\lambda(i)$ denote the number of entries of λ that are equal to i. We thus have that $\lambda(2) + \lambda(4) + \cdots + \lambda(2m)$ is even. Now, let $I = \{i_1 < i_2 < \cdots < i_\ell\}$ be a subset of the index set $\{1, 2, \ldots, m\}$ such that an index $j \in \{1, 2, \ldots, m\}$ is in I if and only if $\lambda(2j)$ is odd. Since $\lambda(2) + \lambda(4) + \cdots + \lambda(2m)$ is even, we may thus deduce that ℓ is even. Write $\ell = 2k$. As above, for sequences s_1 and s_2, write $s_1 < s_2$ if each entry of s_2 is strictly greater than each entry of s_1. We thus have that the chain

$$(2i_1, 2i_2) < (2i_3, 2i_4) < \cdots < (2i_{k-1}, 2i_k)$$

consists of length-2 partitions p with distinct even entries such that the width of p_1 is less than or equal to $2m$, regardless of the parity of k.

Now, suppose that λ has an odd number of a_1's, an odd number of a_2's, and so forth, and an odd number of $a_{2\ell}$'s, with $a_1 < a_2 < \cdots < a_{2\ell}$, and where

$$(a_2, a_1) < (a_4, a_3) < \cdots < (a_{2\ell}, a_{2\ell-1})$$

is a chain of partitions of the form indicated in the Theorem under consideration. A width-$(2m+1)$ partition $\lambda \vdash n$ with an even number of even entries either: (1) Has an even number of twos, an even number of fours, and so forth, or (2) May be constructed uniquely from a width-$(2m+1)$ partition of n with entries in $\{1, 2, 3, \ldots, 2m+1\}$ with an even number of twos, an even number of fours, and so forth, together with a strictly increasing chain of the form

$$(a_2, a_1) < (a_4, a_3) < \cdots < (a_{2\ell}, a_{2\ell-1})$$

as given above.

The expression

$$\frac{x^{2m+1}}{(1-x)(1-x^3)\cdots(1-x^{2m+1})(1-x^4)(1-x^8)\cdots(1-x^{4m})}$$

is the generating function for the sequence that enumerates width-$(2m+1)$ partitions of n with entries in $\{1, 3, \ldots, 2m+1\} \cup \{4, 8, \ldots, 4m\}$. For a partition of this form, for any entries that are multiples of 4, we divide any such entry into equal parts, so as to produce an even number of even entries, so that the

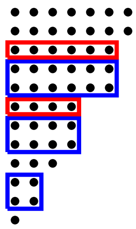

FIGURE 1. An illustration of the decomposition of the even entries of the partition $\left(7^2, 6^3, 4^3, 3, 2^2, 1\right)$.

numerator in (3) gives the required multiplicity according to our decomposition of the even entries of a partition with an even number of even entries. □

Example 2.2. An illustration of a decomposition, based on our proof of Lemma 2.1, of the even entries of a partition is given in Figure 1.

Example 2.3. Letting $m = 4$, we have that $\sum_{n \geq 0} p_{e,e}^{9}(n) x^n$ is equal to the following:

$$\begin{aligned} & \frac{\left(1 + x^6 + x^8 + 2x^{10} + x^{12} + x^{14} + x^{6+14}\right) x^9}{(x;x^2)_5 (x^4;x^4)_4} \\ & = x^9 + x^{10} + x^{11} + 2x^{12} + 3x^{13} + 4x^{14} + 6x^{15} + 8x^{16} + 12x^{17} + \cdots . \end{aligned}$$

So, for example, let us consider the coefficient of x^{14} in the above expansion. This should be equal to $p_{e,e}^{9}(14)$, i.e., the number of width-9 partitions of 14 with an even number of even parts, and we would expect that there should be 4 such partitions. If we examine the Ferrers diagrams presented in Figure 2, then we find that our combinatorial interpretation of the coefficient of x^{14} in the above expansion is correct.

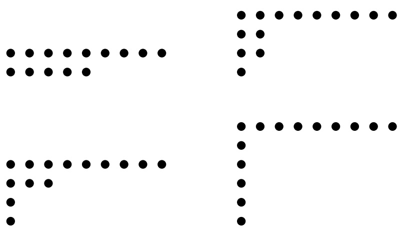

FIGURE 2. Ferrers diagrams.

3. Main results

Before evaluating the generating function $\sum_{n\geq 0} p_{e,e}^{2m+1}(n)x^n$ and proving this evaluation, we recall the definition of the q-binomial coefficient:

$$\binom{n}{m}_q = \frac{\left(q^{m+1};q\right)_\infty \left(q^{-m+n+1};q\right)_\infty}{(q;q)_\infty \left(q^{n+1};q\right)_\infty} = \prod_{i=0}^{m-1} \frac{1-q^{n-i}}{1-q^{i+1}}.$$

Theorem 3.1. *The generating function $\sum_{n\geq 0} p_{e,e}^{2m+1}(n)x^n$ is equal to*

$$\frac{\left(1+\sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} x^{2j(2j+1)}\binom{m}{2j}_{x^2}\right) x^{2m+1}}{(x;x^2)_{m+1}(x^4;x^4)_m}$$

for all $m \in \mathbb{N}$.

Proof. To evaluate the sum

$$\sum_{\lambda<\mu<\cdots<\eta} x^{|\lambda|+|\mu|+\cdots+|\eta|},$$

we evaluate sums of the following forms:

$$\sum_{\lambda} x^{|\lambda|}, \ \sum_{\lambda<\mu} x^{|\lambda|+|\mu|}, \text{ etc.}$$

It is easily seen that:

$$\sum_{\lambda} x^{|\lambda|} = \sum_{i_1=1}^{m-1} \sum_{i_2=i_1+1}^{m} x^{2i_1+2i_2},$$

$$\sum_{\lambda<\mu} x^{|\lambda|+|\mu|} = \sum_{i_1=1}^{m-1} \sum_{i_2=i_1+1}^{m} \sum_{i_3=i_2+1}^{m-1} \sum_{i_4=i_3+1}^{m} x^{2i_1+2i_2+2i_3+2i_4},$$

etc.

Let $p_s(x)$ denote the following polynomial expression, where m is understood to be fixed:

$$\sum_{i_1=1}^{m-1} \sum_{i_2=i_1+1}^{m} \sum_{i_3=i_2+1}^{m-1} \sum_{i_4=i_3+1}^{m} \cdots \sum_{i_{2s-1}=i_{2s-2}+1}^{m-1} \sum_{i_{2s}=i_{2s-1}+1}^{m} x^{2i_1+2i_2+2i_3+2i_4+\cdots+2i_{2s-1}+2i_{2s}}.$$

From the above definition for $p_s(x)$ together with the definition for q-binomial coefficients, it is easily seem, combinatorially or through the use of partial sum-type operators, that

$$p_s\left(\sqrt{x}\right) = x^{s(2s+1)}\binom{m}{2s}_x$$

for all s, m, and x. Since $\sum_{n\geq 0} p_{e,e}^{2m+1}(n)x^n$ equals

$$\frac{\left(1+\sum_{\lambda<\mu<\cdots<\eta} x^{|\lambda|+|\mu|+\cdots+|\eta|}\right) x^{2m+1}}{(x;x^2)_{m+1}(x^4;x^4)_m}$$

according to Lemma 2.1, we may express this generating function as

$$\frac{\left(1+\sum_{j=1}^{\lfloor \frac{m}{2}\rfloor} p_j(x)\right) x^{2m+1}}{(x;x^2)_{m+1}(x^4;x^4)_m},$$

as desired. □

Example 3.2. Letting $m=2$ with respect to Theorem 3.1, we have that

$$\sum_{n\geq 0} p_{e,e}^5(n)x^n = \frac{x^5+x^{11}}{(1-x)(1-x^3)(1-x^4)(1-x^5)(1-x^8)},$$

and we thus have that the number of width-five even partitions of n is equal to the number of partitions with parts in $\{1,3,4,5,8\}$ and of order $n-5$ or $n-11$.

With regard to our proof of Theorem 3.1, a very similar argument may be used to prove the following result.

Theorem 3.3. *The generating function* $\sum_{n\geq 0} p_{o,e}^{2m}(n)x^n$ *is equal to*

$$\frac{(x-1)x^{2m-1}\left(4\sum_{j=1}^{\lfloor \frac{m}{2}\rfloor} \frac{\left(x^{4m+4}\right)^j\left(x^{-2m};x^2\right)_{2j}}{(-1;x)_{j+1}(-1;x^2)_{j+1}(x;x)_j(x^2;x^4)_j}+1\right)}{\left(\frac{1}{x};x^2\right)_{m+1}\left(x^4;x^4\right)_m}$$

for all $m\in\mathbb{N}$.

We may mimic, as below, our multisum construction involved above, in order to evaluate the generating function for $p_{e,e}^{2m}(n)$, for $n\in\mathbb{N}_0$. By analogy with the definition of the polynomial expression $p_s(x)$, define $q_s(x)$ as follows:

$$\sum_{i_1=1}^{m-1}\sum_{i_2=i_1+1}^{m}\sum_{i_3=i_2+1}^{m-1}\sum_{i_4=i_3+1}^{m}\cdots\sum_{i_{2s-1}=i_{2s-2}+1}^{m-1}\sum_{i_{2s}=i_{2s-1}+1}^{m}$$
$$\sum_{i_{2s+1}=i_{2s}+1}^{m} x^{2i_1+2i_2+2i_3+2i_4+\cdots+2i_{2s-1}+2i_{2s}+2i_{2s+1}},$$

noting that the upper parameter of the innermost sum is given as m, as opposed to $m-1$. By direct analogy with the proof of Theorem 3.1, we can show that

$$\sum_{n\geq 0} p_{e,e}^{2m}(n)x^n$$

may be written as

$$\frac{x^{2m}(x^2+x^4+\cdots+x^{2m})}{\left(\frac{x\left(\frac{1}{x};x^2\right)_{m+1}}{x-1}\right)(x^4;x^4)_m}+\frac{x^{2m}\sum_{j=1}^{\lfloor\frac{m-1}{2}\rfloor}q_j(x)}{\left(\frac{x\left(\frac{1}{x};x^2\right)_{m+1}}{x-1}\right)(x^4;x^4)_m}$$

for arbitrary $m\in\mathbb{N}$. We can similarly show that:

$$\sum_{n\geq 0}p_{o,e}^{2m+1}(n)x^n=\frac{x^{2m+1}(x^2+x^4+\cdots+x^{2m})}{(x;x^2)_{m+1}(x^4;x^4)_m}+\frac{x^{2m+1}\sum_{j=1}^{\lfloor\frac{m-1}{2}\rfloor}q_j(x)}{(x;x^2)_{m+1}(x^4;x^4)_m}.$$

The results given above may be used to construct simple proofs for identities concerning fixed-width even/odd partitions.

Corollary 3.4. *The number of width-three even partitions of order at most n is equal to the number of partitions of width at most 4 whose order is in $\{n-3,n-4\}$.*

Proof. Letting $m=1$ with respect to Theorem 3.1, we have that:

$$\sum_{n\geq 0}p_{e,e}^{3}(n)x^n=\frac{x^3}{(1-x)\,(1-x^3)\,(1-x^4)}.$$

Applying the partial sum operator $\frac{1}{1-x}$ to both sides of the above equality, we have that:

$$\frac{1}{1-x}\sum_{n\geq 0}p_{e,e}^{3}(n)x^n=\frac{x^3}{(1-x)^2\,(1-x^3)\,(1-x^4)}.$$

Rewrite the above rational function as follows:

$$\frac{1}{1-x}\sum_{n\geq 0}p_{e,e}^{3}(n)x^n=\frac{x^3+x^4}{(1-x)(1-x^2)\,(1-x^3)\,(1-x^4)}.$$

Corollary 3.4 follows immediately from the above equality. □

We leave it as an open problem to generalize our methods and results so as to be applicable to fixed-width partitions such that the number of parts that are congruent to z_1 modulo z_2 is congruent to z_3 modulo z_4, for fixed values z_1, z_2, z_3, and z_4. Also, we leave it as an open problem to evaluate the following bivariate generating functions:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}p_{e,e}^{k}(n)z^kq^n \quad\text{and}\quad \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}p_{o,e}^{k}(n)z^kq^n.$$

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