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SOME RELATIONS ON PARAMETRIC LINEAR EULER SUMS

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Abstract. Recently, Alzer and Choi [2] introduced and studied a set of the four linear Euler sums with parameters. These sums are parametric extensions of Flajolet and Salvy's four kinds of linear Euler sums [9]. In this paper, by using the method of residue computations, we will establish two explicit combined formulas involving two parametric linear Euler sums $S^{++}_{p,q}(a,b)$ and $S^{+-}_{p,q}(a,b)$ defined by Alzer and Choi, which can be expressed in terms of a linear combinations of products of trigonometric functions, digamma functions and Hurwitz zeta functions.

1. Introduction

Let $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^-$ be the sets of complex numbers, integers, positive integers and negative integers, respectively. We also denote by $\mathbb{N}_0$ the set of non-negative integers and by $\mathbb{N}_0^-$ the set of non-positive integers.

In [2], Alzer and Choi defined the following four distinct kinds of linear Euler sums with parameters:

$$
\begin{aligned}
&S^{++}_{p,q}(a,b) := \sum_{n=1}^{\infty} \frac{H^{(p)}_n(a)}{(n+b)^q}, \quad S^{+-}_{p,q}(a,b) := \sum_{n=1}^{\infty} \frac{H^{(p)}_n(a)}{(n+b)^q}(-1)^{n-1},\\
&S^{-+}_{p,q}(a,b) := \sum_{n=1}^{\infty} \frac{\bar{H}^{(p)}_n(a)}{(n+b)^q}, \quad S^{--}_{p,q}(a,b) := \sum_{n=1}^{\infty} \frac{\bar{H}^{(p)}_n(a)}{(n+b)^q}(-1)^{n-1},
\end{aligned}
\tag{1}
$$

where $a,b \in \mathbb{C} \setminus \mathbb{N}^-$ and $p,q \in \mathbb{C}$ are adjusted so that the involved series converge. Here $H^{(p)}_n(a)$ and $\bar{H}^{(p)}_n(a)$ are the *parametric harmonic numbers* of

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order p and the *alternating parametric harmonic numbers* of order p, respectively, defined by

$$H_n^{(p)}(a) := \sum_{j=1}^{n} \frac{1}{(j+a)^p} \quad (n \in \mathbb{N},\ p \in \mathbb{C},\ a \in \mathbb{C} \setminus \mathbb{N}^-) \tag{2}$$

and

$$\bar{H}_n^{(p)}(a) := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(j+a)^p} \quad (n \in \mathbb{N},\ p \in \mathbb{C},\ a \in \mathbb{C} \setminus \mathbb{N}^-). \tag{3}$$

For convenience, we let $H_0^{(p)}(a) = \bar{H}_0^{(p)}(a) := 0$. Clearly, if $a = b = 0$ in (1)-(3) then the sums become the classical *linear Euler sums*, the classical *harmonic numbers* of order p and the classical *alternating harmonic numbers* of order p (see [9]), respectively, which are defined by

$$\begin{aligned} S_{p,q}^{++} &\equiv S_{p,q}^{++}(0,0) := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}, \quad S_{p,q}^{+-} \equiv S_{p,q}^{+-}(0,0) := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}(-1)^{n-1}, \\ S_{p,q}^{-+} &\equiv S_{p,q}^{-+}(0,0) := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q}, \quad S_{p,q}^{--} \equiv S_{p,q}^{--}(0,0) := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q}(-1)^{n-1}, \end{aligned} \tag{4}$$

$$H_n^{(p)} \equiv H_n^{(p)}(0) := \sum_{k=1}^{n} \frac{1}{k^p}, \quad H_n \equiv H_n^{(1)}, \quad H_0^{(p)} := 0, \tag{5}$$

$$\bar{H}_n^{(p)} \equiv \bar{H}_n^{(p)}(0) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^p}, \quad \bar{H}_n \equiv \bar{H}_n^{(1)}, \quad \bar{H}_0^{(p)} := 0. \tag{6}$$

When taking the limit $n \to \infty$ in (2), (3), (5) and (6) we get the *Hurwitz zeta function, alternating Hurwitz zeta function, Rimann zeta function* and *alternating Riemann zeta function*, respectively:

$$\begin{aligned} \zeta(p; a+1) &:= \lim_{n\to\infty} H_n^{(p)}(a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^p} \quad (\Re(p) > 1,\ a \in \mathbb{C} \setminus \mathbb{N}^-), \\ \bar{\zeta}(p; a+1) &:= \lim_{n\to\infty} \bar{H}_n^{(p)}(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+a)^p} \quad (\Re(p) > 0,\ a \in \mathbb{C} \setminus \mathbb{N}^-), \\ \zeta(p) &\equiv \zeta(p;1) := \lim_{n\to\infty} H_n^{(p)} = \sum_{n=1}^{\infty} \frac{1}{n^p} \quad (\Re(p) > 1), \\ \bar{\zeta}(p) &\equiv \bar{\zeta}(p;1) := \lim_{n\to\infty} \bar{H}_n^{(p)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \quad (\Re(p) > 0). \end{aligned}$$

Similarly, setting $r \in \mathbb{N}$, for any $p_j \in \mathbb{C}$ and $a_j \in \mathbb{C} \setminus \mathbb{N}^-$ $(1 \le j \le r)$ we define the Hurwitz zeta function with r-variables and the alternating Hurwitz zeta function with r-variables by

$$\zeta(p;a_1+1,\ldots,a_r+1):=\sum_{n=1}^{\infty}\frac{1}{(n+a_1)^{p_1}\cdots(n+a_r)^{p_r}}\quad(\Re(p_1+\cdots+p_r)>1),$$

$$\bar{\zeta}(p;a_1+1,\ldots,a_r+1):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+a_1)^{p_1}\cdots(n+a_r)^{p_r}}\quad(\Re(p_1+\cdots+p_r)>0).$$

The study of classical linear Euler sums $S_{p,q}^{++}$, which have been researched since the time of Euler, has a long history. Euler started this line of investigation in the course of a correspondence with Goldbach and he was the first to consider the linear Euler sums $S_{p,q}^{++}$. Euler solved this problem in the case $p=1$ in 1775. Moreover, he conjectured that the linear Euler sums $S_{p,q}^{++}$ would be reducible to zeta values when $p+q$ is odd, and even gave a conjectured general formula, which was proved in [5]. Further, Flajolet and Salvy [9] proved that the three linear Euler sums $S_{p,q}^{+-}, S_{p,q}^{-+}$ and $S_{p,q}^{--}$ can be expressed in terms of alternating Riemann zeta values, when $p+q$ is odd, by using the method of residue calculus. There are many other researches on the Euler sums and Euler type sums. Some related results of this subject can be found in the works of [1,3,6–8,10–14,16,17,21] and references therein. For details and history, please see the books by Srivastava-Choi [15] and Zhao [22].

Recently, some parametric Euler sums have been introduced and studied, see [2,4,18,19] and references therein. For example, in [2], Alzer and Choi studied many interesting features and identities of the four parametric linear Euler sums (1), including their analytic continuations and mingling connections. In [19], Xu defined the parametric digamma function $\Psi(-z;a)$ and used the residue computations to find a large numbers of formulas of infinite series of parametric harmonic numbers. Here the parametric digamma (or Psi) function $\Psi(-z;a)$ is defined by

$$\Psi(-z;a)+\gamma:=\frac{1}{z-a}+\sum_{k=1}^{\infty}\left(\frac{1}{k+a}-\frac{1}{k+a-z}\right)\quad(z\in\mathbb{C},\ a\in\mathbb{C}\setminus\mathbb{N}^-),\tag{7}$$

where γ denotes the Euler-Mascheroni constant defined by

$$\gamma:=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\approx 0.57721566490153286060651\ldots.$$

Clearly, the function $\Psi(-z;a)$ is meromorphic on the entire complex plane with a simple pole at $z=n+a$ for each nonnegative integer n. Moreover, when $a=0$ then the $\Psi(-z;0)$ becomes the classical digamma function defined by

$$\psi(-z)+\gamma:=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k-z}\right)\quad(z\in\mathbb{C},\ a\in\mathbb{C}\setminus\mathbb{N}^-).\tag{8}$$

In particular, from the definitions of (7) and (8), we also obtain the relation

$$\Psi(-z;a)+\gamma=\psi(a-z)-\psi(a+1).$$

In this paper, we extend the tools developed by Flajolet-Salvy [9] and Xu [19] to establish some evaluations for the first and second sums of (1). Our main results are the following two theorems.

Let

$$\begin{aligned} S_{p,q}(a,b) :&= \sum_{n=1}^{\infty} \frac{1}{(n-b)^q} \sum_{k=1}^{n} \frac{1}{(k-a)^p} - (-1)^{p+q} \sum_{n=1}^{\infty} \frac{1}{(n+b)^q} \sum_{k=1}^{n} \frac{1}{(k+a)^p}, \\ &= S^{++}_{p,q}(-a,-b) - (-1)^{p+q} S^{++}_{p,q}(a,b) \end{aligned}$$

and

$$\begin{aligned} L_{p,q}(a,b) :&= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-b)^q} \sum_{k=1}^{n} \frac{1}{(k-a)^p} - (-1)^{p+q} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+b)^q} \sum_{k=1}^{n} \frac{1}{(k+a)^p} \\ &= -S^{+-}_{p,q}(-a,-b) + (-1)^{p+q} S^{+-}_{p,q}(a,b). \end{aligned}$$

Theorem 1.1. *For any $a, b \notin \mathbb{Z}$ with $a \neq b$ and positive integers $p \geq 1$, $q > 1$, if $p = 1$, then*

$$\begin{aligned} S_{1,q}(a,b) = &-\pi \cot(\pi a)\, \zeta(q; a-b) + \frac{1}{a} \zeta(q; 1-b) \\ &+ \frac{(-1)^q}{ab^q} + (-1)^q \zeta(1, q; a+1, b+1) \\ &- \frac{1}{(q-1)!} \lim_{z \to b} \frac{d^{q-1}}{dz^{q-1}} \left\{ \pi \cot(\pi z) \left(\Psi(-z; a) + \gamma \right) \right\}, \end{aligned} \tag{9}$$

if $p > 1$, then

$$\begin{aligned} S_{p,q}(a,b) = &(-1)^{p+1} \zeta(p, a+1) \left(\zeta(q; 1-b) + (-1)^q \zeta(q; 1+b) \right) \\ &+ \frac{(-1)^{p+1}}{a^p} \zeta(q; 1-b) + \frac{(-1)^{p+q+1}}{b^q} \zeta(p; 1+a) \\ &+ \frac{(-1)^{p+q+1}}{a^p b^q} + (-1)^{p+q+1} \zeta(p, q; a+1, b+1) \\ &- \frac{1}{(p-1)!} \sum_{n=0}^{\infty} \frac{d^{p-1}}{dz^{p-1}} \left(\frac{\pi \cot(\pi z)}{(z-b)^q} \right) \Bigg|_{z=n+a} \\ &- \frac{1}{(q-1)!} \frac{d^{q-1}}{dz^{q-1}} \left(\frac{\pi \cot(\pi z) \Psi^{(p-1)}(-z; a)}{(p-1)!} \right) \Bigg|_{z=b}. \end{aligned} \tag{10}$$

Theorem 1.2. *For any $a, b \notin \mathbb{Z}$ with $a \neq b$ and positive integers $p \geq 1$, $q > 1$, if $p = 1$, then*

$$\begin{aligned} L_{1,q}(a,b) = &\frac{(-1)^q}{ab^q} - \frac{1}{a} \bar{\zeta}(q; 1-b) - (-1)^q \bar{\zeta}(1, q; a+1, b+1) \\ &- \frac{\pi}{\sin(\pi a)} \bar{\zeta}(q; a-b) - \frac{1}{(q-1)!} \frac{d^{q-1}}{dz^{q-1}} \left\{ \frac{\pi \left(\Psi(-z; a) + \gamma \right)}{\sin(\pi z)} \right\} \Bigg|_{z=b}, \end{aligned} \tag{11}$$

if $p > 1$, then

$$L_{p,q}(a,b) = (-1)^{p+q} \zeta(p; a+1) \left(\bar{\zeta}(q; b+1) + (-1)^q \bar{\zeta}(q; 1-b) \right)$$

$$
\begin{aligned}
&+(-1)^{p+q}\,\bar{\zeta}(p,q;a+1,b+1)+\frac{(-1)^p}{a^p}\bar{\zeta}\,(q;1-b)\\
&+\frac{(-1)^{p+q+1}}{b^q}\zeta\,(p;a+1)+\frac{(-1)^{p+q+1}}{a^pb^q}\\
&-\sum_{n=0}^{\infty}\frac{1}{(p-1)!}\frac{d^{p-1}}{dz^{p-1}}\left\{\frac{\pi}{(z-b)^q\sin(\pi z)}\right\}\Bigg|_{z=n+a}\\
&-\frac{1}{(q-1)!}\frac{d^{q-1}}{dz^{q-1}}\left\{\frac{\pi\Psi^{(p-1)}(-z;a)}{(p-1)!\sin(\pi z)}\right\}\Bigg|_{z=b}.
\end{aligned}
\tag{12}
$$

2. Some lemmas

In this section, we state some lemmas that will subsequently be used in our proofs of the main results.

We define a complex kernel function $\xi(s)$ with two requirements: (i) $\xi(z)$ is meromorphic in the whole complex plane. (ii) $\xi(z)$ satisfies $\xi(z)=o(z)$ over an infinite collection of circles $|z|=\rho_k$ with $\rho_k\to\infty$. Applying these two conditions on the kernel function $\xi(z)$, Flajolet and Salvy recalled the following residue lemma.

Lemma 2.1 ([9])**.** *Let $\xi(z)$ be a kernel function and let $r(z)$ be a rational function which is $O(z^{-2})$ at infinity. Then*

$$\sum_{\alpha\in T}\operatorname{Res}(r(z)\xi(z),\alpha)+\sum_{\beta\in E}\operatorname{Res}(r(z)\xi(z),\beta)=0,$$

where E is the set of poles of $r(z)$ and T is the set of poles of $\xi(z)$ that are not poles of $r(z)$. Here $\operatorname{Res}(r(z),\alpha)$ denotes the residue of $r(z)$ at $z=\alpha$.

Lemma 2.2 ([19, Thms. 2.1-2.3])**.** *For integers $n\geq 0$ and $|a|<1$, the following Taylor and Laurent series hold true*:

$$
\begin{aligned}
\Psi\,(-z;a)+\gamma &\overset{z\to-n}{=} H_{n-1}^{(1)}\,(a)\\
&\quad+\sum_{j=1}^{\infty}\Big(H_{n-1}^{(j+1)}\,(a)-\zeta\,(j+1;a+1)\Big)\,(z+n)^j \quad (n\geq 1),\\
\Psi\,(-z;a)+\gamma &\overset{z\to n}{=} \frac{1}{z-n-a}+H_n^{(1)}\,(-a)\\
&\quad+\sum_{j=1}^{\infty}\Big((-1)^jH_n^{(j+1)}\,(-a)-\zeta\,(j+1;a+1)\Big)\,(z-n)^j,\\
\Psi\,(-z;a)+\gamma &\overset{z\to n+a}{=} \frac{1}{z-n-a}+H_n-(\psi\,(a+1)+\gamma)\\
&\quad+\sum_{j=1}^{\infty}\Big((-1)^jH_n^{(j+1)}-\zeta\,(j+1)\Big)\,(z-n-a)^j.
\end{aligned}
$$

Lemma 2.3 ([19, Cor. 2.4]). *For integers $p>1$, $n\geq 0$ and $|a|<1$, the following identities hold*:

$$\frac{\Psi^{(p-1)}(-z;a)}{(p-1)!} \overset{z\to -n}{=} (-1)^p \sum_{j=0}^{\infty} \binom{j+p-1}{p-1} \times \left(\zeta(j+p;a+1) - H_{n-1}^{(j+p)}(a)\right)(z+n)^j \quad (n\geq 1),$$

$$\frac{\Psi^{(p-1)}(-z;a)}{(p-1)!} \overset{z\to n}{=} \frac{(-1)^p}{(z-n)^p} \sum_{j=p}^{\infty} \binom{j-1}{p-1} \times \left(\frac{1}{a^j} + \zeta(j;a+1) + (-1)^j H_n^{(j)}(-a)\right)(z-n)^j,$$

$$\frac{\Psi^{(p-1)}(-z;a)}{(p-1)!} \overset{z\to n+a}{=} \frac{1}{(z-n-a)^p} + \frac{(-1)^p}{(z-n-a)^p} \sum_{j=p}^{\infty} \binom{j-1}{p-1} \times \left(\zeta(j) + (-1)^j H_n^{(j)}\right)(z-n-a)^j.$$

Lemma 2.4 ([2, Eqs. (3.3) and (3.8)]). *For $p,q\in\mathbb{N}$ and $a,b\in\mathbb{C}\setminus\mathbb{N}^-$ with $a\neq b$,*

$$\zeta(p,q;a+1,b+1) = (-1)^q \binom{p+q-2}{p-1} \frac{\psi(b+1)-\psi(a+1)}{(a-b)^{p+q-1}} + (-1)^q \sum_{j=1}^{q-1} \frac{1}{j!} \binom{p+q-j-2}{p-1} \frac{\psi^{(j)}(b+1)}{(a-b)^{p+q-j-1}} + (-1)^p \sum_{j=1}^{p-1} \frac{1}{j!} \binom{p+q-j-2}{q-1} \frac{\psi^{(j)}(a+1)}{(b-a)^{p+q-j-1}}, \tag{13}$$

$$\bar{\zeta}(p,q;a+1,b+1) = \sum_{j=0}^{p-1} (-1)^j \binom{q+j-1}{q-1} \frac{\bar{\zeta}(p-j;a+1)}{(b-a)^{q-j}} + \sum_{j=0}^{q-1} (-1)^j \binom{p+j-1}{p-1} \frac{\bar{\zeta}(q-j;b+1)}{(a-b)^{p-j}}, \tag{14}$$

where $\psi^{(j)}(a+1) = (-1)^{j+1} j! \zeta(j+1;a+1)$ for $j\in\mathbb{N}$.

3. Proofs of theorems

Flajolet and Salvy proved that every linear sum $S_{p,q}^{++}$ whose weight $p+q$ is odd is expressible as a polynomial in zeta values by applying the kernel function

$$\frac{1}{2}\pi\cot(\pi z)\frac{\psi^{(p-1)}(-z)}{(p-1)!}$$

to the base function $r(z) = z^{-q}$, please see their article for further reference. Next, we will use similar methods to prove Theorems 1.1 and 1.2.

3.1. Proof of Theorem 1.1

First, we prove (9). Let

$$h(z) := \frac{\pi\cot(\pi z)(\Psi(-z;a)+\gamma)}{(z-b)^q} \quad (a, b \in \mathbb{C}\setminus\mathbb{Z},\ a \neq b,\ q \in \mathbb{N}\setminus\{1\}).$$

We consider the contour integral (In present paper, the orientation of the contour is counterclockwise.)

$$\oint_{(\infty)} h(z)dz,$$

where $\oint_{(\infty)}$ denotes integral along large circles, that is, the limit of integrals $\oint_{|z|=R}$ as $R \to \infty$. Clearly, $\pi\cot(\pi z)(\Psi(-z;a)+\gamma)$ is a kernel function. Hence, $\oint_{(\infty)} h(z)dz = 0$. This function $h(z)$ has poles at $z = n$ $(n \in \mathbb{Z})$, $n + a$ $(n \in \mathbb{N}_0)$ and b. Thus, using Lemma 2.1, we obtain

$$\sum_{n=1}^{\infty}\text{Res}(h(z),-n) + \sum_{n=0}^{\infty}\text{Res}(h(z),n) + \sum_{n=0}^{\infty}\text{Res}(h(z),n+a) + \text{Res}(h(z),b) = 0.$$

At a negative integer $-n$, a nonnegative integer n and a complex $n+a$ $(n \in \mathbb{N}_0)$, the poles are simple and by Lemma 2.2, we find that the residues are

$$\text{Res}(h(z),-n) = \lim_{z\to -n}\frac{\Psi(-z;a)+\gamma}{(z-b)^q} = (-1)^q\frac{H^{(1)}_{n-1}(a)}{(n+b)^q} \quad (n \in \mathbb{N}),$$

$$\text{Res}(h(z),n) = \lim_{z\to n}\frac{\Psi(-z;a)+\gamma}{(z-b)^q} = \frac{-\frac{1}{a}+H^{(1)}_n(-a)}{(n-b)^q} \quad (n \in \mathbb{N}_0),$$

$$\text{Res}(h(z),n+a) = \lim_{z\to n+a}\frac{\pi\cot(\pi z)}{(z-b)^q} = \frac{\pi\cot(\pi a)}{(n+a-b)^q} \quad (n \in \mathbb{N}_0).$$

At a complex b, the pole has order q and the residue is

$$\begin{aligned}\text{Res}(h(z),b) &= \frac{1}{(q-1)!}\lim_{z\to b}\frac{d^{q-1}}{dz^{q-1}}\frac{\pi\cot(\pi z)(\Psi(-z;a)+\gamma)}{(z-b)^q}(z-b)^q \\ &= \frac{1}{(q-1)!}\lim_{z\to b}\frac{d^{q-1}}{dz^{q-1}}\left\{\pi\cot(\pi z)(\Psi(-z;a)+\gamma)\right\}.\end{aligned}$$

Summing these contributions yields the (9).

Next, we prove (10). Similarly, let

$$f(z) := \frac{\pi\cot(\pi z)\Psi^{(p-1)}(-z;a)}{(p-1)!(z-b)^q} \quad (a, b \in \mathbb{C}\setminus\mathbb{Z},\ a \neq b,\ q \in \mathbb{N}\setminus\{1\}).$$

We consider the contour integral

$$\oint_{(\infty)} f(z)dz = 0.$$

This function $f(z)$ has also poles at $z = n$ ($n \in \mathbb{Z}$), $n + a$ ($n \in \mathbb{N}_0$) and b. Applying Lemma 2.3, by direct residue computations, we deduce that for positive integer n,

$$\begin{aligned}
\mathrm{Res}(f(z), -n) &= \lim_{z\to -n} \frac{\Psi^{(p-1)}(-z;a)}{(p-1)!(z-b)^q} \\
&= (-1)^{p+q}\frac{\zeta(p;a+1) - H_{n-1}^{(p)}(a)}{(n+b)^q} \quad (n \in \mathbb{N}), \\
\mathrm{Res}(f(z), n) &= \lim_{z\to n} \frac{\Psi^{(p-1)}(-z;a)}{(p-1)!(z-b)^q} \\
&= (-1)^p \frac{\frac{1}{a^p} + \zeta(p,a+1) + (-1)^p H_n^{(p)}(-a)}{(n-b)^q} \quad (n \in \mathbb{N}_0), \\
\mathrm{Res}(f(z), n+a) &= \frac{1}{(p-1)!} \lim_{z\to n+a} \left\{ \frac{d^{p-1}}{dz^{p-1}} (z-n-a)^p f(z) \right\} \\
&= \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \left(\frac{\pi\cot(\pi z)}{(z-b)^q} \right)\Bigg|_{z=n+a} \quad (n \in \mathbb{N}_0), \\
\mathrm{Res}(f(z), b) &= \frac{1}{(q-1)!} \frac{d^{q-1}}{dz^{q-1}} \left\{ (z-b)^q \cdot \frac{\pi\cot(\pi z)\Psi^{(p-1)}(-z;a)}{(p-1)!(z-b)^q} \right\}\Bigg|_{z=b} \\
&= \frac{1}{(q-1)!} \lim_{z\to b} \frac{d^{q-1}}{dz^{q-1}} \left\{ \frac{\pi\cot(\pi z)\Psi^{(p-1)}(-z;a)}{(p-1)!} \right\}.
\end{aligned}$$

Then, applying Lemma 2.1, we have

$$\sum_{n=1}^{\infty} \mathrm{Res}(f(z), -n) + \sum_{n=0}^{\infty} \mathrm{Res}(f(z), n) + \sum_{n=0}^{\infty} \mathrm{Res}(f(z), n+a) + \mathrm{Res}(f(z), b) = 0.$$

Thus, combining related identities yields the desired result. This completes the proof of the theorem.

3.2. Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1, we need to consider the following contour integrals

$$\begin{aligned}
\oint_{(\infty)} g(z)\,dz &:= \oint_{(\infty)} \frac{\pi\left(\Psi(-z;a) + \gamma\right)}{\sin(\pi z)(z-b)^q} dz = 0, \\
\oint_{(\infty)} G(z)\,dz &:= \oint_{(\infty)} \frac{\pi\Psi^{(p-1)}(-z;a)}{\sin(\pi z)(p-1)!(z-b)^q} dz = 0.
\end{aligned}$$

By direct residue computations, we deduce that

$$\mathrm{Res}(g(z), -n) = (-1)^q (-1)^n \frac{H_{n-1}^{(1)}(a)}{(n+b)^q} \quad (n \in \mathbb{N}),$$

$$\operatorname{Res}\left(g\left(z\right),n\right)=\left(-1\right)^{n}\frac{-\frac{1}{a}+H_{n}^{(1)}\left(-a\right)}{\left(n-b\right)^{q}}\quad\left(n\in\mathbb{N}_{0}\right),$$

$$\operatorname{Res}\left(g\left(z\right),n+a\right)=\frac{\left(-1\right)^{n}\pi}{\sin\left(\pi a\right)\left(n+a-b\right)^{q}}\quad\left(n\in\mathbb{N}_{0}\right),$$

$$\operatorname{Res}\left(g\left(z\right),b\right)=\frac{1}{\left(q-1\right)!}\lim_{z\to b}\frac{d^{q-1}}{dz^{q-1}}\left\{\frac{\pi\left(\Psi\left(-z;a\right)+\gamma\right)}{\sin\left(\pi z\right)}\right\},$$

$$\operatorname{Res}\left(G\left(z\right),-n\right)=\left(-1\right)^{p+q}\left(-1\right)^{n}\frac{\zeta\left(p,a+1\right)-H_{n-1}^{(p)}\left(a\right)}{\left(n+b\right)^{q}}\quad\left(n\in\mathbb{N}\right),$$

$$\operatorname{Res}\left(G\left(z\right),n\right)=\left(-1\right)^{p}\left(-1\right)^{n}\frac{\frac{1}{a^{p}}+\zeta\left(p,a+1\right)+\left(-1\right)^{p}H_{n}^{(p)}\left(-a\right)}{\left(n-b\right)^{q}}\quad\left(n\in\mathbb{N}_{0}\right),$$

$$\operatorname{Res}\left(G\left(z\right),n+a\right)=\frac{1}{\left(p-1\right)!}\frac{d^{p-1}}{dz^{p-1}}\left(\frac{\pi}{\sin\left(\pi z\right)\left(z-b\right)^{q}}\right)|_{z=n+a}\quad\left(n\in\mathbb{N}_{0}\right),$$

$$\operatorname{Res}\left(G\left(z\right),b\right)=\frac{1}{\left(q-1\right)!}\lim_{z\to b}\frac{d^{q-1}}{dz^{q-1}}\left\{\frac{\pi\Psi^{(p-1)}(-z;a)}{\sin\left(\pi z\right)\left(p-1\right)!}\right\}_{z=b}.$$

Then, applying Lemma 2.1, we have

$$\sum_{n=1}^{\infty}\operatorname{Res}(g(z),-n)+\sum_{n=0}^{\infty}\operatorname{Res}(g(z),n)+\sum_{n=0}^{\infty}\operatorname{Res}(g(z),n+a)+\operatorname{Res}(g(z),b)=0$$

and

$$\sum_{n=1}^{\infty}\operatorname{Res}(G(z),-n)+\sum_{n=0}^{\infty}\operatorname{Res}(G(z),n)+\sum_{n=0}^{\infty}\operatorname{Res}(G(z),n+a)+\operatorname{Res}(G(z),b)=0.$$

Hence, we have proved the Theorem 1.2 by a direct calculation.

Remark 3.1. In fact, formulas (10) and (12) can also obtained by differentiating (9) and (11) $p-1$ times with respect to a, respectively.

4. Some examples

Let

$$|\mathbf{r}|_l := r_0 + r_1 + \cdots + r_l \ (r_j \in \mathbb{N}_0).$$

Lemma 4.1 ([20, Thm. 2.2]). *For any integers $k \geq 0,\ m \geq 1$ and complex number $s \in \mathbb{C}\setminus\mathbb{N}_0$, we have*

$$\frac{d^{2k}}{ds^{2k}}\left(\frac{1}{\sin^{2m-1}(\pi s)}\right)$$

$$=\pi^{2k}\sum_{l=0}^{k}\left(\frac{(2m+2l-2)!}{(2m-2)!}\sum_{|\mathbf{r}|_l=k-l}\prod_{h=0}^{l}(2m+2h-1)^{2r_h}\right)\frac{(-1)^{k-l}}{\sin^{2m+2l-1}(\pi s)},$$

$$\frac{d^{2k+1}}{ds^{2k+1}}\left(\frac{1}{\sin^{2m-1}(\pi s)}\right)$$

$$= \pi^{2k+1} \sum_{l=0}^{k} \left(\frac{(2m+2l-1)!}{(2m-2)!} \sum_{|\mathbf{r}|_l = k-l} \prod_{h=0}^{l} (2m+2h-1)^{2r_h} \right) \frac{(-1)^{k+1-l} \cos(\pi s)}{\sin^{2m+2l}(\pi s)}.$$

Lemma 4.2 ([20, Thm. 2.3])**.** *For any integer $k \in \mathbb{N}_0$ and complex number $s \in \mathbb{C} \setminus \mathbb{N}_0$, we have*

$$\begin{aligned} &\frac{d^{2k}}{ds^{2k}} (\cot(\pi s)) \\ &= \pi^{2k} \sum_{0 \le l \le j \le k} \left\{ \binom{2k}{2j} (2l)! - \binom{2k}{2j+1} (2l+1)! \right\} \\ &\quad \times \left\{ \sum_{|\mathbf{r}|_l = j-l} \prod_{h=0}^{l} (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l} \cos(\pi s)}{\sin^{2l+1}(\pi s)}, \\ &\frac{d^{2k+1}}{ds^{2k+1}} (\cot(\pi s)) \\ &= \pi^{2k+1} \sum_{1 \le l \le j \le k+1} \left\{ \binom{2k+2}{2j} (2l-1)! - \binom{2k+2}{2j+1} (2l-1)!(2l+1) \right\} \\ &\quad \times \left\{ \sum_{|\mathbf{r}|_l = j-l} \prod_{h=0}^{l} (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l}}{\sin^{2l}(\pi s)}, \end{aligned}$$

where

$$\binom{n}{k} := \frac{n!}{k!(n-k)!},$$

if $k > n$, then $\binom{n}{k} = 0$.

Setting $p + q \le 5$ in Theorems 1.1 and 1.2, and applying Lemmas 4.1 and 4.2, we can obtain the following examples.

Example 4.3. For any $a, b \notin \mathbb{Z}$ with $a \neq b$,

$$\begin{aligned} S_{1,2}(a,b) &= S_{1,2}^{++}(-a,-b) + S_{1,2}^{++}(a,b) \\ &= -\pi \cot(\pi a) \zeta(2; a-b) + \frac{1}{ab^2} + \frac{1}{a} \zeta(2; 1-b) + \zeta(1,2; a+1, b+1) \\ &\quad + \frac{\pi^2}{\sin^2(\pi b)} (\Psi(-b; a) + \gamma) + \pi \cot(\pi b) \zeta(2; a-b), \\ S_{1,3}(a,b) &= S_{1,3}^{++}(-a,-b) - S_{1,3}^{++}(a,b) \\ &= -\pi \cot(\pi a) \zeta(3; a-b) - \frac{1}{ab^3} + \frac{1}{a} \zeta(3; 1-b) \\ &\quad - \zeta(1,3; a+1, b+1) - \frac{\pi^2}{\sin^2(\pi b)} \zeta(2; a-b) \\ &\quad + \pi \cot(\pi b) \zeta(3; a-b) - \frac{\pi^3 \cot(\pi b)}{\sin^2(\pi b)} (\Psi(-b; a) + \gamma), \end{aligned}$$

$$
\begin{aligned}
S_{1,4}(a,b) &= S_{1,4}^{++}(-a,-b) + S_{1,4}^{++}(a,b) \\
&= -\pi\cot(\pi a)\,\zeta(4;a-b) + \frac{1}{a}\zeta(4;1-b) + \frac{1}{ab^4} + \zeta(1,4;a+1,b+1) \\
&\quad + \frac{\pi^4}{3}\left(\Psi(-b;a)+\gamma\right)\left(2\frac{\cot^2(\pi b)}{\sin^2(\pi b)} + \frac{1}{\sin^4(\pi b)}\right) \\
&\quad + \zeta(2;a-b)\,\frac{\pi^3\cot(\pi b)}{\sin^2(\pi b)} - \zeta(3;a-b)\,\frac{\pi^2}{\sin^2(\pi b)} \\
&\quad + \pi\cot(\pi b)\,\zeta(4;a-b),
\end{aligned}
$$

$$
\begin{aligned}
S_{2,2}(a,b) &= S_{2,2}^{++}(-a,-b) - S_{2,2}^{++}(a,b) \\
&= -\zeta(2;a+1)\left(\zeta(2;1-b)+\zeta(2;1+b)\right) \\
&\quad - \frac{\zeta(2;1-b)}{a^2} - \frac{\zeta(2;a+1)}{b^2} - \frac{1}{a^2b^2} - \zeta(2,2;a+1,b+1) \\
&\quad + \frac{\pi^2}{\sin^2(\pi a)}\cdot\zeta(2;a-b) + 2\pi\cot(\pi a)\zeta(3;a-b) \\
&\quad + \frac{\pi^2}{\sin^2(\pi b)}\zeta(2;a-b) - 2\pi\cot(\pi b)\zeta(3;a-b),
\end{aligned}
$$

$$
\begin{aligned}
S_{2,3}(a,b) &= S_{2,3}^{++}(-a,-b) + S_{2,3}^{++}(a,b) \\
&= -\zeta(2;a+1)\left(\zeta(3;1-b)-\zeta(3;1+b)\right) \\
&\quad - \frac{1}{a^2}\zeta(3;1-b) + \frac{1}{b^3}\zeta(2;1+a) + \frac{1}{a^2b^3} + \zeta(2,3;a+1,b+1) \\
&\quad + \frac{\pi^2}{\sin^2(\pi a)}\zeta(3;a-b) + 3\pi\cot(\pi a)\zeta(4;a-b) \\
&\quad - \frac{\pi^3\cot(\pi b)}{\sin^2(\pi b)}\zeta(2;a-b) + 2\frac{\pi^2}{\sin^2(\pi b)}\zeta(3;a-b) \\
&\quad - 3\pi\cot(\pi b)\,\zeta(4;a-b),
\end{aligned}
$$

$$
\begin{aligned}
S_{3,2}(a,b) &= S_{3,2}^{++}(-a,-b) + S_{3,2}^{++}(a,b) \\
&= \zeta(3;a+1)\left(\zeta(2;1-b)+\zeta(2;1+b)\right) \\
&\quad + \frac{1}{a^3}\zeta(2;1-b) + \frac{1}{b^2}\zeta(3;1+a) + \frac{1}{a^3b^2} + \zeta(3,2;a+1,b+1) \\
&\quad - \frac{\pi^2}{\sin^2(\pi b)}\zeta(3;a-b) + 3\pi\cot(\pi b)\zeta(4;a-b) \\
&\quad - 3\pi\cot(\pi a)\,\zeta(4;a-b) - 2\frac{\pi^2}{\sin^2(\pi a)}\zeta(3;a-b) \\
&\quad - \frac{\pi^3\cot(\pi a)}{\sin^2(\pi a)}\zeta(2;a-b).
\end{aligned}
$$

Example 4.4. For any $a, b \notin \mathbb{Z}$ with $a \neq b$,

$$\begin{aligned}
L_{1,2}(a,b) =&\ -S^{+-}_{1,2}(-a,-b) - S^{+-}_{1,2}(a,b)\\
=&\ \frac{1}{ab^2} - \frac{1}{a}\bar{\zeta}(2;1-b) - \bar{\zeta}(1,2;a+1,b+1)\\
&+ (\Psi(-b;a)+\gamma)\frac{\pi^2\cot(\pi b)}{\sin(\pi b)} + \frac{\pi}{\sin(\pi b)}\zeta(2;a-b)\\
&- \frac{\pi}{\sin(\pi a)}\bar{\zeta}(2;a-b),\\
L_{1,3}(a,b) =&\ -S^{+-}_{1,3}(-a,-b) + S^{+-}_{1,3}(a,b)\\
=&\ -\frac{1}{ab^3} - \frac{1}{a}\bar{\zeta}(3;1-b) + \bar{\zeta}(1,3;a+1,b+1) - \frac{\pi}{\sin(\pi a)}\bar{\zeta}(3;a-b)\\
&+ \frac{\pi}{\sin(\pi b)}\zeta(3;a-b) - \frac{\pi^2\cot(\pi b)}{\sin(\pi b)}\zeta(2;a-b)\\
&- \frac{\pi^3}{2}\left(\frac{\cot^2(\pi b)}{\sin(\pi b)} + \frac{1}{\sin^3(\pi b)}\right)(\Psi(-b;a)+\gamma),\\
L_{1,4}(a,b) =&\ -S^{+-}_{1,4}(-a,-b) - S^{+-}_{1,4}(a,b)\\
=&\ \frac{1}{ab^4} - \frac{1}{a}\bar{\zeta}(4;1-b) - \bar{\zeta}(1,4;a+1,b+1) - \frac{\pi}{\sin(\pi a)}\bar{\zeta}(4;a-b)\\
&+ \frac{\pi^3}{2}\left\{\frac{\cot^2(\pi b)}{\sin(\pi b)} + \frac{1}{\sin^3(\pi b)}\right\}\zeta(2;a-b) + \frac{\pi}{\sin(\pi b)}\zeta(4;a-b)\\
&+ \frac{\pi^4}{6}\left\{\frac{\cot^3(\pi b)}{\sin(\pi b)} + 5\frac{\cot(\pi b)}{\sin^3(\pi b)}\right\}(\Psi(-b;a)+\gamma)\\
&- \pi^2\frac{\cot(\pi b)}{\sin(\pi b)}\zeta(3;a-b),\\
L_{2,1}(a,b) =&\ -S^{+-}_{2,1}(-a,-b) - S^{+-}_{2,1}(a,b)\\
=&\ \zeta(2;a+1)\left\{\bar{\zeta}(1;1-b) - \bar{\zeta}(1;b+1)\right\} - \bar{\zeta}(2,1;a+1,b+1)\\
&+ \frac{1}{a^2}\bar{\zeta}(1;1-b) + \frac{1}{a^2 b} + \frac{1}{b}\zeta(2;a+1) - \frac{\pi}{\sin(\pi b)}\zeta(2;a-b)\\
&+ \frac{\pi}{\sin(\pi a)}\bar{\zeta}(2;a-b) + \frac{\pi^2\cot(\pi a)}{\sin(\pi a)}\bar{\zeta}(1;a-b),\\
L_{2,2}(a,b) =&\ -S^{+-}_{2,2}(-a,-b) + S^{+-}_{2,2}(a,b)\\
=&\ \zeta(2;a+1)\left\{\bar{\zeta}(2;b+1) + \bar{\zeta}(2;1-b)\right\} + \bar{\zeta}(2,2;a+1,b+1)\\
&+ \frac{1}{a^2}\bar{\zeta}(2,1-b) - \frac{1}{b^2}\zeta(2;a+1) - \frac{1}{a^2b^2} + \frac{2\pi}{\sin(\pi a)}\bar{\zeta}(3;a-b)\\
&+ \frac{\pi^2\cot(\pi a)}{\sin(\pi a)}\bar{\zeta}(2;a-b) - \frac{2\pi}{\sin(\pi b)}\zeta(3;a-b)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\pi^2\cot(\pi b)}{\sin(\pi b)}\zeta(2;a-b),\\
L_{2,3}(a,b)=&-S^{+-}_{2,3}(-a,-b)-S^{+-}_{2,3}(a,b)\\
=&-\zeta(2;a+1)\left\{\bar\zeta(3;b+1)-\bar\zeta(3;1-b)\right\}+\frac{1}{a^2}\bar\zeta(3;1-b)\\
&+\frac{1}{b^3}\zeta(2;a+1)+\frac{1}{a^2b^3}-\bar\zeta(2,3;a+1,b+1)\\
&+\frac{3\pi}{\sin(\pi a)}\bar\zeta(4;a-b)+\frac{\pi^2\cot(\pi a)}{\sin(\pi a)}\bar\zeta(3;a-b)\\
&-\frac{\pi^3\left(\cos^2(\pi b)+1\right)}{2\sin^3(\pi b)}\zeta(2;a-b)+2\frac{\pi^2\cot(\pi b)}{\sin(\pi b)}\zeta(3;a-b)\\
&-\frac{3\pi}{\sin(\pi b)}\zeta(4;a-b),\\
L_{3,2}(a,b)=&-S^{+-}_{3,2}(-a,-b)-S^{+-}_{3,2}(a,b)\\
=&-\zeta(3;a+1)\left\{\bar\zeta(2;b+1)+\bar\zeta(2;1-b)\right\}-\bar\zeta(3,2;a+1,b+1)\\
&-\frac{1}{a^3}\bar\zeta(2;1-b)+\frac{1}{b^2}\zeta(3;a+1)+\frac{1}{a^3b^2}-\frac{\pi^2\cot(\pi b)}{\sin(\pi b)}\zeta(3;a-b)\\
&+\frac{3\pi}{\sin(\pi b)}\zeta(4;a-b)-\frac{3\pi}{\sin(\pi a)}\bar\zeta(4;a-b)\\
&-2\frac{\pi^2\cot(\pi a)}{\sin(\pi a)}\bar\zeta(3;a-b)-\frac{\pi^3\left(\cos^2(\pi b)+1\right)}{2\sin^3(\pi b)}\bar\zeta(2;a-b).
\end{aligned}
$$

Remark 4.5. Note that $\zeta(p,q;a+1,b+1)$ and $\bar\zeta(p,q;a+1,b+1)$ can be evaluated in terms of the digamma functions and Hurwitz zeta functions by using (13) and (14).

5. Other results

Theorem 5.1. *For any* $\alpha,\beta\notin\mathbb{Z},\ \alpha\neq\beta$ *and* $\alpha,\beta\neq n+a,\ a\in\mathbb{C}\backslash\mathbb{N}_0\ (n\in\mathbb{N}_0)$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H^{(1)}_{n-1}(a)}{(n-\alpha)(n-\beta)}+\sum_{n=0}^{\infty}\frac{H^{(1)}_{n}(-a)}{(n+\alpha)(n+\beta)}\\
&-\frac{1}{a}\frac{\psi(\beta)-\psi(\alpha)}{\beta-\alpha}+\pi\cot(\pi a)\frac{\psi(\beta+a)-\psi(\alpha+a)}{\beta-\alpha}\\
&+\frac{\pi}{\beta-\alpha}\left\{\cot(\pi\beta)\left(\Psi(\beta;a)+\gamma\right)-\cot(\pi\alpha)\left(\Psi(\alpha;a)+\gamma\right)\right\}=0.
\end{aligned}
\tag{15}
$$

Proof. To prove this identity, we consider

$$F(z):=\frac{\pi\cot(\pi z)\left(\Psi(-z;a)+\gamma\right)}{(z+\alpha)(z+\beta)},$$

where $\alpha,\beta\notin\mathbb{Z},\ \alpha\neq\beta,$ and $\alpha,\beta\neq n+a,\ n\in\mathbb{N}$.

Note that the function in the contour integration only has poles at $z = n$ $(n \in \mathbb{Z})$, $n+a$ $(n \in \mathbb{N}_0)$ and $-\alpha, -\beta$. Applying Lemma 2.2 and direct residue computations, we deduce that

$$\begin{aligned}
\mathrm{Res}\left(F\left(z\right), -n\right) &= \frac{H_{n-1}^{(1)}\left(a\right)}{\left(n-\alpha\right)\left(n-\beta\right)} \quad (n \in \mathbb{N}),\\
\mathrm{Res}\left(F\left(z\right), n\right) &= \frac{-\frac{1}{a} + H_n^{(1)}\left(-a\right)}{\left(n+\alpha\right)\left(n+\beta\right)} \quad (n \in \mathbb{N}_0),\\
\mathrm{Res}\left(F\left(z\right), n+a\right) &= \frac{\pi\cot\left(\pi a\right)}{\left(n+a+\alpha\right)\left(n+a+\beta\right)} \quad (n \in \mathbb{N}_0),\\
\mathrm{Res}\left(F\left(z\right), -\alpha\right) &= \frac{\pi\cot\left(\pi\alpha\right)\left(\Psi(-\alpha;a)+\gamma\right)}{\alpha-\beta},\\
\mathrm{Res}\left(F\left(z\right), -\beta\right) &= \frac{\pi\cot\left(\pi\beta\right)\left(\Psi(-\beta;a)+\gamma\right)}{\beta-\alpha}.
\end{aligned}$$

Hence, applying Lemma 2.1 we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathrm{Res}\left(F\left(z\right), -n\right) + \sum_{n=0}^{\infty}\mathrm{Res}\left(F\left(z\right), n\right) + \sum_{n=0}^{\infty}\mathrm{Res}\left(F\left(z\right), n+a\right)\\
&+ \mathrm{Res}\left(F\left(z\right), -\alpha\right) + \mathrm{Res}\left(F\left(z\right), -\beta\right) = 0.
\end{aligned}$$

Thus, summing these contributions yields the desired evaluation. □

Observe that

$$\pi\cot(\pi a) = \frac{1}{a} - 2a\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}$$

and the limit

$$\lim_{a\to 0}\frac{1}{a}\sum_{n=0}^{\infty}\left\{\frac{1}{(n+a+\alpha)(n+a+\beta)} - \frac{1}{(n+\alpha)(n+\beta)}\right\} = \frac{\zeta(2;\beta)-\zeta(2;\alpha)}{\beta-\alpha}.$$

Hence, setting $a \to 0$ in Theorem 5.1 leads to the corollary.

Corollary 5.2. *For any complexes α, β with $\alpha \neq \beta$ and $\alpha, \beta \notin \mathbb{Z}$, we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_{n-1}}{(n-\alpha)(n-\beta)} + \sum_{n=1}^{\infty}\frac{H_n}{(n+\alpha)(n+\beta)}\\
&= \frac{\zeta(2;\alpha)-\zeta(2;\beta)}{\beta-\alpha} + \frac{\pi\cot(\pi\alpha)(\psi(\alpha)+\gamma) - \pi\cot(\pi\beta)(\psi(\beta)+\gamma)}{\beta-\alpha}.
\end{aligned}$$

Theorem 5.3. *For any integer $p > 1$ and any $\alpha, \beta \notin \mathbb{Z}$, $\alpha \neq \beta$ and $\alpha, \beta \neq n+a$, $a \in \mathbb{C}\setminus\mathbb{N}_0$ $(n \in \mathbb{N}_0)$,*

$$\sum_{n=1}^{\infty}\frac{H_n^{(p)}(-a)}{(n+\alpha)(n+\beta)} - (-1)^p\sum_{n=1}^{\infty}\frac{H_{n-1}^{(p)}(a)}{(n-\alpha)(n-\beta)} \tag{16}$$

$$+ (-1)^p \zeta(p; a+1) \frac{\pi \cot(\pi\alpha) - \pi \cot(\pi\beta)}{\beta - \alpha}$$
$$+ \frac{(-1)^p}{a^p} \frac{\psi(\beta) - \psi(\alpha)}{\beta - \alpha} + \frac{1}{(p-1)!} \sum_{n=0}^{\infty} \frac{d^{p-1}}{dz^{p-1}} \frac{\pi \cot(\pi z)}{(z+\alpha)(z+\beta)} \Big|_{z=n+a}$$
$$- \frac{\pi \cot(\pi\alpha) \Psi^{(p-1)}(\alpha; a)}{\beta - \alpha} + \frac{\pi \cot(\pi\beta) \Psi^{(p-1)}(\beta; a)}{\beta - \alpha} = 0.$$

Proof. The theorem results from applying the kernel function

$$\xi(z) := \frac{\pi \cot(\pi z)\, \Psi^{(p-1)}(-z; a)}{(p-1)!}$$

to the base function $r(z) := 1/((z+\alpha)(z+\beta))$, and performing the residue computation. We leave the details to the interested reader. □

Setting $p = 2$ and 3 in (16) yields the following corollary.

Corollary 5.4. *We have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}(-a)}{(n+\alpha)(n+\beta)} - \sum_{n=2}^{\infty} \frac{H_{n-1}^{(2)}(a)}{(n-\alpha)(n-\beta)}$$
$$= -\zeta(2, a+1)\, \zeta(1,1; 1-\alpha, 1-\beta) - \frac{1}{a^2} \zeta(1,1; \alpha, \beta) + \frac{\pi \cot(\pi\alpha)}{\beta - \alpha} \zeta(2; a+\alpha)$$
$$- \frac{\pi \cot(\pi\beta)}{\beta - \alpha} \zeta(2; a+\beta) + \frac{\pi^2}{\sin^2(\pi a)} \zeta(1,1; a+\alpha, a+\beta)$$
$$+ \frac{\pi \cot(\pi a)}{\beta - \alpha} \left\{ \zeta(2; a+\alpha) - \zeta(2; a+\beta) \right\},$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}(-a)}{(n+\alpha)(n+\beta)} + \sum_{n=2}^{\infty} \frac{H_{n-1}^{(3)}(a)}{(n-\alpha)(n-\beta)}$$
$$= -\zeta(p, a+1)\, \zeta(1,1; 1-\alpha, 1-\beta) + \frac{1}{a^3} \zeta(1,1; \alpha, \beta)$$
$$+ \zeta(p, a+1)\, \zeta(1,1; 1+\alpha, 1+\beta)$$
$$+ \frac{1}{\beta - \alpha} \left\{ \begin{array}{l} \pi \cot(\pi a)\, \zeta(3; a+\beta) - \pi \cot(\pi a)\, \zeta(3; a+\alpha) + \frac{\pi^2}{\sin^2(\pi a)} \zeta(2; a+\beta) \\ - \frac{\pi^2}{\sin^2(\pi a)} \zeta(2; a+\alpha) + \frac{\pi^3 (\alpha - \beta) \cot(\pi a)}{\sin^2(\pi a)} \zeta(1,1; a+\alpha, a+\beta) \end{array} \right\}$$
$$+ \frac{\pi \cot(\pi\beta)}{\beta - \alpha} \zeta(3; a+\beta) - \frac{\pi \cot(\pi\alpha)}{\beta - \alpha} \zeta(3; a+\alpha).$$

Remark 5.5. In fact, formula (16) can also established by differentiating (15) $p-1$ times with respect to a. In addition, it is possible to establish some other relations involving parametric nonlinear Euler sums by using the techniques described here. For example, considering the

$$\oint_{(\infty)} \frac{\pi \cot(\pi z) \Psi^{(m-1)}(-z; a_1) \Psi^{(p-1)}(-z; a_2)}{(m-1)!(p-1)!(z-b)^q} dz = 0$$

and

$$\oint_{(\infty)} \frac{\pi\Psi^{(m-1)}(-z;a_1)\Psi^{(p-1)}(-z;a_2)}{(m-1)!(p-1)!\sin(\pi z)(z-b)^q}dz = 0$$

we may derive some results of parametric quadratic Euler sums.

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