Bull. Korean Math. Soc. **60** (2023), No. 4, pp. 1131–1139
https://doi.org/10.4134/BKMS.b220595
pISSN: 1015-8634 / eISSN: 2234-3016

DYNAMICS OF RANDOM DYNAMICAL SYSTEMS

Enkhbayar Azjargal, Zorigt Choinkhor, and Nyamdavaa Tsegmid

Abstract. In this paper, we introduce the concept of ω-expansive of random map on compact metric spaces $\mathcal{P}$. Also we introduce the definitions of positively, negatively shadowing property and shadowing property for two-sided RDS. Then we show that if φ is ω-expansive and has the shadowing property for ω, then φ is topologically stable for ω.

1. Introduction

The shadowing property and expansivity plays an important role in the study of dynamical systems. In the last few years, many dynamists have studied various shadowing properties and expansivity, and its relationship with topological stability. The research related to them involves many aspects of the modern theory of dynamical systems, such as the theory of stability (see [1, 5–7]), etc. The theory of random dynamical system studies the action of random maps, drawn from a collection with prescribed probability, on a state space. Some results about various properties for RDS can be seen in [2,8]. The concept of expansivity for dynamical systems has been generalized to RDS in [3]. In [4], the authors defined a RDS and random sets in an uniform space and studied expansivity of an uniform random operator.

In this paper we shall consider the shadowing property, expansivity and stability for a measurable random dynamical system on a compact metric space, with a θ-shift map. In this section we give the definition of expansivity and definitions of a positively (negatively) shadowing and shadowing in RDS. In Section 2, we prove some essential properties of the shadowing property, expansivity for RDS and stability.

Let $\mathcal{P}$ be a compact metric space with metric d. A homeomorphism $f : \mathcal{P} \to \mathcal{P}$ is *expansive* if there is a constant $e > 0$ such that $x \neq y$ $(x, y \in \mathcal{P})$ implies $d(f^n(x), f^n(y)) > e$ for some integer n. A sequence points $\{x_i\}_{i\in\mathbb{Z}}$ of a compact metric space $\mathcal{P}$ is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{Z}$. Given $\varepsilon > 0$ a δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ is said to be ε-*shadowed* by a point

Received August 29, 2022; Accepted January 27, 2023.
2020 *Mathematics Subject Classification.* Primary 37H30, 37H12, 37B05.
Key words and phrases. ω-expansive, shadowing property, topologically stable.
This work was financially supported by MNUE.

$x \in \mathcal{P}$ if $d(f^i(x), x_i) < \varepsilon$ for every $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f can be ε-shadowed by some point of $\mathcal{P}$ [1, 5].

We assume that $\mathcal{P}$ is a compact metric space, Γ is a finite family of continuous maps from $\mathcal{P}$ into itself and Q is the probability measure on the σ-field of Γ. The space Ω is the full shift, $\Omega = \Gamma^{\mathbb{Z}}$, which is the set of all possible two sided infinitely long sequences of deterministic transformations, i.e.,

$$\Omega = \{\omega : (\ldots, f_{-1}, f_0, f_1, \ldots, f_k, \ldots) \mid f_k \in \Gamma\}.$$

Let $\mathbb{P}$ be probability measure on σ-field of Ω.

Definition 1 ([8, Definition 2.1])**.** $(\Omega, F, \mathbb{P}, \theta)$ is a metric dynamical system if $(\Omega, F, \mathbb{P})$ is a probability space and $\theta(t) : \Omega \to \Omega$, $t \in \mathbb{Z}$ is a family of measure-preserving transformations such that

1. $\theta(0) = \mathrm{id}$, $\theta(s) \circ \theta(t) = \theta(s + t)$ for every $s, t \in \mathbb{Z}$.
2. The mapping $(t, \omega) \to \theta(t)\omega$ is measurable.
3. $\theta(t)\mathbb{P} = \mathbb{P}$ for every $t \in \mathbb{Z}$.

Definition 2 ([8, Definition 2.2])**.** A measurable random dynamical system (RDS) on the complete separable metric space $(\mathcal{P}, d)$ over a metric dynamical system $(\Omega, F, \mathbb{P}, \theta)$ is a map $\mathbb{Z} \times \Omega \times \mathcal{P} \to \mathcal{P} : (t, \omega, i) \to \varphi(t, \omega)i$, with the following properties:

1. The map $(t, \omega, i) \to \varphi(t, \omega)i$ is $B(\mathbb{Z}) \otimes F \otimes B(\mathcal{P})$, $B(\mathcal{P})$-measurable.
2. The map $i \to \varphi(t, \omega)i$ satisfies the cocycle property:
$$\varphi(0, \omega) = \mathrm{id}, \quad \varphi(s + t, \omega) = \varphi(s, \theta(t)\omega) \circ \varphi(t, \omega)$$
for every $s, t \in \mathbb{Z}$ and $\omega \in \Omega$.

From the definition, the RDS is driven by the base flow and for one particular noise realization ω, one can treat $i \to \varphi(t, \omega)i$ as a non-autonomous dynamical system, which defines one-point motion.

Remark 1.1. In this paper, $\mathbb{P}$ is the Bernoulli measure defined on the cylinder set,

$$\mathbb{P}([\alpha_0, \alpha_1, \ldots, \alpha_k]) = Q(\alpha_0)Q(\alpha_1) \cdots Q(\alpha_k),$$

i.e., maps at different steps are chosen independently with the same probability measure Q and θ is a shift map.

Since θ is a shift map, we have

$$\begin{cases} \varphi(n, \omega)x = f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0(x), & n \geq 1; \\ \varphi(0, \omega)x = x, & n = 0; \\ \varphi(n, \omega)x = f_n \circ f_{n+1} \circ \cdots \circ f_{-1}(x), & n \leq -1. \end{cases}$$

The ω-*orbit* of a state (point) $x \in \mathcal{P}$ is the sequence $\{\varphi(n, \omega)x\}_{n \in \mathbb{Z}}$.

Definition 3. Let $(\mathcal{P}, d)$ be a compact metric space and $\omega \in \Omega$. A random map $\varphi : \mathcal{P} \to \mathcal{P}$ is said to be ω-*expansive*, if there exists a constant $e > 0$ (called an expansivity constant) such that for any $x, y \in \mathcal{P}$, $x \neq y$, $d(\varphi(n,\omega)x, \varphi(n,\omega)y) > e$ for some $n \in \mathbb{Z}$. Equivalently, if for $x, y \in \mathcal{P}$, $d(\varphi(n,\omega)x, \varphi(n,\omega)y) \leqslant e$ for all $n \in \mathbb{Z}$, then $x = y$.

Remark 1.2. If in the above definition $\begin{cases} f_n = f, & n \geq 0; \\ f_n = f^{-1}, & n \leq -1 \end{cases}$ and θ-shift map, then ω-expansivity of random map is equivalent to expansivity of f on $\mathcal{P}$.

Definition 4. For $\delta > 0$, a (δ, ω)-*positive* (*negative*) *pseudo orbit* is a sequence $\{x_k\}_{k\in\mathbb{Z}}$ of points (states) in X satisfying

$$d(f_k(x_k), x_{k+1}) < \delta \ (d(f_{-k-1}(x_{-k}), x_{-k-1}) < \delta)$$

for all $k \in \mathbb{N}_0$, where f_k is k-th component of ω, $\omega \in \Omega$.

Definition 5. We say that φ has the *positively* (*negatively*) *shadowing property*, for ω if $\forall\varepsilon > 0$, $\exists\delta > 0$ such that each (δ, ω)-positive (negative) pseudo orbit for φ is ε-shadowed by some point of $\mathcal{P}$.

Definition 6. We say that φ has the *shadowing property* for ω if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every (δ, ω)-positive and negative pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ of φ, there exists $y \in \mathcal{P}$ satisfying $d(\varphi(i,\omega)y, x_i) < \varepsilon$ for all $i \in \mathbb{Z}$

Remark 1.3. If in the above definition $\begin{cases} f_n = f, & n \geq 0; \\ f_n = f^{-1}, & n \leq -1 \end{cases}$ and θ-shift map, then shadowing property of random map for ω is equivalent to shadowing property of f on $\mathcal{P}$.

2. The main result

Theorem 2.1. *Let $f : \mathcal{P} \to \mathcal{P}$ be a homeomorphism on the compact metric space $(\mathcal{P}, d)$. If φ has the positively shadowing property for $\omega = \{f_i\} \in \Gamma^{\mathbb{Z}}$ satisfying $f_{-k} = f_{k-1}$, $k \in \mathbb{N}$, then φ has the negatively shadowing property for ω.*

Proof. Suppose that φ has the positively shadowing property for ω. Let $\varepsilon > 0$. Take $\delta > 0$ such that (δ, ω)-positively pseudo orbit for φ is ε-shadowed by some point of X. Let $\{x_{-k}\}_{k\in\mathbb{N}_0}$ be a (δ, ω)-negatively pseudo orbit for φ. Define a new sequence $\{y_k\}_{k\in\mathbb{N}_0}$ by $y_k = x_{-k}$. Since $f_{-k} = f_{k-1}$ for all $k \in \mathbb{N}_0$, $\{y_k\}_{k\in\mathbb{N}_0}$ is (δ, ω)-positively pseudo orbit for φ. Then $\{y_k\}_{k\in\mathbb{N}_0}$ is ε-shadowed by some point of X. Therefore φ has the negatively shadowing property for ω. □

Theorem 2.2. *Let $f : \mathcal{P} \to \mathcal{P}$ be a homeomorphism on the compact metric space (X, d). $\Omega = \Gamma^{\mathbb{Z}}$, here $\Gamma = \{f, f^{-1}\}$. If f has the shadowing property and $\omega = \begin{cases} f, & \text{for } \ -k \leq i \leq k-1; \\ f^{-1}, & \text{otherwise,} \end{cases}$ then φ has the positively and negatively shadowing property for ω.*

Proof. Let $\varepsilon > 0$ be given. Then there exists $\varepsilon_1 > 0$ such that $\varepsilon_1 < \frac{\varepsilon}{2}$ and $d(x, y) < \varepsilon_1$ implies

$$\max\{d(f^{-i}(x), f^{-j}(y)) \mid i = 0, \dots, k\} < \frac{\varepsilon}{2}.$$

Let $\omega = (\cdots f^{-1} \underbrace{f \cdots f}_{k} \underbrace{f \cdots f}_{k} f^{-1} \cdots)$. Then choose $\delta_1 > 0$ such that $\forall\delta_1$-pseudo orbit for f is $\frac{\varepsilon_1}{2}$-shadowed by some $y \in \mathcal{P}$. Also choose $\delta_2 > 0$ such that $\forall\delta_2$-pseudo orbit for f^{-1} is $\frac{\varepsilon_1}{2}$-shadowed by some $z \in \mathcal{P}$. We can take $\delta = \min\{\delta_1, \delta_2\}$.

Claim: Each (δ, ω)-pseudo orbit for φ is ε-shadowed by some point of $\mathcal{P}$.

Let $\{x_n\}_{n\in\mathbb{Z}}$ be a (δ, ω)-pseudo orbit for φ. Then

$$\begin{aligned} &d(f(x_i), x_{i+1}) < \delta, \quad 0 \leqslant i \leqslant k, \\ &d(f(x_{i+1}), x_i) < \delta, \quad -k \leqslant i \leqslant -1, \\ &d(f^{-1}(x_i), x_{i+1}) < \delta, \quad k+1 \leqslant i, \\ &d(f^{-1}(x_{i+1}), x_i) < \delta, \quad i \leqslant -k-1. \end{aligned}$$

Therefore if y is shadowing point by f, then $d(f^k(x_k), y) < \varepsilon_1$ implies

$$d(f^{k-l}(x_k), f^{-l}(y)) < \frac{\varepsilon}{2}.$$

So

$$d(x_{k-l}, f^{-l}(y)) \leqslant d(x_{k-l}, f^{k-l}(y)) + d(f^{k-l}(y), f^{-l}(y)) < \varepsilon_1 + \frac{\varepsilon}{2} < \varepsilon.$$

If $f^{-k}(y) = a$, then $d(\varphi(n, \omega)a, x_n) < \varepsilon$, $n \in \mathbb{N}_0$.

Therefore φ has the positively shadowing property. By Theorem 2.1, φ has the negatively shadowing property. □

Lemma 2.3. *Let $\varphi : \mathcal{P} \to \mathcal{P}$ be an ω-expansive random map with the shadowing property. If $\varepsilon < e/2$ and δ corresponds to ε as in Definition 5, then there is a unique $x \in \mathcal{P}$ which ε-shadowed a given (δ, ω)-pseudo orbit.*

Proof. If y also ε-traces $\{x_n\}_{n\in\mathbb{Z}}$, then

$$d(\varphi(n, \omega)x, \varphi(n, \omega)y) \leq d(\varphi(n, \omega)x, x_n) + d(x_n, \varphi(n, \omega)y) < 2\varepsilon < e, \forall n \in \mathbb{Z}$$

and $x = y$ by expansivity. □

Theorem 2.4. *Let $\varphi : \mathcal{P} \to \mathcal{P}$ be an ω-expansive random map with the positively and negatively shadowing property for ω. Then φ has the shadowing property for ω.*

Proof. For a given ω, let e be an expansive constant of the ω-expansive. Let $\{x_n\}_{n\in\mathbb{Z}} \subset \mathcal{P}$. Take $0 < \varepsilon < \frac{e}{2}$. By the assumption, we can choose $\delta_1 > 0$ such that a (δ_1, ω)-positive pseudo orbit $\{x_n\}_{n\geq 0}$ for φ is ε-shadowed by some $y \in \mathcal{P}$. Also choose $\delta_2 > 0$ such that a (δ_2, ω)-negative pseudo orbit $\{x_n\}_{n\leq 0}$ for φ is

ε-shadowed by some $z \in \mathcal{P}$. We can take $\delta = \min\{\delta_1, \delta_2\}$. Then $\{x_n\}_{n\in\mathbb{Z}}$ is a (δ, ω)-positive and negative pseudo orbit of φ for ω.

$$d(\varphi(n,\omega)y, \varphi(n,\omega)z) \leq d(\varphi(n,\omega)y, x_n) + d(x_n, \varphi(n,\omega)z) < 2\varepsilon < e, \forall n \in \mathbb{Z}$$

and $y = z$ by ω-expansivity. Therefore $\{x_n\}_{n\in\mathbb{Z}}$ is ε-shadowed by some point of $\mathcal{P}$ and φ has the shadowing property for ω. □

For $\omega = (f_i)_{i\in\mathbb{Z}}$, we define $\omega^k = (g_n)_{n\in\mathbb{Z}}$ by

$$g_n = f_{(n+1)k-1} \circ f_{(n+1)k-2} \circ \cdots \circ f_{nk}, \quad n \geqslant 0,$$
$$g_n = f_{nk} \circ f_{nk+1} \circ \cdots \circ f_{nk+k-1}, \quad n \leqslant -1.$$

Theorem 2.5. *Let k be a positive integer and $\omega \in \Gamma^{\mathbb{Z}}$. If φ is ω^k-expansive, then φ is ω-expansive.*

Proof. Let $\omega^k = \{g_i\}_{i\in\mathbb{Z}}$ and e be an expansive constant of the ω^k-expansive. Then for any $x, y \in \mathcal{P}$, $x \neq y$, there exists $n \in \mathbb{Z}$ such that

$$d(\varphi(n,\omega^k)x, \varphi(n,\omega^k)y) > e.$$

For $n \geqslant 0$,

$$\begin{aligned} d(\varphi(n,\omega^k)x, \varphi(n,\omega^k)y) &= d(g_{n-1} \circ g_{n-2} \circ \cdots \circ g_0(x), g_{n-1} \circ g_{n-2} \circ \cdots \circ g_0(y)) \\ &= d(f_{nk-1} \circ \cdots \circ f_0(x), f_{nk-1} \circ \cdots \circ f_0(y)) \\ &= d(\varphi(nk-1,\omega)x, \varphi(nk-1,\omega)y) > e. \end{aligned}$$

For $n < 0$,

$$d(g_n \circ g_{n+1} \circ \cdots \circ g_{-1}(x), g_n \circ g_{n+1} \circ \cdots \circ g_{-1}(y)) = d(\varphi(nk,\omega)x, \varphi(nk,\omega)y) > e.$$

Then φ is ω-expansive. □

Remark 2.6. In general, the converse of Theorem 2.5 is not true. Indeed, we can choose $\omega \in \Gamma^{\mathbb{Z}}$ such that each g_n is the identity maps on $\mathcal{P}$.

Theorem 2.7. *Let k be a positive integer and $\omega \in \Gamma^{\mathbb{Z}}$. If φ has the shadowing property for ω, then φ has the shadowing property for ω^k.*

Proof. Let $\varepsilon > 0$ be given. Since φ has the shadowing property for ω, there exists a $\delta > 0$ such that every (δ, ω)-pseudo orbit for φ is ε-shadowed by some point of $\mathcal{P}$. Let $\{y_i\}_{i\in\mathbb{Z}}$ be a (δ, ω^k)-pseudo orbit for φ. Then

$$d(g_n(y_n), y_{n+1}) < \delta, \quad n \geqslant 0,$$
$$d(g_n(y_{n+1}), y_n) < \delta, \quad n \leqslant -1,$$

where

$$g_n = f_{(n+1)k-1} \circ f_{(n+1)k-2} \circ \cdots \circ f_{nk}, \quad n \geqslant 0,$$
$$g_n = f_{nk} \circ f_{nk+1} \circ \cdots \circ f_{nk+k-1}, \quad n \leqslant -1.$$

Put $x_{nk} = y_n$ and for $1 \leq j \leq k-1$

$$x_{nk+j} = f_{nk+j-1} \circ \cdots \circ f_{nk}(y_n), \quad n \geqslant 0,$$

$$x_{(n+1)k-j} = f_{(n+1)k-j} \circ \cdots \circ f_{(n+1)k-1}(y_n), \quad n \leqslant -1.$$

We show that $\{x_i\}_{i\in\mathbb{Z}}$ is a (δ,ω)-pseudo orbit for φ. For any j, $0 \leqslant j \leqslant k-2$

$$d(f_{nk+j}(x_{nk+j}), x_{nk+j+1}) = 0, \quad n \geqslant 0.$$

For any j, $1 \leqslant j \leqslant k-1$

$$d(f_{(n+1)k-j}(x_{(n+1)k-j+1}), x_{(n+1)k-j}) = 0, \quad n \leqslant -1,$$

by the construction of $\{x_i\}$. Now for $j = k-1$

$$\begin{aligned} d(f_{nk+k-1}(x_{nk+k-1}), x_{nk+k}) &= d(f_{nk+k-1}(f_{nk+k-2} \circ \cdots \circ f_{nk})(y_n), y_{n+1}) \\ &= d(g_n(y_n), y_{n+1}) < \delta, \quad n \geqslant 0, \end{aligned}$$

for $j = 0$

$$\begin{aligned} d(f_{nk}(x_{nk+1}), x_{nk}) &= d(f_{nk}(f_{nk+1} \circ \cdots \circ f_{nk+k-1})(y_{n+1}), y_n) \\ &= d(g_n(y_{n+1}), y_n) < \delta, \quad n \leqslant -1. \end{aligned}$$

By φ has shadowing property, $\{x_i\}_{i\in\mathbb{Z}}$ is ε-traced by some $y \in \mathcal{P}$, i.e.,

$$d(\varphi(n,\omega)y, x_n) < \varepsilon.$$

In particular for $n = km$,

$$d(\varphi(km,\omega)y, x_{km}) = d(\varphi(m,\omega^k)y, y_m) < \varepsilon, \quad m \in \mathbb{Z}.$$

Thus φ has shadowing property for ω^k. □

Let $(\mathcal{P}, d)$ be a compact metric space. Define $\Gamma_1 = \{f \in \Gamma \mid f\text{-continuous}\}$. Then (Γ_1, d_1) is a metric space, where

$$d_1(f,g) = \sup_{x\in\mathcal{P}} d(f(x), g(x)), \quad f, g \in \Gamma_1.$$

We define a metric d_2 on $\Gamma_1^{\mathbb{Z}}$,

$$d_2(\omega_1,\omega_2) = \sup_{n\in\mathbb{Z}} d_1(f_n, g_n),$$

where $\omega_1 = \{f_i\}_{i\in\mathbb{Z}}$, $\omega_2 = \{g_i\}_{i\in\mathbb{Z}}$.

Definition 7. A random map φ is *topologically stable* for $\omega \in \Gamma_1^{\mathbb{Z}}$, if for every $\varepsilon > 0$, there exists a number $\delta \in (0,1)$ such that for ω^* with $d_2(\omega,\omega^*) < \delta$ there is a continuous map h such that for all $x \in \mathcal{P}$, $d(h(x),x) < \varepsilon$ and $d_1(\varphi(n,\omega)h, \varphi(n,\omega^*)) < \varepsilon$ for $\forall n \in \mathbb{Z}$.

Theorem 2.8. *If a random map φ on compact metric space $\mathcal{P}$ is ω-expansive and has shadowing property for ω, then φ is topologically stable for $\omega \in \Gamma_1^{\mathbb{Z}}$.*

Proof. Let $e > 0$ be an ω-expansive constant of φ, $\omega = \{f_i\}_{i\in\mathbb{Z}}$. Fix $0 < \varepsilon < \frac{e}{3}$. Let $0 < \delta < \min\{\frac{e}{3}, 1\}$ be such that every (δ,ω)-pseudo orbit can be ε-shadowed by some point of $\mathcal{P}$. Let $\{x_i\}_{i\in\mathbb{Z}}$ be (δ,ω)-shadowed by $x \in \mathcal{P}$. By Lemma 2.3, there exists a unique $x \in \mathcal{P}$ which ε-shadows a given (δ,ω)-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$.

Let ω^* be $\omega^* = \{f_i^*\}_{i\in\mathbb{Z}} \in \Gamma_1^{\mathbb{Z}}$ such that $d_2(\omega, \omega^*) < \delta$. Then for a sequence $y^* = \{\varphi(n, \omega^*)y\}_{n\in\mathbb{Z}}$, we have

$$d(f_n(\varphi(n,\omega^*)y), \varphi(n+1,\omega^*)y) = d(f_n(\varphi(n,\omega^*)y), f_n^*(\varphi(n,\omega^*)y)) < \delta,\ n \geqslant 0,$$
$$d(f_n(\varphi(n+1,\omega^*)y), \varphi(n,\omega^*)y) = d(f_n(\varphi(n+1,\omega^*)y), f_n^*(\varphi(n+1,\omega^*)y)) < \delta,\ n \leqslant -1.$$

Hence y^* is a (δ, ω)-pseudo orbit. Thus there exists a unique $h(y) \in \mathcal{P}$ which is the ε-shadowing point for y^*. We can define a map $h : \mathcal{P} \longrightarrow \mathcal{P}$ with

$$d(\varphi(n,\omega)h(y), \varphi(n,\omega^*)y) < \varepsilon \tag{2.1}$$

for all $n \in \mathbb{Z}$ and $y \in \mathcal{P}$. Letting $n = 0$, we have $d(h(y), y) < \varepsilon$ for each $y \in \mathcal{P}$.

Finally, we show that h is continuous. Let $\lambda > 0$. Then we can choose $N > 0$ such that $|n| \leqslant N$,

$$d(\varphi(n,\omega)x, \varphi(n,\omega)y) < e \Rightarrow d(x,y) < \lambda. \tag{2.2}$$

Suppose (2.2) does not hold. Let α be a finite open cover of $\mathcal{P}$ with diameter less than e. Then there exists an $\varepsilon_1 > 0$ such that for each $n > 0$, there exist $x_n, y_n \in \mathcal{P}$ with $d(x_n, y_n) > \varepsilon_1$ and $A_{n,k} \in \alpha$ $(-n \leqslant k \leqslant n)$ with

$$d(\varphi(k,\omega)x_n, \varphi(k,\omega)y_n) \in A_{n,k}.$$

Since $\mathcal{P}$ is compact, there exist $x, y \in \mathcal{P}$ such that $x_n \to x,\ y_n \to y$ and $x \neq y$. For fixed k, consider the sets $A_{n,k}$ for $n > 0$. Then infinitely many of $A_{n,k}$ coincide to some $A_k \in \alpha$. Since $x_n, y_n \in A_0$ for infinitely many $n > 0$, we have $x, y \in \overline{A_0}$. Similarly, for each k, $\varphi(k,\omega)x_n, \varphi(k,\omega)y_n \in A_k$ for infinitely many $n > 0$. We have $\varphi(k,\omega)x, \varphi(k,\omega)y \in \overline{A_k}$. Since $\operatorname{diam}(A_k) = \operatorname{diam}(\overline{A_k}) < e$, we have

$$d(\varphi(k,\omega)x, \varphi(k,\omega)y) < e, \quad \forall k \in \mathbb{Z}$$

which gives a contradiction to the fact that φ is ω-expansive.

Since $\omega^* = \{f_i^*\}_{i\in\mathbb{Z}}$ and f_i^* is an invertible and continuous map on $\mathcal{P}$, we can choose η such that $d(x,y) < \eta$ implies

$$d(\varphi(n,\omega^*)x, \varphi(n,\omega^*)y) < \frac{e}{3} \quad \text{for} \quad |n| < N. \tag{2.3}$$

If $d(x,y) < \eta$, then by (2.1), (2.3) we have

$$\begin{aligned}
&d(\varphi(n,\omega)h(x), \varphi(n,\omega)h(y)) \\
&\leqslant d(\varphi(n,\omega)h(x), \varphi(n,\omega^*)x) + d(\varphi(n,\omega^*)x, \varphi(n,\omega^*)y) \\
&\quad + d(\varphi(n,\omega^*)y, \varphi(n,\omega)h(y)) \\
&\leqslant \varepsilon + \frac{e}{3} + \varepsilon < e, \quad |n| \leqslant N.
\end{aligned}$$

Therefore $d(x,y) < \eta$ implies $d(h(x), h(y)) < \lambda$ and the continuity of h is proved. □

Example 2.9. Let the space $\mathcal{P}$ be $\{0,1\}^{\mathbb{Z}}$ with the metric d defined by $d(x,y) = 2^{-m}$ if m is the largest natural number with $x_j = y_j, \forall |j| < m$, and $d(x,y) = 1$ if $x_0 \neq y_0$. Let $\sigma : \mathcal{P} \to \mathcal{P}$ be a shift homeomorphism defined by $(\sigma x)_j = x_{j+1}$, where $x = (x_j)_{-\infty}^{\infty}$. If we choose $\omega = \{f_i\}_{i\in\mathbb{Z}}$, $f_i \in \Gamma = \{\sigma, \sigma^2, \ldots, \sigma^k\}$, then φ is not ω-expansive.

Proof. Let $\varepsilon > 0$. Then there exists a natural m such that $\frac{1}{2^m} < \varepsilon$. We can choose $x = \{x_k\}_{k\in\mathbb{Z}}$, $y = \{y_k\}_{k\in\mathbb{Z}}$ such that $x_k = y_k$ for all $k \geq -m$ and $x_k \neq y_k$ for all $k < -m$.

Then $\forall n \in \mathbb{Z}$, $d(\varphi(n,\omega)x, \varphi(n,\omega)y) < \varepsilon$ and $x \neq y$. Therefore φ is not ω-expansive. ☐

Example 2.10. In Example 2.9, we choose $\omega = \{f_i\}_{i\in\mathbb{Z}}, \begin{cases} f_i = \sigma, & i \geq 0; \\ f_i = \sigma^{-1}, & i \leq -1. \end{cases}$ Then φ is ω-expansive and has the shadowing property. Indeed, for $\varepsilon > 0$ choose the smallest natural number m such that $2^{-m} < \varepsilon$. Let $\{x^i\}_{i\in\mathbb{Z}}$ be a $(2^{-(m+1)}, \omega)$-pseudo orbit for φ and $x^i = (\ldots, x^i_{-1}, x^i_0, x^i_1, \ldots)$. Then when $|k| \leq m$, we have

$$x^i_{k+1} = (\sigma x^i)_k = x^{i+1}_k, \qquad i \geq 0$$

and

$$x^i_k = (\sigma^{-1} x^{i+1})_k = x^{i+1}_{k-1}, \qquad i \leq -1.$$

Define a point x by $x = (\ldots, x^{-1}_0, x^0_0, x^1_0, \ldots)$. Then when $|k| \leq m$, we have

$$(\sigma^i x)_k = x^{i+k}_0 = x^{i+k-1}_1 = \cdots = x^i_k, \quad i \geq 0,$$
$$(\sigma^i x)_k = x^{i+k}_0 = x^{i+k-1}_1 = \cdots = x^i_k, \quad i \leq -1.$$

Therefore, $d(\sigma^i(x), x^i) \leq 2^{-(m+1)} < 2^{-m} < \varepsilon$ for $i \in \mathbb{Z}$, i.e., the point x is an ε-shadowing point of $\{x^i\}_{i\in\mathbb{Z}}$.

Acknowledgements. The authors thank the anonymous reviewers for their helpful suggestions. This work has been supported by Mongolian National University of Education.

References

[1] N. Aoki and K. Hiraide, *Topological theory of dynamical systems*, North-Holland Mathematical Library, 52, North-Holland, Amsterdam, 1994.

[2] L. Arnold, *Random dynamical systems*, Springer Monographs in Mathematics, Springer, Berlin, 1998. https://doi.org/10.1007/978-3-662-12878-7

[3] H. Crauel and M. Gundlach, *Stochastic Dynamics*, Springer, New York, 1999. https://doi.org/10.1007/b97846

[4] I. J. Kadhim and A. H. Khalil, *On Expansive random operators over a uniform random dynamical systems*, Eur. J. Sci. Res. **142** (2016), no. 4, 334–342.

[5] S. Y. Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Mathematics, 1706, Springer, Berlin, 1999.

[6] D. Thakkar and R. T. Das, *Topological stability of a sequence of maps on a compact metric space*, Bull. Math. Sci. **4** (2014), no. 1, 99–111. https://doi.org/10.1007/s13373-013-0045-z

[7] P. Walters, *On the pseudo-orbit tracing property and its relationship to stability*, in The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977), 231–244, Lecture Notes in Math., 668, Springer, Berlin, 1978.

[8] F. X.-F. Ye and H. Qian, *Stochastic dynamics II: Finite random dynamical systems, linear representation, and entropy production*, Discrete Contin. Dyn. Syst. Ser. B **24** (2019), no. 8, 4341–4366. https://doi.org/10.3934/dcdsb.2019122

Enkhbayar Azjargal
Department of Mathematics
Mongolian National University of Education
Baga toiruu-14, Ulaanbaatar 14191-0068, Mongolia
Email address: azjargal@msue.edu.mn

Zorigt Choinkhor
Department of Mathematics
Mongolian National University of Education
Baga toiruu-14, Ulaanbaatar 14191-0068, Mongolia
Email address: ch.zorigt8113@gmail.com

Nyamdavaa Tsegmid
Department of Mathematics
Mongolian National University of Education
Baga toiruu-14, Ulaanbaatar 14191-0068, Mongolia
Email address: nyamdavaa.ts@msue.edu.mn