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SEMI-SYMMETRIC STRUCTURE JACOBI OPERATOR FOR REAL HYPERSURFACES IN THE COMPLEX QUADRIC

Imsoon Jeong, Gyu Jong Kim, and Changhwa Woo

Abstract. In this paper, we introduce the notion of *semi-symmetric structure Jacobi operator* for Hopf real hypersufaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. Next we prove that there does not exist any Hopf real hypersurface in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ with semi-symmetric structure Jacobi operator. As a corollary, we also get a non-existence property of Hopf real hypersurfaces in the complex quadric Q^m with either symmetric (parallel), or recurrent structure Jacobi operator.

Introduction

The complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is a compact Hermitian symmetric space of rank 2 which is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ (see [5, 6, 11, 17, 19]). Q^m is equipped with two remarkable geometric structures: a Kähler structure (J, g) and a parallel rank 2 subbundle $\mathfrak{A}$ of the endomorphism bundle End (TQ^m), which consists of all the real structures on the tangent space of Q^m. Q^1 is isometric to the round 2-sphere S^2. For $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of rank two and its maximal sectional curvature is equal to 4. Q^2 is isometric to the Riemannian product $S^2 \times S^2$. Thus in this paper, we assume $m \geq 3$.

A real hypersurface M in Q^m is an immersed submanifold of real codimension 1. Then the Kahler structure (J, g) on Q^m induces on M an almost contact metric structure (ϕ, ξ, η, g) in the following way: Let us denote by N a unit normal vector field on M in Q^m, and take the Reeb vector field on M as $\xi = -JN$. If $\eta(X) = g(X, \xi)$ for any X tangent to M, where in this case g is the restriction to M of the Riemannian metric of Q^m, we can write

$$JX = \phi X + \eta(X)N,$$

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where ϕX denotes the tangential component of JX. We say that a real hypersurface M is a Hopf hypersurface if the Reeb vector field ξ of M is principal, that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$, where S denotes the shape operator of M. It is known that the Reeb flow on M is geodesic if and only if ξ is a principal curvature vector of M everywhere. In particular, when the Reeb curvature function $\alpha = g(S\xi, \xi)$ is identically vanishing, we say that M has a vanishing geodesic Reeb flow. Otherwise, a real hypersurface M has a non-vanishing geodesic Reeb flow.

In this paper the geometric properties of real hypersurfaces M in complex quadric Q^m, which are tubes of radius r $(0 < r < \frac{\pi}{2})$ around the totally geodesic $\mathbb{C}P^l$ in Q^m, when $m = 2l$ or tubes of radius r $(0 < r < \frac{\pi}{2\sqrt{2}})$ around the totally geodesic Q^{m-1} in Q^m, are presented. The condition of isometric Reeb flow is equivalent to the commuting condition of the shape operator S with the structure tensor ϕ of M.

The Reeb flow on a real hypersurface M in a Kähler manifold $(\widetilde{M}, J, g)$ is isometric if M satisfies the property $\mathcal{L}_\xi g = 0$, where $\mathcal{L}_\xi$ is the Lie derivative in the direction of ξ. Okumura [11] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S\,(U_1 U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^m$ for some $k \in \{0, \ldots, m-1\}$.

Moreover, in a paper due to Suh [20], we find the following result for the complex quadric $Q^m = SO_{m+2}/SO_2 SO_m$:

Theorem A. *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. Then the Reeb flow on M is isometric if and only if m is even, say $m = 2l$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^l \subset Q^m$.*

In this paper, we consider the structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ of the real hypersurface M, where R is the Riemannian curvature tensor and ξ is the Reeb vector field of M.

Many geometers considered the parallelism of the tensor field R_ξ of type (1,1) on M. Among them, Suh [24] considered the notion of *parallel structure Jacobi operator* of real hypersurfaces in Q^m, that is, $\nabla_X R_\xi = 0$ for any $X \in TM$ and obtained a non-existence property in the following:

Theorem B. *There does not exist any Hopf real hypersurface in Q^m, $m \geq 3$, with parallel structure Jacobi operator.*

In this paper, we consider a weaker condition on a tensor type (1,1) on M in the complex quadric Q^m, namely, semi-symmetry. Actually, in [1] a tensor field F of type $(1, s)$ on a Riemannian manifold is said to be *semi-symmetric* if $R \cdot F = 0$, where the Riemannian curvature tensor R of M acts as a derivation on F. Geometers have proved various results concerning the semi-symmetric conditions of several tensors on real hypersurfaces in complex space forms (see [1, 4, 10, 12, 16]).

Recently, Lee, Suh, and Woo ([9]) applied the notion of semi-symmetry of the Ricci tensor, what we call *Ricci semi-symmetry*, to a real hypersurface in complex quadric Q^m and proved the following:

Theorem C. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with semi-symmetric Ricci tensor. Then M is locally congruent to one of the following*:

(1) *A tube of radius r over $\mathbb{C}P^k$ immersed as a totally geodesic submanifold in Q^m.*

(2) *For $m \geq 4$, M has 4-distinct constant principal curvatures*

$$\alpha, \quad \beta = \gamma = 0, \quad \lambda = \frac{-\alpha + \sqrt{\alpha^2 + 4(m-3)\left\{(m-2)\alpha^2 + 3(m-3)\right\}}}{2(m-3)}$$

and

$$\mu = \frac{-\alpha - \sqrt{\alpha^2 + 4(m-3)\left\{(m-2)\alpha^2 + 3(m-3)\right\}}}{2(m-3)}$$

whose corresponding principal curvature spaces satisfy $\xi \in T_\alpha$, $A\xi, AN \in T_{\beta=\gamma}$, T_λ and T_μ have multiplicities 1, 2, $m-2$ and $m-2$, respectively, where the Reeb function α is constant on M.

(3) *M has 4-distinct constant principal curvatures given by*

$$\alpha, \quad \beta = \gamma = 0, \quad \lambda = -\frac{1}{\alpha}, \quad \text{and} \quad \mu = -\frac{\alpha}{\alpha^2 + 2}$$

whose corresponding principal curvature spaces and multiplicities are the same as in (2).

(4) *For $m = 3$, M has 3-distinct principal curvatures given by*

$$\alpha = 0, \quad \beta = \gamma = 0, \quad \lambda = \frac{h + \sqrt{h^2 + 12}}{2}, \quad \text{and} \quad \mu = \frac{h - \sqrt{h^2 + 12}}{2},$$

where h denotes the trace of S.

Motivated by these works, in this paper we want to give a classification of semi-symmetric structure Jacobi operator for Hopf real hypersurfaces in the complex quadric Q^m, $m \geq 3$, and prove the following theorem.

Main Theorem. *There does not exist any Hopf real hypersurface M with semi-symmetric structure Jacobi operator in the complex quadric Q^m, $m \geq 3$.*

By the results given in [2] and [3], respectively, we know that if a tensor field is symmetric (parallel) or recurrent, it naturally satisfies semi-symmetry. Hence, we obtain the following corollary:

Corollary ([24])**.** *There does not exist any connected Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with parallel or recurrent structure Jacobi operator.*

This paper is composed as follows: In Section 1, we give differential geometric structures of the complex quadric Q^m and some fundamental identities. In Section 2, we recall important lemmas for real hypersurfaces of the complex quadric Q^m. Finally, in Section 3, we prove a key lemma which states that the unit normal vector field N on M is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal under our assumption. In both cases, we have a contradiction which gives a complete proof of our theorem.

1. The complex quadric

$\mathbb{C}P^{m+1}$ stands for the $(m+1)$-dimensional complex projective space of constant holomorphic sectional curvature 4 with respect to the Fubini-Study metric. A point $[z]$ in $\mathbb{C}P^{m+1}$ is the complex span of z, i.e., $[z]=\{\lambda z \mid \lambda \in \mathbb{C}\}$, where z is a nonzero vector of $\mathbb{C}^{m+2}$. For each $[z]$ in $\mathbb{C}P^{m+1}$ the tangent space $T_{[z]}\mathbb{C}P^{m+1}$ is identified with the orthogonal complement $\mathbb{C}^{m+2} \ominus [z]$ of $[z]$ in $\mathbb{C}^{m+2}$.

The m-dimensional complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ is a complex hypersurface defined by the quadratic equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $(z_1, \ldots, z_{m+2}) \in \mathbb{C}P^{m+1}$, which is isometric to the real Grassmannian of oriented two-planes of $\mathbb{R}^{m+2}$ and is a compact Hermitian symmetric space of rank two.

At each $[z]$ in Q^m the tangent space $T_{[z]}Q^m$ can be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus ([z] \oplus [\rho])$ of $[z] \oplus [\rho]$ in $\mathbb{C}^{m+2}$, where ρ is a unit normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$. The shape operator A_ρ of Q^m with respect to the unit normal vector ρ is given by

$$A_\rho w = \bar{w}$$

for all $w \in T_{[z]}Q^m$. The shape operator $A = A_\rho$ restricted to $T_{[z]}Q^m$ is a complex conjugation and acts as an involution, $A^2 = I$. If $V(A_\rho)$ is the $(+1)$-eigenspace, we have the following decomposition for A_ρ

$$T_{[z]}Q^m = V(A_\rho) \oplus JV(A_\rho).$$

The Gauss equation for the complex hypersurface $Q^m \subset CP^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be expressed in terms of the Riemannian metric g, the complex structure J and a generic real structure A:

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ + g(AY,Z)AX \\ & - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY\end{aligned}$$

for any $X, Y, Z \in T_z Q^m$, $z \in Q^m$.

A nonzero tangent vector W at a point of Q^m is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for Q^m, $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic vectors.

- If there exists a conjugation A such that $W \in V(A) := \mathrm{Eig}(A, 1)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
- If there exist a real structure A and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular and is called $\mathfrak{A}$-isotropic.

For every unit vector field W tangent to Q^m, there is a complex conjugation A and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. Singular vectors correspond to the values $t = 0$ and $t = \pi/4$.

2. Preliminaries

In this section, we recall some important lemmas for real hypersurfaces of the complex quadric Q^m, which have been proven in [7,13–15,17,20–23,25,26].

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure on M and by ∇ the induced Riemannian connection on M. Note that $\xi = -JN$, where N is a unit normal vector field of M. The vector field ξ is known as the Reeb vector field of M and η is the 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \bigoplus R\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and we have $\phi\xi = 0$. At each point $z \in M$ we define

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathcal{U}_z\},$$

which is the maximal $\mathcal{U}_z$-invariant subspace of T_zM. If the integral curves of ξ are geodesics in M, the hypersurface M is called a Hopf hypersurface. The integral curves of ξ are geodesics in M if and only if ξ is a principal curvature vector of M everywhere. If we assume that M is a Hopf hypersurface, then we have $S\xi = \alpha\xi$, where S denotes the shape operator of real hypersurfaces M with Reeb curvature $\alpha = g(S\xi, \xi)$.

By the Kähler structure J of the complex quadric Q^m, one can write:

$$JX = \phi X + \eta(X)N$$

for any tangent vector field X on M, where $\phi X = \tan(JX)$.

Then it naturally satisfies the following relations:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0.$$

The Gauss and Weingarten formulas for M are given as $\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N$ and $\bar{\nabla}_X N = -S(X)$ for any $X, Y \in T_zM$ and $N \in T_z^{\perp}M$, $z \in M$.

The curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m is given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ & - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z + g(AY,Z)AX \end{aligned}$$

$$
\begin{aligned}
&- g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY \\
&+ g(SY,Z)SX - g(SX,Z)SY
\end{aligned}
$$

for any $X, Y, Z \in TM$.

Since Q^m has a real structure A, we decompose AX into its tangential and normal components for a fixed $A \in \mathfrak{A}_{[z]} := \{A_{\lambda\rho} \mid \lambda \in S^1 \subset \mathbb{C}\}$ and $X \in T_{[z]}M$, $[z](=: x) \in Q^m$: $AX = BX + \rho(X)N$, where BX is the tangential component of AX and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = g(JX, A\xi).$$

By taking the covariant derivative of A, we have $(\bar{\nabla}_X A)Y = q(X)JAY$ for any $X, Y \in TM$, where $q(X)$ is any one-form [19].

Lemma 2.1 ([20]). *For each $x \in M$, we have*:

(i) *If N_x is $\mathfrak{A}$-principal, then $\mathcal{Q}_x = \mathcal{C}_x$.*

(ii) *If N_x is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that $N_x = \cos(t)Z_1 + \sin(t)JZ_2$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_x = \mathcal{C}_x \ominus \mathbb{C}(JX + Y)$.*

Moreover, at each point $x \in M$, we can choose $A \in \mathfrak{A}_x$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see [18, Proposition 3]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$
\begin{cases}
\xi = \sin(t)Z_2 - \cos(t)JZ_1, \\
AN = \cos(t)Z_1 - \sin(t)JZ_2, \\
A\xi = \sin(t)Z_2 + \cos(t)JZ_1.
\end{cases}
$$

This implies $g(\xi, AN) = 0$ and $g(A\xi, \xi) = -g(AN, N) = -\cos(2t)$ on M. We now assume that M is a Hopf real hypersurface. Then the shape operator S of M in Q^m satisfies $S\xi = \alpha\xi$ with the Reeb function $\alpha = g(S\xi, \xi)$ on M.

By virtue of the Codazzi equation, we obtain the following lemma.

Lemma 2.2 ([23]). *Let M be a Hopf real hypersurface in Q^m, $m \ge 3$. Then we obtain*

$$d\alpha(X) = d\alpha(\xi)\eta(X) + 2g(A\xi, \xi)g(X, AN)$$

and

$$
\begin{aligned}
&2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + g(X, AN)g(Y, A\xi) \\
&- g(Y, AN)g(X, A\xi) - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\
&- 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X) = 0
\end{aligned}
$$

for any tangent vector fields X and Y on M.

By virtue of the Codazzi equation, we get the following two lemmas.

Lemma 2.3 ([20]). *Let M be a Hopf real hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then the Reeb function α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

Lemma 2.4 ([20]). *Let M be a Hopf real hypersurface in Q^m, $m \geq 3$, such that the normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then the Reeb function α is constant.*

Hereafter, unless otherwise stated, X, Y and Z denote any tangent vector fields on M.

3. Semi-symmetric structure Jacobi operator

Suppose that the structure Jacobi operator of M is semi-symmetric, that is, M satisfies the condition $(R(X,Y)R_\xi)Z = 0$. We see that the given condition is equivalent to

$$R(X,Y)(R_\xi Z) = R_\xi(R(X,Y)Z). \tag{$*$}$$

The structure Jacobi operator R_ξ is defined by $R_\xi(X) = R(X,\xi)\xi$, where R denotes the Riemannian curvature tensor on M. Then from the Gauss equation, it can be written as

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi + \beta(AX)^T - g(AX,\xi)A\xi - g(AX,N)(AN)^T \\ &\quad + \alpha SX - g(SX,\xi)S\xi, \end{aligned} \tag{3.1}$$

where we have put $\alpha = g(S\xi,\xi)$, $\beta = g(A\xi,\xi)$ and $(AY)^T$ (resp., $(AN)^T$) denotes the tangential part of AY (resp., AN).

Since the structure Jacobi operator R_ξ comes from the Riemannian curvature tensor R of M in Q^m, we have the following theorem [3]:

Theorem 3.1. *Let M^n be an n-dimensional almost contact metric manifold. Then the structure Jacobi operator R_ξ is semi-symmetric if and only if the structure Jacobi operator vanishes, that is, $R_\xi = 0$.*

Proof. Suppose the structure Jacobi operator is semi-symmetric. Then

$$R(X,Y)R_\xi Z - R_\xi\big(R(X,Y)Z\big) = \big(R(X,Y)\cdot R_\xi\big)Z = 0.$$

In particular, for $Y = Z = \xi$, we obtain $R_\xi^2 X = 0$. Since R_ξ is symmetric, there exists a basis $\{e_j\}$ for $j \in \{1,\dots,n\}$ such that $R_\xi(e_j) = \lambda_j e_j$, where λ_j denotes the corresponding eigenvalue associated to e_j. This gives us

$$R_\xi^2(e_j) = {\lambda_j}^2 e_j = 0.$$

This means that $\lambda_j = 0$ for all $j \in \{1,\dots,n\}$. Then we have that the structure Jacobi operator vanishes. The converse holds trivially. □

So the condition of semi-symmetric structure Jacobi operator is equivalent to the vanishing of the structure Jacobi operator. From now on, we may consider vanishing structure Jacobi operator instead of structure Jacobi operator for real hypersurfaces in the complex quadric Q^m. Since $R_\xi = 0$, it yields that the structure Jacobi operator is of Codazzi type, from the result of [26].

Lemma 3.2. *Let M be a real hypersurface in the complex quadric Q^m, $m \geq 3$, with vanishing structure Jacobi operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

By virtue of Lemma 3.2, we know that the normal vector field N is a singular tangent vector field. We will introduce the following proposition, and prove a key lemma which plays an important role in the proof of our Main Theorem in the introduction.

On the other hand, from [20], a tube M of radius r, $0 < r < \frac{\pi}{2}$, around the totally geodesic $\mathrm{C}P^l$ in Q^m, $m = 2l$ is denoted by $\mathcal{T}_A$. The model space $\mathcal{T}_A$ has four distinct constant principal curvatures as follows. From now on, we may put $m = 2l$.

TABLE 1.

Principal curvature	Eigenspace	Multiplicity
$\alpha = 2\cot(2r)$	$T_\alpha = \mathcal{F} = \mathcal{R}\xi$	1
$\gamma = 0$	$T_\gamma = \mathcal{C} \ominus \mathcal{Q}$	2
$\delta = -\tan(r)$	$T_\delta = T\mathbb{C}P^k \ominus (\mathcal{C} \ominus \mathcal{Q})$	$l-2$
$\sigma = \cot(r)$	$T_\sigma = \nu\mathbb{C}P^k \ominus \mathbb{C}\nu M$	$l-2$

From the table and Lemmas 2.3 and 2.4 in Section 2, it follows that:

Proposition A. *Let M be the tube of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^l$ in Q^m $(m = 2l)$, $l \geq 2$. Then the following statements hold*:

(i) *M is a Hopf hypersurface.*

(ii) *The tangent bundle TM and the normal bundle νM of M consist of $\mathfrak{A}$-isotropic singular tangent vectors of Q^m.*

(iii) *M has four distinct constant principal curvatures. Their values and corresponding principal curvature spaces and multiplicities are given in Table 1. The real structure A determined by the $\mathfrak{A}$-isotropic unit normal vector at $[z]$ maps $T_{[z]}\mathbb{C}P^k \ominus \big(\mathcal{C}_{[z]} \ominus \mathcal{Q}_{[z]}\big)$ onto $\nu_{[z]}\mathbb{C}P^k \ominus \mathbb{C}\nu_{[z]}M$, and vice versa.*

(iv) *The shape operator S of M and the structure tensor field ϕ of M commute with each other, that is, $S\phi = \phi S$.*

(v) *The Reeb flow on M is an isometric flow.*

By virtue of Proposition A, we can prove the following lemma.

Lemma 3.3. *There does not exist any real hypersurface in the complex quadric Q^m, $m \geq 3$, with vanishing structure Jacobi operator and $\mathfrak{A}$-isotropic normal vector field.*

Proof. Assume that $R_\xi = 0$ and the unit normal vector field N is $\mathfrak{A}$-isotropic for a Hopf real hypersurface M in the complex quadric Q^m. Then the normal vector field N of M can be written as

$$N = \frac{1}{\sqrt{2}}\,(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes the $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}\,(Z_1 - JZ_2)\,, \;\; AJN = -\frac{1}{\sqrt{2}}\,(JZ_1 + Z_2)\,, \;\; \text{and} \;\; JN = \frac{1}{\sqrt{2}}\,(JZ_1 - Z_2)\,.$$

Then it gives

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0, \quad \text{and} \quad g(AN, N) = 0$$

which means that these vector fields AN and $A\xi$ are tangent to M. By virtue of these formulas for $\mathfrak{A}$-isotropic unit vector field and as $g(JAX, \xi) = -g(AX, J\xi) = -g(AX, N)$, the structure Jacobi operator R_ξ can be rearranged as follows:

$$R_\xi X = X - \eta(X)\xi - g(AX, \xi)A\xi - g(X, AN)AN + \alpha SX - \alpha^2\eta(X)\xi.$$

Case I. α does not vanish, that is, $\alpha \neq 0$.

Then, by using $\phi A\xi = -AN$ and $\phi AN = A\xi$, we see that

$$\begin{aligned} 0 = \phi R_\xi X - R_\xi \phi X &= \alpha(\phi S - S\phi)X - g(AX, \xi)\phi A\xi + g(A\phi X, \xi)A\xi \\ &\quad - g(AX, N)\phi AN + g(A\phi X, N)AN \\ &= \alpha(\phi S - S\phi)X, \end{aligned}$$

where we have used $\phi A\xi = -AN$ and $A\xi = \phi AN$.

This gives $(\phi S - S\phi)X = 0$ for any vector field X on M.

Case II. α vanishes, that is, $\alpha = 0$.

$$0 = R_\xi X = X - \eta(X)\xi - g(AX, \xi)A\xi - g(X, AN)AN. \tag{3.2}$$

Substituting X by $S\phi X$ into the equation (3.2), then we have

$$\begin{aligned} 0 = R_\xi S\phi X &= S\phi X - \eta(S\phi X)\xi - g(AS\phi X, \xi)A\xi - g(S\phi X, AN)AN \\ &= S\phi X, \end{aligned} \tag{3.3}$$

where we have used $SAN = 0$ and $SA\xi = 0$ (see [27, Lemma 6.1]).

Using symmetry of S and skew-symmetry of ϕ in (3.3), we have

$$\phi SX = 0. \tag{3.4}$$

Then by virtue of (3.3) and (3.4), we have $S\phi X - \phi SX = 0$ for any vector field X on M. That is, the shape operator S commutes with the structure tensor field ϕ. Thus M has isometric Reeb flow.

Let us consider the converse problem, whether the structure Jacobi operator R_ξ for a real hypersurface of type $\mathcal{T}_A$ satisfies the condition $(*)$ or does not.

Under the condition of vanishing structure Jacobi operator $R_\xi = 0$, it follows that

$$\begin{aligned} 0 &= R_\xi X \\ &= X - \eta(X)\xi - g(AX,\xi)A\xi - g(X,AN)AN + \alpha SX - \alpha^2\eta(X)\xi. \end{aligned} \tag{3.5}$$

Taking $X = W \in T_\delta$ and using [27, Lemma 6.1], then we have

$$R_\xi W = (1+\alpha\delta)W = \tan^2 rW. \tag{3.6}$$

Since the radius r should be positive, R_ξ cannot vanish. This gives us a contradiction. Accordingly, we get a complete proof of our lemma. □

As a sequel, let us consider that N is $\mathcal{A}$-principal. Lee and Suh [8] proved:

Proposition B ([8])**.** *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then M has an $\mathfrak{A}$-principal singular normal vector field N if and only if M is a contact real hypersurface with constant mean curvature and non-vanishing Reeb function in Q^m.*

In order to give the non-existence property for real hypersurfaces with vanishing structure Jacobi operator and $\mathfrak{A}$-principal N in Q^m, we need the following lemma and proposition.

Lemma A ([8])**.** *Let M be a real hypersurface in the complex quadric Q^m, $m \geq 3$, with $\mathfrak{A}$-principal singular normal vector field N. Then we obtain*:

(i) $AX = BX$,
(ii) $A\phi X = -\phi AX$,
(iii) $A\phi SX = -\phi SX$ *and* $q(X) = 2g(SX,\xi)$,
(iv) $ASX = SX - 2g(SX,\xi)\xi$ *and* $SAX = SX - 2\eta(X)S\xi$

for any $X \in T_zM$, $z \in M$, where $BX = (AX)^T$ denotes the tangential part of the vector field AX on M in Q^m.

On the other hand, from Suh [21], we know that the model space $\mathcal{T}_B$ has three distinct constant principal curvatures as follows.

Proposition C ([21])**.** *Let $\mathcal{T}_B$ be the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the m-dimensional sphere S^m which is embedded in Q^m as a real form of Q^m. Then the following facts hold ($m = 2l$):*

(i) *$\mathcal{T}_B$ is a Hopf hypersurface.*
(ii) *The normal bundle of $\mathcal{T}_B$ consists of $\mathfrak{A}$-principal singular vector fields.*
(iii) *$\mathcal{T}_B$ has three distinct constant principal curvatures.*

principal curvature	*eigenspace*	*multiplicity*
$\sigma = 0$	$T_\sigma = JV(A) \cap \mathcal{C}$	$l-1$
$\delta = \sqrt{2}\tan(\sqrt{2}r)$	$T_\delta = V(A) \cap \mathcal{C}$	$l-1$
$\alpha = -\sqrt{2}\cot(\sqrt{2}r)$	$T_\alpha = \mathbb{R}\xi$	1

(iv) $S\phi + \phi S = \tau\phi$, $\tau = -\frac{2}{\alpha}$ (*$\mathcal{T}_B$ is a contact hypersurface*).

Now, we want to check whether the model space of type $\mathcal{T}_B$ satisfies the vanishing structure Jacobi operator. Thus we will prove the following lemma:

Lemma 3.4. *There does not exist any real hypersurface in the complex quadric Q^m, $m \geq 3$, with vanishing structure Jacobi operator and $\mathfrak{A}$-principal normal vector field.*

Proof. By Proposition B, under the condition of Hopf and $\mathfrak{A}$-principal normal vector field, M becomes a contact real hypersurface. Thus we consider the converse problem, whether the structure Jacobi operator R_ξ of $\mathcal{T}_B$ satisfies the condition of semi-symmetry.

We consider a Hopf real hypersurface M in Q^m with $\mathfrak{A}$-principal unit normal vector field N. Then N satisfies $AN = N$ for a complex conjugation $A \in \mathfrak{A}$. It implies that AY is tangent to M for all $Y \in T_xM$, $x \in M$ (in particular, $A\xi = -AJN = JAN = JN = -\xi \in T_xM$). Then the structure Jacobi operator R_ξ on M is given by

$$R_\xi Y = Y - 2\eta(Y)\xi - AY + \alpha SY - \alpha^2\eta(Y)\xi. \tag{3.7}$$

Using Proposition C and taking $Y = W \in T_\delta$ into (3.8), we have

$$\begin{aligned} R_\xi W &= W - 2\eta(W)\xi - AW + \alpha SW - \alpha^2\eta(W)\xi \\ &= \alpha\delta W \\ &= -2W. \end{aligned} \tag{3.8}$$

Thus the structure Jacobi operator R_ξ does not vanish, this gives a contradiction. So give a complete proof of our lemma. □

Remark 3.5. By virtue of Lemma A(iv), we know that $T_\sigma \subset J(V(A))$ and $T_\delta \subset V(A)$, because $ASY = SY$ for $Y \in T_{\delta \neq 0}$. Moreover, $A\xi = -\xi$ and $AN = N$, accordingly, we have $T_\delta \oplus [N] = V(A)$ and $T_\sigma \oplus [\xi] = JV(A)$.

Summing up all of the facts mentioned above, we complete the proof of our Main Theorem in the introduction.

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Imsoon Jeong
Department of Mathematics Education
Cheongju University
Chungcheongbuk-do 28503, Korea
Email address: `isjeong@cju.ac.kr`

Gyu Jong Kim
Department of Mathematics Education
Woosuk University
Jeollabuk-do, 55338, Korea
Email address: `hb2107@naver.com`

Changhwa Woo
Department of Applied Mathematics
Pukyong National University
Busan 48513, Korea
Email address: `legalgwch@pknu.ac.kr`