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ON THE GREATEST COMMON DIVISOR OF BINOMIAL COEFFICIENTS

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Abstract. Let $n \geqslant 2$ be an integer, we denote the smallest integer b such that $\gcd\left\{\binom{n}{k} : b < k < n-b\right\} > 1$ as $b(n)$. For any prime p, we denote the highest exponent α such that $p^\alpha \mid n$ as $v_p(n)$. In this paper, we partially answer a question asked by Hong in 2016. For a composite number n and a prime number p with $p \mid n$, let $n = a_m p^m + r$, $0 \leqslant r < p^m$, $0 < a_m < p$. Then we have

$$v_p\Big(\gcd\Big\{\binom{n}{k} : b(n) < k < n-b(n),\ (n,k) > 1\Big\}\Big) = \begin{cases} 1, & a_m = 1 \text{ and } r = b(n), \\ 0, & \text{otherwise.} \end{cases}$$

1. Introduction

Let n and k be nonnegative integers. The binomial coefficient $\binom{n}{k}$ is defined by $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ if $0 \leqslant k \leqslant n$, and $\binom{n}{k} := 0$ otherwise. For any finite set $S = \{a_1, a_2, \ldots, a_n\}$ of integers, we denote the greatest common divisor of all the elements of S by $(a_1, a_2, \ldots, a_n)$ or $\gcd S$. Since the problem of the greatest common divisor of binomial coefficients was first studied by Ram [14] in 1909, many mathematicians have made contributions to this topic. Ram proved that

$$\gcd\left\{\binom{n}{k} : 0 < k < n\right\} = \begin{cases} p, & \text{if } n = p^m \text{ is a power of a prime } p, \\ 1, & \text{otherwise.} \end{cases}$$

In 1985, Joris, Oestreicher and Steinig [5] gave an explicit formula for $\gcd\left\{\binom{n}{k} : r \leqslant k \leqslant s\right\}$ for any $0 \leqslant r \leqslant s \leqslant n$, but it is too complicated to be stated here.

Let p be a prime and n be a positive integer. We denote the highest exponent α such that $p^\alpha \mid n$ as $v_p(n)$, and $v_p(n)$ is called the p-adic valuation of n. We denote the sum of the digits of n in the p-adic as $\sigma_p(n)$. Mendelsohn [12] proved that $\gcd\left\{\binom{2n}{2k-1} : 1 \leqslant k \leqslant n\right\} = 2^{1+v_2(n)}$. In 1972, Albree [1] generalized the result of Mendelsohn by showing that if p is a prime, then

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$\gcd\left\{\binom{pn}{k} : 1 \leqslant k \leqslant pn,\ p \nmid k\right\} = p^{1+v_p(n)}$. McTague proved in [10] that

$$v_p\left(\gcd\left\{\binom{2n}{2k} : 0 < k < n\right\}\right) = \begin{cases} 1, & \text{if } 2n = p^i + p^j \text{ for some } 0 \leqslant i \leqslant j, \\ 0, & \text{otherwise,} \end{cases}$$

and showed in [11] that

$$v_p\left(\gcd\left\{\binom{n}{qk} : 0 < k < \frac{n}{q}\right\}\right) = \begin{cases} 1, & \text{if } \sigma_p(n) \leqslant q, \\ 0, & \text{otherwise,} \end{cases}$$

for any integers n with $0 < q < n$ and $p \equiv 1 \pmod q$.

For $n \geqslant 2$, we denote by $b(n)$ the smallest integer b such that $\gcd\left\{\binom{n}{k} : b < k < n - b\right\} > 1$. In 2004, Granville [15] showed that $b(n) = n - p^m$, where p^m is the largest prime power not greater than n, and Soulé [15] proved that $b(n) \leqslant \frac{n}{4}$.

In 2016, Hong [4] proved that

$$\gcd\left\{\binom{mn}{k} : 1 \leqslant k \leqslant mn,\ (m,k) = 1\right\} = m \prod_{\text{prime } p \mid (m,n)} p^{v_p(n)}.$$

In addition, Hong asked the following interesting questions in 2016. And one of the formulas, F_n, was resolved by Xiao, Yuan and Lin [16] in 2022: Let p^m be the largest prime power not greater than n. Then $F_n = p$. In fact, in 2004, Kaplan and Levy [6] already gave an explicit formula of F_n.

Problem (Hong [4]). *Let $n \geqslant 2$ be an integer. Find the explicit formula for*

$$F_n := \gcd\left\{\binom{n}{k} : b(n) < k < n - b(n)\right\},$$

$$G_n := \gcd\left\{\binom{n}{k} : b(n) < k < n - b(n),\ (n,k) = 1\right\}$$

and

$$H_n := \gcd\left\{\binom{n}{k} : b(n) < k < n - b(n),\ (n,k) > 1\right\},$$

respectively.

In 2001, Baker, Harman and Pintz [3] proved that there exists a prime number p in $[n - n^{0.525}, n]$ when n is large enough. Although the number of prime powers in a given interval must be no less than the number of primes, Panaitopol [13] showed us in the same year that the distributions of primes and prime powers are of the same order, i.e., $\pi(n) \sim \pi^*(n)$, where $\pi(n)$ is the number of primes not greater than n, and $\pi^*(n)$ is the number of prime powers not greater than n. Hence $b(n) \ll n^{0.525}$.

The main work of the present paper is to give partial conclusions related to G_n and H_n. We give an explicit formula for $v_p(G_n)$ when $p \nmid n$ and $v_p(H_n)$ when $p \mid n$. The main result of this paper is as follows.

Theorem 1.1. *Let $n \geqslant 2$ and $n \neq 6$ be a positive integer, p be a prime with $p < n$ and $p \nmid n$. Put $n = a_m p^m + r$, $0 < r < p^m$, $0 < a_m < p$. We have*

$$v_p(G_n) = \begin{cases} 1, & a_m = 1 \text{ and } r = b(n), \\ 0, & \text{otherwise}, \end{cases}$$

if one of the following two conditions holds.

(1) $b(n) \leqslant \sqrt{n}$;

(2) *n is large enough.*

Theorem 1.2. *Let n be a composite number, p be a prime with $p \mid n$. Put $n = a_m p^m + r$, $0 \leqslant r < p^m$, $0 < a_m < p$. We have*

$$v_p(H_n) = \begin{cases} 1, & a_m = 1 \text{ and } r = b(n), \\ 0, & \text{otherwise}. \end{cases}$$

2. Preliminaries

In this section, we repeat some relevant lemmas from references without proofs, which are needed in the proof of Theorems 1.1 and 1.2, thus making our exposition self-contained.

Lemma 2.1 (Kummer [9, P116])**.** *For any integers $0 \leqslant k \leqslant n$ and any prime p, $v_p\binom{n}{k}$ is equal to the number of carries when adding k to $n-k$ in base p. Equivalently, $v_p\binom{n}{k}$ is also equal to the number of borrows when subtracting k from n in base p.*

Lemma 2.2 (Soulé [15, Equations (5) and (6)])**.** *For any positive integer $n \geqslant 2$, we have $b(n) \leqslant \frac{n}{4}$ and $b(n) \ll n^{0.525}$.*

Lemma 2.3 (Koblitz [7, Exercise 1.2.14])**.** *Let n and k be integers with $0 \leqslant k \leqslant n$, p be a prime. Then*

$$v_p\binom{n}{k} = \frac{\sigma_p(k) + \sigma_p(n-k) - \sigma_p(n)}{p-1}.$$

Lemma 2.4 (Xiao, Yuan and Lin [16, Lemma 2.3])**.** *Let $n > 2$ and a be positive integers with $a < \frac{n}{2}$, p be a prime and $b(n,p) := n - p^{\lfloor \log_p n \rfloor}$. If $0 \leqslant b(n,p) \leqslant a$, then $\sigma_p(k) + \sigma_p(n-k) \geqslant p + \sigma_p(b(n,p))$ for every positive integers k with $a < k < n-a$.*

Lemma 2.5 (The equivalent forms of the Prime Number Theorem, [2, Theorem 4.4])**.** *Let x be a real number. The first Chebyshev function is given by*

$$\vartheta(x) = \sum_{\substack{p \leqslant x \\ p \text{ is prime}}} \ln p.$$

Then

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$$

The following lemma is simple but necessary.

Lemma 2.6. *Let $n \geqslant 2$ and $n \neq 6$ be a positive integer. Then there exists an integer k such that $b(n) < k < n - b(n)$ and $(n,k) = 1$.*

Proof. For $n = 2$ or 4, we have $b(2) = b(4) = 0$, take $k = 1$, then $b(n) < k < n - b(n)$ and $(n,k) = 1$.

If $n > 2$ and $n \neq 4$, 6, by Lemma 2.2, it suffices to find an integer k such that $\frac{n}{4} < k \leqslant \frac{n}{2}$ and $(n,k) = 1$. We have

- For $n \equiv 1 \pmod 2$, take $k = \frac{n-1}{2}$, then $(n,k) = 1$ and $\frac{n}{4} < \frac{n-1}{2} < \frac{n}{2}$.
- For $n \equiv 2 \pmod 4$ and $n \neq 6$, take $k = \frac{n}{2} - 2$, then $(n,k) = 1$ and $\frac{n}{4} < \frac{n}{2} - 2 < \frac{n}{2}$.
- For $n \equiv 0 \pmod 4$ and $n \neq 4$, take $k = \frac{n}{2} - 1$, then $(n,k) = 1$ and $\frac{n}{4} < \frac{n}{2} - 1 < \frac{n}{2}$.

This completes the proof. □

3. Proof of Theorem 1.1

Although $b(n) \ll n^{0.525}$, Panaitopol [13] showed that we cannot compress the upper bound of $b(n)$ lower because $\pi(n) \sim \pi^*(n)$. We have verified with the help of a computer that $b(n) < \sqrt{n}$ holds for $2 \leqslant n < 10^{11}$. Next we give a proof of Theorem 1.1.

Proof of Theorem 1.1(1). We have $n = a_m p^m + r$, $0 < a_m < p$, $0 < r < p^m$ and $(n,p) = 1$. We divide the proof into three cases.

Case 1. $a_m > 1$. Let $k = p^m$. Then $k < \frac{n}{2}$ and $(n,k) = 1$. Since $n < p^{m+1}$, we have $k > n^{\frac{m}{m+1}} \geqslant n^{1/2} \geqslant b(n)$ and

$$v_p\binom{n}{k} = v_p\binom{a_m p^m + r}{p^m} = 0$$

by Kummer's Theorem. Hence $v_p(G_n) = 0$.

Case 2. $a_m = 1$ and $r > b(n)$. Let $k = r$. Then $k < \frac{n}{2}$ and $(n,k) = 1$. Now

$$v_p\binom{n}{k} = v_p\binom{p^m + r}{r} = 0,$$

and $v_p(G_n) = 0$ again.

Case 3. $a_m = 1$ and $r = b(n)$. Then $n = p^m + b(n)$. By Lemmas 2.3 and 2.4 that

$$\begin{aligned} v_p(G_n) &= \min\left\{v_p\binom{n}{k} : b(n) < k < n - b(n),\ (n,k) = 1\right\} \\ &= \min\left\{\frac{\sigma_p(k) + \sigma_p(n-k) - \sigma_p(n)}{p-1} : b(n) < k < n - b(n),\ (n,k) = 1\right\} \\ &\geqslant \frac{p + \sigma_p(b(n)) - \sigma_p(n)}{p-1} = \frac{p-1}{p-1} = 1. \end{aligned}$$

Therefore, it suffices to find a positive integer k such that $b(n) < k < n - b(n)$, $(n,k) = 1$ and $v_p\binom{n}{k} = 1$. We divide the process into three subcases.

SUBCASE 3.1. $m \geqslant 3$. Since $n < p^{m+1}$, we have $p^{m-1} > n^{\frac{m-1}{m+1}} \geqslant n^{2/4} \geqslant b(n)$. Take $k = p^{m-1}$. Then we have

$$v_p\binom{n}{k} = v_p\binom{p^m \qquad + b(n)}{p^{m-1}} = 1.$$

Therefore $v_p(G_n) = 1$ in this subcase.

SUBCASE 3.2. $m = 2$. Then $n = p^2 + r$. Since $r = b(n) \leqslant \sqrt{n} = \sqrt{p^2 + r}$, we obtain $r \leqslant \frac{1}{2}\bigl(1 + \sqrt{4p^2+1}\bigr) < p + 1$. Notice that $r \neq p$, hence $r < p$. Take $k = p$. Then we have

$$v_p\binom{n}{k} = v_p\binom{p^2 \quad + b(n)}{p} = 1$$

and $v_p(G_n) = 1$.

SUBCASE 3.3. $m = 1$. Then $n = p + r$, $0 < r < p$. By Lemma 2.6, there exists an integer k with $b(n) = r < k < p = n - b(n)$ such that $(n,k) = 1$ except for $n = 6$. Hence

$$v_p\binom{n}{k} = v_p\binom{p + r}{k} = 1$$

and $v_p(G_n) = 1$ again. □

Proof of Theorem 1.1(2). Let $n = a_m p^m + r$, $0 < a_m < p$, $0 < r < p^m$. We divide the proof into four cases.

CASE 1. $a_m > 1$ and $m \geqslant 2$. Let $k = p^m$. Then $(n,k) = 1$ and $k < \frac{n}{2}$. Since $n < p^{m+1}$, we have $k > n^{\frac{m}{m+1}} \geqslant n^{2/3} > n^{0.525} \gg b(n)$ and

$$v_p\binom{n}{k} = v_p\binom{a_m p^m + r}{p^m} = 0.$$

Hence $v_p(G_n) = 0$.

CASE 2. $a_m > 1$ and $m = 1$. Let $n = ap + r$, $1 < a < p$, $0 < r < p$. We divide this cases into two subcases.

SUBCASE 2.1. $a \leqslant p^{0.475} - 1$. Then $p \geqslant [(a+1)p]^{0.525} > n^{0.525} \gg b(n)$. Take $k = p$. Then we have $(n,k) = 1$ and $b(n) < k < \frac{n}{2}$. Now

$$v_p\binom{n}{k} = v_p\binom{ap + r}{p} = 0$$

and $v_p(G_n) = 0$.

SUBCASE 2.2. $a > p^{0.475} - 1$. Since $b(n) \ll n^{0.525} < p^{1.05}$, we have $n - b(n) > (p^{0.475} - 1)p - b(n) \gg p^{1.475} - p^{1.05} - p$. By Lemma 2.5,

$$\prod_{\substack{q \leqslant x \\ q \text{ is prime}}} q \sim \exp(x),$$

we have,

$$\prod_{p^{0.05}<q\leqslant p^{0.475}-p^{0.05}-1} q = \prod_{q\leqslant p^{0.475}-p^{0.05}-1} q \Big/ \prod_{q\leqslant p^{0.05}} q$$
$$\sim \exp\left(p^{0.475} - 2p^{0.05} - 1\right)$$
$$\gg p^2 > n.$$

It follows that there exists a prime q with $(q,n) = 1$ and $b(n) \ll p^{1.05} < qp \leqslant p^{1.475} - p^{1.05} - p \ll n - b(n)$. Take $k = qp$. Then we have

$$v_p\binom{n}{k} = v_p\binom{ap+r}{qp} = 0$$

and $v_p(G_n) = 0$ again.

CASE 3. $a_m = 1$ and $r > b(n)$. Let $k = r$. Then $r < p^m$ and $(n,k) = 1$. Now

$$v_p\binom{n}{k} = v_p\binom{p^m + r}{r} = 0,$$

and $v_p(G_n) = 0$ again.

CASE 4. $a_m = 1$ and $r = b(n)$. Then $n = p^m + b(n)$, $b(n) < p^m$ and $v_p(G_n) \geqslant 1$ by Lemmas 2.3 and 2.4. We divide this cases into two subcases.

SUBCASE 4.1. $m \geqslant 3$. Take $k = p^{m-1}$. Then $n < 2p^m$,

$$k > \left(\frac{n}{2}\right)^{\frac{m-1}{m}} \geqslant \left(\frac{n}{2}\right)^{2/3} \gg n^{0.525} \gg b(n).$$

Since

$$v_p\binom{n}{k} = v_p\binom{p^m + b(n)}{p^{m-1}} = 1$$

by Kummer's Theorem, we have $v_p(G_n) = 1$.

SUBCASE 4.2. $m = 2$. $n = p^2 + b(n)$. Since $b(n) \ll n^{0.525}$, $p = \sqrt{n - b(n)} \sim \sqrt{n}$, we have

$$\prod_{n^{0.025}<q<p} q = \prod_{q\leqslant p-1} q \Big/ \prod_{q\leqslant n^{0.025}} q \sim \exp\left(n^{0.5} - n^{0.025} - 1\right) \gg n.$$

It follows that there exist a prime q with $(q,n) = 1$ and $n^{0.025} < q < p$. Take $k = qp$. Then we have

$$v_p\binom{n}{k} = v_p\binom{p^2 + b(n)}{qp} = 1.$$

SUBCASE 4.3. $m = 1$. Then there exists an integer k with $b(n) = r < k < p = n - b(n)$ such that $(n,k) = 1$ by Lemma 2.6. Hence

$$v_p\binom{n}{k} = v_p\binom{p + r}{k} = 1$$

and $v_p(G_n) = 1$. □

4. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

Proof of Theorem 1.2. Let $n = a_m p^m + r$, $0 < a_m < p$, $0 \leqslant r < p^m$. We first consider the case when $m \geqslant 2$. We divide the proof into three cases.

Case 1. $a_m > 1$. Let $s = \lfloor \frac{a_m}{2} \rfloor \geqslant 1$. Then $2s \leqslant a_m \leqslant 2s+1$, and hence $2sp^m \leqslant n < (2s+2)p^m$. Take $k = sp^m$. Then $\frac{n}{2} \geqslant k = sp^m > \frac{sn}{2s+2} \geqslant \frac{n}{4} \geqslant b(n)$ by Lemma 2.2. Thus

$$v_p\binom{n}{k} = v_p\binom{a_m p^m + r}{sp^m} = 0$$

and $v_p(H_n) = 0$.

Case 2. $a_m = 1$ and $r > b(n)$. Let $k = r$. Then $k < \frac{n}{2}$ and $(n,k) \geqslant p$. Now

$$v_p\binom{n}{k} = v_p\binom{p^m + r}{r} = 0,$$

and $v_p(H_n) = 0$ again.

Case 3. $a_m = 1$ and $r = b(n)$. Then $v_p(H_n) \geqslant 1$ by Lemmas 2.3 and 2.4. Since $n = p^m + b(n) \leqslant p^m + \frac{n}{4}$, we have $\frac{n}{4} \leqslant \frac{p^m}{3}$. Take $k = (p-1)p^{m-1}$. Then $b(n) \leqslant \frac{p^m}{3} < k < p^m = n - b(n)$. Now

$$v_p\binom{n}{k} = v_p\binom{p^m + b(n)}{(p-1)p^{m-1}} = 1$$

and $v_p(H_n) = 1$.

Next, we consider the case when $m = 1$. Let $n = ap + r$, $0 < a < p$, $0 \leqslant r < p$. Since $p \mid r$, we have $r = 0$ and $n = ap$. If $a = 1$, then $n = p$ is a prime, there is no integer k with $b(n) < k < n - b(n)$ and $(n,k) > 1$. If $a > 2$, then $\frac{a}{4} + 1 < \frac{3}{4}a$, thus there exists an integer s such that $\frac{a}{4} < s < \frac{3}{4}a$. If $a = 2$, we can take $s = 1$. Let $k = sp$. Then $b(n) \leqslant \frac{n}{4} < k < \frac{3}{4}n \leqslant n - b(n)$. We have

$$v_p\binom{n}{k} = v_p\binom{ap}{sp} = 0,$$

and $v_p(H_n) = 0$.

This completes the proof. □

5. Examples and remarks

In this section, we will point out the difficulties of $v_p(G_n)$ when $p \mid n$ and $v_p(H_n)$ when $p \nmid n$.

For the prime $p \mid n$, if $b(n) = 0$, since $(n,k) = 1$, then

$$v_p\binom{n}{k} = v_p\left(\frac{n}{k}\binom{n-1}{k-1}\right) = v_p(n) + v_p\binom{n-1}{k-1} \geqslant v_p(n).$$

Notice that $v_p\binom{n}{1} = v_p(n)$, we have $v_p(G_n) = v_p(n)$.

If $b(n)>0$, we have $b(n-1)=b(n)-1$. Because

$$v_p(G_n)=v_p(n)+\min\left\{v_p\binom{n-1}{k-1}:b(n)<k<n-b(n),\ (n,k)=1\right\},$$

let $n'=n-1$, $k'=k-1$, and then

$$\begin{aligned}&\min\left\{v_p\binom{n-1}{k-1}:b(n)<k<n-b(n),\ (n,k)=1\right\}\\ =&\min\left\{v_p\binom{n'}{k'}:b(n')<k'<n'-b(n')-1,\ (n'+1,k'+1)=1\right\}\\ =&\min\left\{v_p\binom{n'}{k'}:b(n')<k'<n'-b(n'),\ (n'+1,k'+1)=1\right\}.\end{aligned}$$

Hence, it is necessary to find the explicit formula for

$$\gcd\left\{\binom{n}{k}:b(n)<k<n-b(n),\ (n+1,k+1)=1\right\}. \tag{1}$$

Although the format of Equation (1) is similar to G_n, it is much more complicated than G_n. And Lemma 2.6 pointing out the coprime integer probably won't help in this case.

Example 5.1. It is easy to check that $v_3(G_{18})=v_3\binom{18}{7}=2$ and $v_2(G_{18})=v_2\binom{18}{5}=3$, where $7=\frac{18}{2}-2$ is coprime to 18 by Lemma 2.6 but $v_2\binom{18}{7}=4$.

That means the minimum value of $v_2\binom{18}{k}$ is obtained at $k=5$, not at $k=7$.

For the prime $p\nmid n$, $v_p(H_n)$ is not easy to determine. The following proposition indicates that $v_2(H_n)$ is determined by the minimum prime factor of n when n is of the form $n=2^a+1$.

Proposition 5.2. *Let a be a positive integer and $n=2^a+1$ be a composite. Then $v_2(H_n)=\lfloor\log_2 p_0\rfloor+1$, where p_0 is the minimum prime factor of n.*

Proof. Let $p_0=2^{\alpha_1}+\cdots+2^{\alpha_s}$, $\alpha_1>\cdots>\alpha_s=0$, $s>1$. For any integer k with $b(n)=1<k<n-1=n-b(n)$ and $(n,k)>1$, let $k=2^e k'$, where $e\geqslant 0$ and k' be an odd. Hence there exists a prime $p\mid n$ such that $p\mid k'$. Suppose that $k'=2^{\beta_1}+\cdots+2^{\beta_r}$, $\beta_1>\cdots>\beta_r=0$, thus $\beta_1=\lfloor\log_2 k'\rfloor\geqslant\lfloor\log_2 p\rfloor\geqslant\lfloor\log_2 p_0\rfloor=\alpha_1$. We divide the proof into two cases.

Case 1. k is even, i.e., $e\geqslant 1$. Let $k_0=2^{a-\alpha_1-1}p_0=2^{a-1}+\cdots+2^{a-1+\alpha_s-\alpha_1}$. Then

$$v_2\binom{n}{k_0}=v_2\binom{2^a\qquad\qquad\qquad\qquad +1}{2^{a-1}+\cdots+2^{a-1+\alpha_s-\alpha_1}}=a-(a-1+\alpha_s-\alpha_1)=\alpha_1+1.$$

Since $e+\beta_1\leqslant a-1$, we have

$$v_2\binom{n}{k}=v_2\binom{2^a\qquad\qquad\qquad +1}{2^{\beta_1+e}+\cdots+2^{\beta_r+e}}=a-e\geqslant\beta_1+1\geqslant\alpha_1+1.$$

Case 2. k is odd, i.e., $e=0$. Now $n-k$ is even and then $v_2\binom{n}{k}=v_2\binom{n}{n-k}\geqslant\alpha_1+1$ by Case 1.

Therefore, $v_2(H_n) = \alpha_1 + 1 = \lfloor \log_2 p_0 \rfloor + 1$. This completes the proof. $\square$

Specially, let $a = 2^t$, i.e., n is a Fermat number. Although we know the prime factors of n are of the form $d \equiv 1 \pmod{2^{t+2}}$ [8, p. 59], the specific form of the minimum prime factor of n has not yet been determined. Hence we just obtain $v_2(H_n) \geqslant t+3$.

In addition, even if the degree of p is 1, we still cannot effectively determine $v_p(H_n)$.

Example 5.3. Let p be a prime and $n = 2p+1$ be a composite. Then $v_p\binom{n}{k} = 1$ for all k satisfying $(n,k) > 1$.

Example 5.4. Let $p > 5$ be a prime, $n = 6p+5$ be a composite. By Kummer's Theorem, we have $v_p\binom{n}{k} = 0$ if and only if $k = ap + r$, $0 \leqslant a \leqslant 6$, $0 \leqslant r \leqslant 5$ with $0 < k < n$. A trivial verification shows that $(n,k) = 1$, 7, 13 or 19 for all $k = ap + r$, $0 \leqslant a \leqslant 6$, $0 \leqslant r \leqslant 5$ with $0 < k < n$. Hence

$$v_p(H_n) = \begin{cases} 1, & \text{if } (n, 7 \times 13 \times 19) = 1, \\ 0, & \text{if } (n, 7 \times 13 \times 19) > 1. \end{cases}$$

In fact, if $(n, 7 \times 13 \times 19) = 1$, then $(n,k) = 1$ for all $k = ap + r$, $0 \leqslant a \leqslant 6$, $0 \leqslant r \leqslant 5$ with $0 < k < n$. Thus $v_p(H_n) = 1$. If one of 7, 13 and 19 divides n, let $k = p+2$, $p+3$ and $p+4$, respectively. Thus we have $(n,k) > 1$ and $v_p(H_n) = 0$.

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