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COPURE PROJECTIVE MODULES OVER FGV-DOMAINS AND GORENSTEIN PRÜFER DOMAINS

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Dedicated to memory of the late Professor Muhammad Zafrullah

Abstract. In this paper, we prove that a domain R is an FGV-domain if every finitely generated torsion-free R-module is strongly copure projective, and a coherent domain is an FGV-domain if and only if every finitely generated torsion-free R-module is strongly copure projective. To do this, we characterize G-Prüfer domains by G-flat modules, and we prove that a domain is G-Prüfer if and only if every submodule of a projective module is G-flat. Also, we study the $D + M$ construction of G-Prüfer domains. It is seen that there exists a non-integrally closed G-Prüfer domain that is neither Noetherian nor divisorial.

1. Introduction

Throughout this paper, R is always a commutative ring with $0 \neq 1$. For an R-module M, the dual module $\mathrm{Hom}_R(M, R)$ of M is denoted by M^*. For a domain R, the quotient field of R is denoted by $qf(R)$.

Let n be a fixed non-negative integer. Recall from [14] that an R-module M is said to be *n-copure projective* if $\mathrm{Ext}^1_R(M, N) = 0$ for any R-module N with $\mathrm{fd}_R(N) \leq n$. In particular, M is called *copure projective* if $n = 0$, and M is called *strongly copure projective* if $\mathrm{Ext}^i_R(M, N) = 0$ for any flat R-module N and any $i \geq 1$. In [35, Corollary 3.16], it is shown that a domain R is G-Dedekind if and only if every ideal of R is strongly copure projective. Comparing with this result, our original motivation of this paper is to characterize G-Prüfer domains by strongly copure projective ideals.

It is well-known that a domain R is a Dedekind domain if and only if every ideal of R is projective. The concept of Dedekind domains has been extended to Gorenstein homological algebra, which are called Gorenstein Dedekind domains in [23]. Recall that a domain R is called *Gorenstein Dedekind* (G-Dedekind) if the Gorenstein global dimension of R is at most one. It is shown in [21,

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Corollary 1.3] that a domain R is G-Dedekind if and only if every ideal of R is G-projective. Hence every Dedekind domain is G-Dedekind. But the converse case is not true because the G-Dedekind domain is not necessarily integrally closed. The first example of a non-integrally closed G-dedekind domain is given in [21, Example 1.12] by Hu and Wang. After that, the *Gorenstein Prüfer* (G-Prüfer) domain is also studied in [30], which is defined to be a coherent domain of weak Gorenstein global dimension at most one. Although the concepts of G-Prüfer domains and G-Dedekind domains originate from the Gorenstein homological algebra, these "low-dimensional" domains can be well characterized by divisorial ideals of multiplicative ideal theory. The notion of divisorial ideal is a classical one, and it was introduced in 1930s. Let A be a fractional ideal of a domain R. Denote $A^{-1} = \{x \in qf(R) \mid xA \subseteq R\}$. Then an ideal I of R is called *divisorial* if $(I^{-1})^{-1} = I$. It is proved in [30, Theorem 4.2] that a domain R is G-Prüfer if and only if R is a coherent FGV-domain, where a domain R is called an *FGV-domain* in [36] if every nonzero finitely generated ideal is divisorial. Thus every G-Dedekind domain is G-Prüfer because a G-Dedekind domain is precisely a Noetherian divisorial domain by [32, Theorem 11.7.7], where a domain R is called *divisorial* in [18] if every nonzero ideal is divisorial. However, a G-Prüfer domain need not be G-Dedekind nor integrally closed. This kind of G-Prüfer domains is constructed in [33, Example 2]. In recent years, the G-Prüfer domains have attracted many research attentions, and some nice properties of them are further obtained in [17, 20, 22, 34]. For example, it follows from [17, Theorem 2.5] that a domain R is G-Prüfer if and only if every finitely generated submodule of a projective R-module is G-projective, and it is shown in [22, Corollary 2.6] that a domain of R is G-Prüfer if and only if every finitely generated ideal of R is G-projective. So the G-Prüfer domain can be viewed as a counterpart of the Prüfer domain in Gorenstein homological algebra.

In Section 1 of this paper, we first study the copure projective modules over an FGV-domain. We prove in Theorem 1 that a domain R is an FGV-domain if every finitely generated torsion-free R-module is strongly copure projective. By Example 3, it follows that the converse case of Theorem 1 is not true. It is natural to ask when the converse case of Theorem 1 holds. To do this, we characterize G-Prüfer domains by G-flat modules, and we prove in Theorem 6 that a domain R is G-Prüfer if and only if every submodule of a projective R-module is G-flat, if and only if every ideal of R is G-flat. By this result, we proved that a coherent domain R is an FGV-domain if every finitely generated torsion-free R-module is strongly copure projective, if and only if every finitely generated ideal of R is strongly copure projective. Thus a coherent domain is G-Prüfer if and only if every finitely generated ideal of R is strongly copure projective.

In Section 2, we study localizations of G-Prüfer domains by localizations of injective modules. It is well-known that the localization of an injective module is not necessarily injective. But for a coherent ring, we have that the localization of an FP-injective module is also FP-injective (Theorem 9), where an R-module M is called *FP-injective* in [31] if $\mathrm{Ext}_R^1(N, M) = 0$ for any finitely presented R-module N. By this result, we get in Corollary 11 that a coherent domain is G-Prüfer if and only if R_P is G-Prüfer for any $P \in \mathrm{Max}(R)$.

In Section 3, we study the classical $D+M$ constructions of G-Prüfer domains. For a classical $D + M$ construction $RDTF$ with D not a field, we prove that $R = D + M$ is G-Prüfer if and only if D is G-Prüfer and $qf(D) = F$. It is seen that there are G-Prüfer domains that are neither G-Dedekind nor integrally closed (Example 16). In particular, we give an example of non-integrally closed G-Prüfer domains (i.e., a coherent FGV-domain) that is neither Noetherian nor divisorial.

We next recall some notations and terminology in the Gorenstein homological algebra. An R-module M is called *Gorenstein projective* (G-projective) in [10] if there exists an exact sequence of projective R-modules

$$\mathbf{P} : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $M \cong \ker(P^0 \to P^1)$ and the functor $\mathrm{Hom}(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective R-module. Dually, an R-module M is called *Gorenstein injective* (G-injective) if there exists an exact sequence of injective R-modules

$$\mathbf{E} : \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

with $M \cong \ker(E^0 \to E^1)$ and the functor $\mathrm{Hom}(E, -)$ leaves $\mathbf{E}$ exact whenever E is an injective R-module. For an R-module M, the Gorenstein injective and projective dimension of M are denoted by $\text{G-id}_R(M)$ and $\text{G-pd}_R(M)$, respectively. It is shown in [5, Theorem 1.1] that for a ring R,

$$\{\text{G-pd}_R(M) \mid M \text{ is an } R\text{-module}\} = \{\text{G-id}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

This common value is called the *Gorenstein global dimension* of R and denoted by G-gl.dim(R). Accordingly, an R-module M is called *Gorenstein flat* (G-flat) in [11] if there exists an exact sequence of flat R-modules

$$\mathbf{F} : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \ker(F^0 \to F^1)$ and the functor $E \otimes_R -$ leaves $\mathbf{F}$ exact whenever E is an injective R-module. The *Gorenstein flat dimension* of an R-module M is defined in terms of Gorenstein flat resolutions, denoted by $G\text{-fd}_R(M)$. As in [5], the *weak Gorenstein global dimension* of a ring R is defined as

$$w.\text{G-gl.dim}(R) = \sup\{G\text{-fd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

We now proceed to state and prove our main results.

2. Copure projective modules over FGV-domains

We start by the following observation for FGV-domains.

Theorem 1. *Let R be a domain. Assume that every finitely generated torsion-free R-module is strongly copure projective. Then*

(1) *every finitely generated torsion-free R-module is reflexive, and*
(2) *R is an FGV-domain.*

Proof. (1) Let M be a finitely generated torsion-free R-module. Then M is a strongly copure projective module, and hence $\mathrm{Ext}_R^1(M,R)=0$. Let

$$0 \longrightarrow A \longrightarrow P \longrightarrow M \longrightarrow 0$$

be an exact sequence of R-modules where P is finitely generated and projective. Applying the functor $\mathrm{Hom}_R(-,R)$ to this sequence, we have the following exact sequence

$$0 \longrightarrow M^* \longrightarrow P^* \longrightarrow A^* \longrightarrow 0.$$

Since P^* is finitely generated, A^* is finitely generated. Hence A^* is a finitely generated torsion-free R-module. By the hypothesis, A^* is strongly coupure projective. And so $\mathrm{Ext}_R^1(A^*,R)=0$. Thus we have the following exact sequence

$$0 \longrightarrow A^{**} \longrightarrow P^{**} \longrightarrow M^{**} \longrightarrow 0.$$

Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & & \big\| {\scriptstyle\cong} & & {\scriptstyle\beta}\big\downarrow & & \\
0 & \longrightarrow & A^{**} & \longrightarrow & P^{**} & \longrightarrow & M^{**} & \longrightarrow & 0
\end{array}$$

By Snake Lemma, we have $\mathrm{cok}\beta=0$. So $0 \longrightarrow \ker(\beta) \longrightarrow M \longrightarrow M^{**} \longrightarrow 0$ is an exact sequence R-modules. Since M is a finitely generated torsion-free module, $\mathrm{rank}(M)=\mathrm{rank}(M^{**})$ by [32, Theorem 7.1.2(1)]. It forces the rank of $\ker(\beta)$ to be zero. Thus $\ker(\beta)$ is a torsion module. Also since $\ker(\beta)$ is a submodule of M, $\ker(\beta)$ is torsion-free. Hence $\ker(\beta)=0$. Thus $M\cong M^{**}$. It follows that M is a reflexive module.

(2) Let I be a finitely generated ideal of R. Then I is reflexive by (1). Hence $I=I_v$, and so R is an FGV-domain. □

Corollary 2. *The following statements are equivalent for a domain R.*

(1) *R is a Prüfer domain.*
(2) *R is integrally closed and every finitely generated torsion-free R-module is strongly copure projective.*
(3) $w.\mathrm{gl.dim}(R)<\infty$ *and every finitely generated torsion-free R-module is strongly copure projective.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) are obvious because every projective module is strongly copure projective.

(2) ⇒ (1) By Theorem 1, R is an FGV-domain. Since R is integrally closed, R is a Prüfer domain by [36, Corollary 8].

(3) ⇒ (1) Since $w.\mathrm{gl.dim}(R)<\infty$, every strongly copure projective is projective by [14, Proposition 3.4]. Thus by (3), every finitely generated torsion-free R-module is projective. So R is a Prüfer domain. □

Now we give an example to show that the converse case of Theorem 1 is not true.

Example 3. Let $\mathbb{Q}$ (resp., $\mathbb{R}$, $\mathbb{C}$) be the field of rational numbers (resp., real numbers, complex numbers). Set $A = \mathbb{C}[X, \mathbb{Q}^+]$, where $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r \geq 0\}$. Then $P = \{f \in A \mid f$ has zero constant term$\}$ is a prime ideal of A. Set $S = A \backslash P$ and $R = \mathbb{R} + P_S$. It follows from [33, Example 11] that R is a non-integrally closed FGV-domain with $w.\mathrm{gl.dim}(R) = 2$. Hence every strongly copure projective R-module is projective by [14, Proposition 3.4]. If every finitely generated torsion-free module over R is strongly copure projective, then every finitely generated torsion-free R-module is projective. It means that R is Prüfer and hence integrally closed. Which is impossible.

It is natural to ask when the converse of Theorem 1 holds. We next characterize G-Prüfer domains by G-flat modules. By this characterization, we prove that a coherent domain R is an FGV-domain if and only if every finitely generated torsion-free R-module is strongly projective.

Let R be a ring. As in [8], $\mathrm{IFD}(R)$ is defined to be $\sup\{\mathrm{fd}_R E \mid E$ is an injective R-module$\}$. In fact, there is an important relationship between $\mathrm{IFD}(R)$ and $w.G\text{-}\mathrm{gl.dim}(R)$.

Lemma 4. *Let R be a ring and let n be a positive integer. Then $w.G\text{-}\mathrm{gl.dim}(R) \leq n$ if and only if $\mathrm{IFD}(R) \leq n$.*

Proof. See [24, Theorem 2.12]. □

Lemma 5. *Let R be domain with $w.G\text{-}\mathrm{gl.dim}(R) \leq 1$. Then R is coherent.*

Proof. Since $w.G\text{-}\mathrm{gl.dim}(R) \leq 1$, $\mathrm{IFD}(R) \leq 1$ by Lemma 4. Let a be a non-zero non-unit of R. Then $\mathrm{IFD}(R/(a)) = 0$ by [20, Lemma 2.3]. Hence $R/(a)$ is an IF-ring. And so $R/(a)$ is a coherent ring by [25, Proposition 2.3]. By [20, Lemma 2.1], it follows that R is a coherent domain. □

It is shown in [20, Corollary 2.4] that a domain is G-Prüfer if and only if every finitely generated ideal is G-flat. However, the proof of (1) ⇒ (2) of [20, Corollary 2.4] depends on the condition that the class of Gorenstein flat modules is projectively resolving. Next we can give a new proof based on Bennis' results in [3]. It is seen that a domain R is G-Prüfer if and only if every submodule of a projective R-module is G-flat, if and only if every submodule of a flat R-module is G-flat.

Let $\mathscr{X}$ be a class of R-modules. As in [3], the class $\mathscr{X}$ is said to be *closed under extension* if for every short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$A, C \in \mathscr{X}$ implies $B \in \mathscr{X}$. The class $\mathscr{X}$ is said to be *projectively resolving* if $\mathscr{X}$ contains all projective R-module, and for every short exact sequence of R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $C \in \mathscr{X}$, $A \in \mathscr{X}$ and $B \in \mathscr{X}$

are equivalent. For a ring R, we denote the class of all G-flat R-modules by $\mathscr{GF}(R)$. It is proved in [3, Theorem 2.3] that $\mathscr{GF}(R)$ is closed under extension if and only if $\mathscr{GF}(R)$ is projectively resolving. Thus by [3, Proposition 2.2], $\mathscr{GF}(R)$ is projectively resolving if R is a coherent ring. By using this result, we now start to study G-flat properties over G-Prüfer domains.

Theorem 6. *The following statements are equivalent for a domain R.*

(1) *R is a G-Prüfer domain.*
(2) $\mathrm{IFD}(R) \leq 1$.
(3) $w.G$-gl.dim$(R) \leq 1$.
(4) *Every submodule of a projective R-module is G-flat.*
(5) *Every finitely generated torsion-free R-module is G-flat.*
(6) *Every ideal of R is G-flat.*
(7) *Every finitely generated ideal of R is G-flat.*
(8) *Every submodule of a flat R-module is G-flat.*
(9) *Every torsion-free R-module is G-flat.*
(10) *Every submodule of a G-flat R-module is G-flat.*

Proof. (1) $\Leftrightarrow$ (3) This follows from Lemma 5.

(2) $\Leftrightarrow$ (3) This follows from Lemma 4.

(3) $\Rightarrow$ (4) Let N be a submodule of a projective R-module P. Then there exists an exact sequence

$$0 \longrightarrow N \longrightarrow P \longrightarrow C \longrightarrow 0,$$

where $C = P/N$. Since $w.G$-gl.dim$(R) \leq 1$, it means that G-fd$(C) \leq 1$. Hence we have the following exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

where A and B are G-flat R-modules. Since P is projective, there is a homomorphism $g : P \to B$ such that the following diagram of exact rows are commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow g & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Thus by Schanuel Lemma, we get the following exact sequence

$$(*) \qquad 0 \longrightarrow N \longrightarrow A \oplus P \longrightarrow B \longrightarrow 0.$$

Since A is G-flat, $A \oplus P$ is G-flat by [19, Proposition 3.2]. Since $w.G$-gl.dim$(R) \leq 1$, R is a coherent domain by Lemma 5. Hence R is GF-closed by [3, Proposition 2.2 (1)]. So the class $\mathscr{GF}(R)$ is projectively resolving by [3, Theorem 2.3]. It is seen in $(*)$ that N is G-flat.

(4) $\Leftrightarrow$ (5) It is clear because every finitely generated torsion-free R-module can be embedded in a free R-module.

(5) $\Rightarrow$ (6) $\Rightarrow$ (7) It is clear.

(7) $\Rightarrow$ (2) Let E be an injective R-module. Let I be a finitely generated ideal of R. Then by (7), I is a G-flat ideal of R. Hence $\mathrm{Tor}_1^R(I,E) = 0$ by [3, Lemma 2.4]. Consider an exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. Then $\mathrm{Tor}_2^R(R/I,E) \cong \mathrm{Tor}_1^R(I,E) = 0$. Hence $\mathrm{fd}_R(E) \le 1$. So $\mathrm{IFD}(R) \le 1$.

(4) $\Rightarrow$ (8) Let N be a submodule of a flat R-module F. Set $C := F/N$. Pick an exact sequence $0 \longrightarrow A \longrightarrow P \longrightarrow C \longrightarrow 0$, where P is projective and $A = \ker(P \to C)$. Then by (4), A is G-flat. Now consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Then by Schanuel Lemma, we get the following exact sequence

$$0 \longrightarrow A \longrightarrow N \oplus P \longrightarrow F \longrightarrow 0.$$

By [3, Lemma 2.5], it follows that $N \oplus P$ is G-flat. (4) $\Leftrightarrow$ (1) gives that R is coherent. Thus by [19, Theorem 3.7], it follows that N is G-flat.

(8) $\Leftrightarrow$ (10) follows from [3, Lemma 2.4].

(8) $\Rightarrow$ (9) Let M be a torsion-free R-module and let K be the quotient field of R. Then $M \to K \otimes_R M$ is monomorphic. Since $K \otimes_R M$ is a linear space over K, $K \otimes_R M$ is isomorphic to a direct sum of some K. Hence $K \otimes_R M$ is a flat R-module. Thus M is G-flat by (8).

(9) $\Rightarrow$ (6) It is obvious. $\square$

Recall from [31] that an R-module M is called *FP-injective* if $\mathrm{Ext}_R^1(N,M) = 0$ for any finitely presented R-module N. Accordingly, the *FP-injective dimension* of M, denoted by $\text{FP-id}_R(M)$, is defined to be the smallest $n \ge 0$ such that $\mathrm{Ext}_R^{n+1}(N,M) = 0$ for all finitely presented R-modules N (if no such n exists, set $\text{FP-id}_R(M) = \infty$). By FP-injective dimension, an *n-FC-ring* in [9] is defined to be a coherent ring with $\text{FP-id}_R(R) \le n$. The 0-FC rings are the so-called *FC-rings*.

Theorem 7. *The following statements are equivalent for a coherent domain R.*

(1) *Every finitely generated torsion-free R-module is strongly copure projective.*
(2) *Every submodule of a projective R-module is strongly copure projective.*
(3) *Every finitely generated ideal of R is strongly copure projective.*
(4) *R is an FGV-domain.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (4) Let I be a finitely generated ideal of R. Then I is a finitely presented ideal because R is a coherent domain. Hence by (3), I is a finitely

presented copure projective ideal of R. So I is copure flat by [14, Proposition 3.7(1)]. Consider the exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

For any injective R-module E, we have $\mathrm{Tor}_2^R(R/I, E) \cong \mathrm{Tor}_1^R(I, E) = 0$. It follows that $\mathrm{fd}_R(E) \leq 1$. Hence $\mathrm{IFD}(R) \leq 1$. And so R is a G-Prüfer domain by Theorem 6. By [30, Theorem 4.2], it follows that R is an FGV-domain.

(4) $\Rightarrow$ (1) Since R is a coherent FGV-domain, R is a G-Prüfer domain by [30, Theorem 4.2]. Let M be a finitely generated torsion-free module. Then M is G-flat by Theorem 6. Since R is a coherent domain, M is a finitely presented module. Thus M is a finitely presented G-flat module. Also since R is a G-Prüfer domain, R is a 1-FC domain by [30, Theorem 4.2]. So R is a finitely presented (strongly) copure flat module by [13, Theorem 2.12]. By [14, Proposition 3.8], it follows that M is (strongly) copure projective. □

3. Localization of injective modules and G-Prüfer domains

In this section, we study the localization of G-Prüfer domains by using Theorem 6. Firstly, we have the following observation for the flat dimensions of injective modules.

Proposition 8. *Let R be a ring and let P be a prime ideal of R. Then $\mathrm{IFD}(R_P) \leq \mathrm{IFD}(R)$.*

Proof. Without loss of generality, we assume that $\mathrm{IFD}(R) = n < \infty$. Let E be an injective R_P-module. Then E as an R-module is also injective. Hence $\mathrm{fd}_R(E) \leq n$. Since R_P is a flat R-module, we have $\mathrm{fd}_{R_P}(E) = \mathrm{fd}_R(E)$ by [32, Corollary 3.8.6]. It follows that $\mathrm{fd}_{R_P}(E) \leq n$. So $\mathrm{IFD}(R_P) \leq n = \mathrm{IFD}(R)$. □

It is well-known that the localization of an injective module is not necessarily injective (See [7, Theorem 25] and [6, Example 1]). But we have the following result for a coherent ring.

Theorem 9. *Let R be a coherent ring. If E is an FP-injective R-module, then E_S is an FP-injective R_S-module for any multiplicatively closed set S of R.*

Proof. Let S be a multiplicatively closed set of R. Then R_S is a coherent ring. Let I_S be a finitely generated ideal of R_S, where I is a finitely generated ideal of R. Consider the exact sequence of R-modules $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. Applying the functor $\mathrm{Hom}_R(-, E)$ to this exact sequence, we get the following exact sequence

$$\mathrm{Hom}_R(R, E) \to \mathrm{Hom}_R(I, E) \to \mathrm{Ext}_R^1(R/I, E) \to 0.$$

Thus we have an exact sequence of R_S-modules

$$\mathrm{Hom}_R(R, E)_S \to \mathrm{Hom}_R(I, E)_S \to \mathrm{Ext}_R^1(R/I, E)_S \to 0.$$

Also since $0 \to I_S \to R_S \to R_S/I_S \to 0$ is an exact sequence of R_S-module, we have the following exact sequence of R_S-modules

$$\mathrm{Hom}_{R_S}(R_S, E_S) \to \mathrm{Hom}_{R_S}(I_S, E_S) \to \mathrm{Ext}^1_{R_S}(R_S/I_S, E_S) \to 0.$$

Consider the following diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(R,E)_S & \longrightarrow & \mathrm{Hom}_R(I,E)_S & \longrightarrow & \mathrm{Ext}^1_R(R/I,E)_S & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta_R} & & \downarrow{\scriptstyle \theta_I} & & \downarrow{\scriptstyle \theta_1} & & \\
\mathrm{Hom}_{R_S}(R_S,E_S) & \longrightarrow & \mathrm{Hom}_{R_S}(I_S,E_S) & \longrightarrow & \mathrm{Ext}^1_{R_S}(R_P/I_S,E_S) & \longrightarrow & 0
\end{array}$$

It is clear that θ_R is an isomorphism. Since R is a coherent ring, I is a finitely presented ideal. Hence θ_I is an isomorphism by [32, Theorem 2.6.16(1)]. By Five Lemma, it follows that θ_1 is also an isomorphism. Since E is FP-injective and R/I is finitely presented, we have $\mathrm{Ext}^1_R(R/I, E) = 0$. So $0 = \mathrm{Ext}^1_R(R/I, E)_S \cong \mathrm{Ext}^1_{R_S}(R_S/I_S, E_S)$. Thus E_S is an FP-injective R_S-module by [31, Lemma 3.1]. □

Theorem 10. *The following statements are equivalent for a coherent ring R.*

(1) $\mathrm{IFD}(R) \leq n$.
(2) $\sup\{\mathrm{IFD}(R_P) \mid P \in \mathrm{Spec}(R)\} \leq n$.
(3) $\sup\{\mathrm{IFD}(R_P) \mid P \in \mathrm{Max}(R)\} \leq n$.

Proof. (1) $\Rightarrow$ (2) This follows from Proposition 8.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Suppose that $\sup\{\mathrm{IFD}(R_P) \mid P \in \mathrm{Max}(R)\} \leq n$. Consider the following exact sequence of R-modules,

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0,$$

where $F_0, F_1, \ldots, F_{n-1}$ are flat. Let P be any maximal ideal of R. Then we have the following exact sequence of R_P-modules:

$$0 \longrightarrow (F_n)_P \longrightarrow (F_{n-1})_P \longrightarrow \cdots \longrightarrow (F_0)_P \longrightarrow E_P \longrightarrow 0,$$

where $(F_0)_P$, $(F_1)_P$,…,$(F_{n-1})_P$ are flat R_P-modules. Since R is coherent, E_P is an FP-injective R_P-module by Theorem 9. Since $\mathrm{IFD}(R_P) \leq n$, $\mathrm{fd}_{R_P}(E_P) \leq n$ by [13, Theorem 3.8]. Hence $(F_n)_P$ is a flat R_P-module. Since P in $\mathrm{Max}(R)$ is arbitrary, we conclude that F_n is a flat R-module. Thus $\mathrm{IFD}(R) \leq n$. □

Corollary 11. *A coherent domain R is G-Prüfer if and only if R_P is G-Prüfer for any $P \in \mathrm{Max}(R)$.*

Proof. By Theorem 6, R is G-Prüfer if and only if $\mathrm{IFD}(R) \leq 1$. By Theorem 6 and Theorem 10, it follows that R is G-Prüfer if and only if R is coherent and R_P is G-Prüfer for any $P \in \mathrm{Max}(R)$. □

Corollary 12. *A domain R is G-Dedekind if and only if R is Noetherian and R_P is G-Dedekind for any $P \in \mathrm{Max}(R)$.*

Proof. By [30, Corollary 4.3], R is a G-Dedekind domain if and only if R is a Noetherian G-Prüfer domain. Thus by Corollary 11, R is G-Dedekind if and only if R is Noetherian and R_P is G-Dedekind for any $P \in \mathrm{Max}(R)$. $\square$

We do not know that the coherent condition in Corollary 11 is superfluous. Hence we have the following question.

Quesition 13. Is a domain R G-Prüfer if R_P is G-Prüfer for each $P \in \mathrm{Max}(R)$?

By Example 14, it is seen that the Noetherian condition in Corollary 12 is essential.

Example 14. Let R be an almost Dedekind domain but not a Dedekind domain ([12, Example 3.4.1]). Then R is not Noetherian and R_P is Dedekind for any $P \in \mathrm{Max}(R)$. Hence R_P is G-Dedekind for any $P \in \mathrm{Max}(R)$. But R is not G-Dedekind by [32, Theorem 11.7.7] because R is not Noetherian. So the Noetherian condition in Corollary 11 is essential.

4. The $D + M$ construction of G-Prüfer domains

Let

$$\begin{array}{ccc} R & \longrightarrow & T \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ D & \longrightarrow & F \end{array}$$

be a pullback of rings, where T is a domain, M is a maximal ideal of T, $F = T/M$, π is the natural projection, and D is a proper subring of F. Then $R = \pi^{-1}(D)$ is a proper subring of T, and the commutative diagram $RDTF$ is called a *Milnor square*. In particular, if $T = V$ is a valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V, then the Milnor square is called *classical $D + M$ construction*.

In [33, Theorem 1(1)], it is shown that for a classical $D + M$ construction with D a field, R is a G-Prüfer domain if and only if $[V/M : R/M] = 2$ and M is a principal ideal of V. In this section, we study the classical $D + M$ construction in the case D is not a field.

Theorem 15. *In the classical "$D + M$" construction with D not a field, R is a G-Prüfer domain if and only if D is a G-Prüfer domain and $qf(D) = F$.*

Proof. By [30, Theorem 4.2], a G-Prüfer domain is precisely a coherent FGV-domain. By [27, Theorem 3.1(2)] and [28, Corollary 3.8(a)], it follows that R is an FGV-domain if and only if D is an FGV-domain with $qf(D) = F$. Thus by [15, Proposition 4.6]([16, Theorem 4.7]), R is a G-Prüfer domain if and only if D is a G-Prüfer domain and $qf(D) = F$. $\square$

By [33, Theorem 1(3)], we can construct abundant non-integrally closed G-Dedekind domains in multiplicative ideal theory. For example, $\mathbb{R} + X\mathbb{C}[[X]]$, $\mathbb{R} + X\mathbb{C}[X]$, $\mathbb{Q} + \mathbb{Q}(\sqrt{2})[[X]]$ and so on, where $\mathbb{Q}$ (resp., $\mathbb{R}$, $\mathbb{C}$) is the field of rational numbers (resp., real numbers, complex numbers). Via these G-Dedekind domains, we can give abundant examples of G-Prüfer domains that are neither G-Dedekind nor integrally closed.

Example 16. Choose any non-integrally closed G-Dedekind domain D as above. Then D is a G-Prüfer domain by [30, Corollary 4.3]. Set $qf(D) = Q$. Then $R = D + XQ[[X]]$ is a G-Prüfer domain by Theorem 15. Since D is not a field, R is a non-Noetherian domain. Hence R is not G-Dedekind by [32, Theorem 11.7.7]. Since D is not integrally closed in Q, R is not integrally closed [32, Theorem 8.6.6]. Thus R is a G-Prüfer domain that is neither G-Dedekind nor integrally closed.

Remark 17. It is easy to check that the Krull dimensions of G-Prüfer domains as in Example 16 are two. So, it is natural to ask whether there is a G-Prüfer domain R with $\dim(R) > 2$. In fact, if we take the valuation group $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (n summands) as in [33, Example 2], then R is a non-integrally closed G-Prüfer domain with $\dim(R) = n$. By [27, Corollary 3.5], this kind of G-Prüfer domain must be a non-Noetherian divisorial domain.

By [32, Theorem 11.7.7], a G-Dedekind domain is precisely a Noetherian divisorial domain. It is natural to ask whether there is a non-integrally closed G-Prüfer domain that is neither Noetherian nor divisorial. At last, we construct this kind of G-Prüfer domain.

Example 18. Choose A a non-integrally closed Noetherian local domain of dimension one whose integral closure $\overline{A}$ is not a finitely generated A-module. (See [29, E. 3.2, p. 206] or [1, Example 5]). Then by [26, Theorem 14.16], there exists an analytically ramified local 1-Gorenstein domain D such that $A \subset D \subset Q$ where $Q = qf(D)$. Hence D is divisiorial by [4, Proposition 1.5]. Since D is analytically ramified, the closure $\overline{D}$ is not finitely generated as D-module by [26, Theorem 10.2]. So D is a non-integrally closed Noetherian local divisorial domain of dimension one with non-finitely generated integral closure $\overline{D}$. Let $R = D + XQ[[X]]$. Then R is not divisiorial by [2, Example 2.11]. Since D is G-Dedekind, D is G-Prüfer. Hence R is G-Prüfer by Theorem 15. Since D is not a field, R is non-Noetherian. Thus R is a non-integrally closed G-Prüfer domain that is neither divisorial nor Noetherian.

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References

[1] D. D. Anderson, S. Xing, and M. Zafrullah, *Almost discrete valuation domains*, Comm. Algebra **49** (2021), no. 1, 173–184. https://doi.org/10.1080/00927872.2020.1797069

[2] S. Bazzoni, *Divisorial domains*, Forum Math. **12** (2000), no. 4, 397–419. https://doi.org/10.1515/form.2000.011

[3] D. Bennis, *Rings over which the class of Gorenstein flat modules is closed under extensions*, Comm. Algebra **37** (2009), no. 3, 855–868. https://doi.org/10.1080/00927870802271862

[4] D. Bennis, *A note on Gorenstein global dimension of pullback rings*, Int. Electron. J. Algebra **8** (2010), 30–44.

[5] D. Bennis and N. Mahdou, *Global Gorenstein dimensions*, Proc. Amer. Math. Soc. **138** (2010), no. 2, 461–465. https://doi.org/10.1090/S0002-9939-09-10099-0

[6] F. Couchot, *Localization of injective modules over valuation rings*, Proc. Amer. Math. Soc. **134** (2006), no. 4, 1013–1017. https://doi.org/10.1090/S0002-9939-05-08350-4

[7] E. C. Dade, *Localization of injective modules*, J. Algebra **69** (1981), no. 2, 416–425. https://doi.org/10.1016/0021-8693(81)90213-1

[8] N. Ding and J. Chen, *The flat dimensions of injective modules*, Manuscripta Math. **78** (1993), no. 2, 165–177. https://doi.org/10.1007/BF02599307

[9] N. Ding and J. Chen, *Coherent rings with finite self-FP-injective dimension*, Comm. Algebra **24** (1996), no. 9, 2963–2980. https://doi.org/10.1080/00927879608825724

[10] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. https://doi.org/10.1007/BF02572634

[11] E. E. Enochs, O. M. G. Jenda, and B. Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), no. 1, 1–9.

[12] M. Fontana, E. Houston, and T. Lucas, *Factoring ideals in integral domains*, Lecture Notes of the Unione Matematica Italiana, 14, Springer, Heidelberg, 2013. https://doi.org/10.1007/978-3-642-31712-5

[13] X. Fu and N. Ding, *On strongly copure flat modules and copure flat dimensions*, Comm. Algebra **38** (2010), no. 12, 4531–4544. https://doi.org/10.1080/00927870903428262

[14] X. Fu, H. Zhu, and N. Ding, *On copure projective modules and copure projective dimensions*, Comm. Algebra **40** (2012), no. 1, 343–359. https://doi.org/10.1080/00927872.2010.531337

[15] S. Gabelli and E. Houston, *Coherentlike conditions in pullbacks*, Michigan Math. J. **44** (1997), no. 1, 99–123. https://doi.org/10.1307/mmj/1029005623

[16] S. Gabelli and E. Houston, *Ideal theory in pullbacks*, in Non-Noetherian commutative ring theory, 199–227, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[17] Z. Gao and F. Wang, *All Gorenstein hereditary rings are coherent*, J. Algebra Appl. **13** (2014), no. 4, 1350140, 5 pp. https://doi.org/10.1142/S0219498813501405

[18] W. Heinzer, *Integral domains in which each non-zero ideal is divisorial*, Mathematika **15** (1968), 164–170. https://doi.org/10.1112/S0025579300002527

[19] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193. https://doi.org/10.1016/j.jpaa.2003.11.007

[20] K. Hu, J. W. Lim, and D. C. Zhou, *Flat dimensions of injective modules over domains*, Bull. Korean Math. Soc. **57** (2020), no. 4, 1075–1081. https://doi.org/10.4134/BKMS.b190757

[21] K. Hu and F. Wang, *Some results on Gorenstein Dedekind domains and their factor rings*, Comm. Algebra **41** (2013), no. 1, 284–293. https://doi.org/10.1080/00927872.2011.629268

[22] K. Hu, F. Wang, and L. Xu, *A note on Gorenstein Prüffer domains*, Bull. Korean Math. Soc. **53** (2016), no. 5, 1447–1455. https://doi.org/10.4134/BKMS.b150760

[23] N. Mahdou and M. Tamekkante, *On (strongly) Gorenstein (semi) hereditary rings*, Arab. J. Sci. Eng. **36** (2011), no. 3, 431–440. https://doi.org/10.1007/s13369-011-0047-7

[24] N. Mahdou and M. Tamekkante, *On (weak) Gorenstein global dimensions*, Acta Math. Univ. Comenian. (N.S.) **82** (2013), no. 2, 285–296.

[25] N. Mahdou, M. Tamekkante, and S. Yassemi, *On (strongly) Gorenstein von Neumann regular rings*, Comm. Algebra **39** (2011), no. 9, 3242–3252. https://doi.org/10.1080/00927872.2010.501773

[26] E. Matlis, 1-*dimensional Cohen-Macaulay rings*, Lecture Notes in Mathematics, Vol. 327, Springer-Verlag, Berlin, 1973.

[27] A. Mimouni, *TW-domains and strong Mori domains*, J. Pure Appl. Algebra **177** (2003), no. 1, 79–93. https://doi.org/10.1016/S0022-4049(02)00171-8

[28] A. Mimouni, *Integral domains in which each ideal is a W-ideal*, Comm. Algebra **33** (2005), no. 5, 1345–1355. https://doi.org/10.1081/AGB-200058369

[29] M. Nagata, *Local Rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers (a division of John Wiley & Sons, Inc.), New York, 1962.

[30] L. Qiao and F. Wang, *A Gorenstein analogue of a result of Bertin*, J. Algebra Appl. **14** (2015), no. 2, 1550019, 13 pp. https://doi.org/10.1142/S021949881550019X

[31] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. (2) **2** (1970), 323–329. https://doi.org/10.1112/jlms/s2-2.2.323

[32] F. Wang and H. Kim, *Foundations of commutative rings and their modules*, Algebra and Applications, 22, Springer, Singapore, 2016. https://doi.org/10.1007/978-981-10-3337-7

[33] S. Xing, *Some examples in Gorenstein multiplicative ideal theory*, Comm. Algebra **50** (2022), no. 8, 3188–3200. https://doi.org/10.1080/00927872.2022.2028163

[34] S. Xing and H. Kim, *Generalized coherent domains of self-weak injective dimension at most one*, Comm. Algebra **47** (2019), no. 5, 1908–1916. https://doi.org/10.1080/00927872.2018.1524006

[35] T. Xiong, *Rings of copure projective dimension one*, J. Korean Math. Soc. **54** (2017), no. 2, 427–440. https://doi.org/10.4134/JKMS.j160014

[36] M. Zafrullah, *The v-operation and intersections of quotient rings of integral domains*, Comm. Algebra **13** (1985), no. 8, 1699–1712. https://doi.org/10.1080/00927878508823247

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