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HYPERSURFACES WITH PRESCRIBED MEAN CURVATURE IN MEASURE METRIC SPACE

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Abstract. For any given function f, we focus on the so-called prescribed mean curvature problem for the measure $e^{-f(|x|^2)}dx$ provided that $e^{-f(|x|^2)} \in L^1(\mathbb{R}^{n+1})$. More precisely, we prove that there exists a smooth hypersurface M whose metric is $ds^2 = d\rho^2 + \rho^2 d\xi^2$ and whose mean curvature function is

$$\frac{1}{n}\frac{u^p}{\rho^\beta}e^{f(\rho^2)}\psi(\xi)$$

for any given real constants p, β and functions f and ψ where u and ρ are the support function and radial function of M, respectively. Equivalently, we get the existence of a smooth solution to the following quasilinear equation on the unit sphere $\mathbb{S}^n$,

$$\sum_{i,j}(\delta_{ij} - \frac{\rho_i\rho_j}{\rho^2 + |\nabla\rho|^2})(-\rho_{ji} + \frac{2}{\rho}\rho_j\rho_i + \rho\delta_{ji}) = \psi\frac{\rho^{2p+2-n-\beta}e^{f(\rho^2)}}{(\rho^2 + |\nabla\rho|^2)^{\frac{p}{2}}}$$

under some conditions. Our proof is based on the powerful method of continuity. In particular, if we take $f(t) = \frac{t}{2}$, this may be prescribed mean curvature problem in Gauss measure space and it can be seen as an embedded result in Gauss measure space which will be needed in our forthcoming papers on the differential geometric analysis in Gauss measure space, such as Gauss-Bonnet-Chern theorem and its application on positive mass theorem and the Steiner-Weyl type formula, the Plateau problem and so on.

1. Introduction

In the theory of classical differential geometry, the existence of hypersurfaces with certain curvature function is a classical topic and such problem can be described as follows: given a function f defined on the unit sphere $\mathbb{S}^n$, does there

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exist a hypersurface $M \subseteq \mathbb{R}^{n+1}$ such that curvature function of M satisfies the following equation

$$\sigma_k = f(\xi), \tag{1}$$

where σ_k is the normalized elementary symmetric function of order k of the principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_n$ of M, that is,

$$\sigma_k = \frac{1}{C_n^k} \sum_{i_1<i_2<i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} \tag{2}$$

for any fixed $k \in \{1, 2, \ldots, n\}$. In particular, if $k = 1$, σ_1 is the mean curvature. If $k = n$, σ_n is the Gaussian curvature and equation (1) is associated to the classical Minkowski problem which was posed and solved by Minkowski [32, 33] provided $\frac{1}{f}$ is the density of the delta measure or a continuous function. The works of Minkowski were extended by Aleksandrov and Jessen and Fenchel independently provided $\frac{1}{f}d\xi$ is a Borel measure defined on the unit sphere $\mathbb{S}^n$, see Schneider [42]. Based on the theory of Monge-Ampère equation, Lewy [30], Nirenberg [35], Cheng and Yau [10], Pogorelov [39] and Caffarelli [6, 7] analyzed the existence of convex solutions to equation (1) in Hölder or Sobolev Space. For the intermediate case, it follows from classical Steiner formula that $\sigma_k = \frac{dW_{n-k}}{d\sigma}$ where dW_{n-k} and $d\sigma$ are the infinitesimal of $(n-k)$-th quermassintegrals measure and surface measure of M, respectively, for any fixed $1 \leq k \leq n$, see Santalo [40] or Burago and Zalgaller [5]. Therefore, σ_k is called the k-th mean curvature and the equation (1) is associated to the so-called Christoffel-Minkowski problem which was solved by Guan and Ma [24].

By using the co-area formula, Federer [17] introduced curvature measures under the hypothesis of positive reach. In convex frame, the curvature measure of a convex body was deduced by Schneider [42]. Based on the potential theory, Wang et al. [13, 14] introduced some notions of k-th mean curvature measure. Some interesting σ_k curvatures were also introduced by Case et al. [8, 9]. In polar coordinate system, the second fundamental form $b = (b_{ij})_{n\times n}$ and the metric $g = (g_{ij})_{n\times n}$ of the hypersurface can be written in the term of the radial function of the hypersurface. Since the principal curvatures $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of $g^{-1}b$, we can see that k-th mean curvature can be written in the term of the radial function of the hypersurface. In this direction, Guan, Li and Li [23] provided the following interesting formula of k-th curvature measure:

$$C_{n-k}^M(E) = \int_E \sigma_k \sqrt{\det g}\, d\xi \tag{3}$$

for any Borel $E \subseteq \mathbb{S}^n$ and gave the existence of hypersurface which was prescribed k-th mean curvature measure. Some associated and interesting results can also be seen in Guan and Spruck [26], Guan and Guan [22] and Guan, Ren and Wang [25].

As an interesting and important measure in probability theory, several aspects of the theory of differential geometric analysis associated to the Gauss measure have been developed, see Bogachev [2]. In particular, the Brunn-Minkowski inequality for the Gauss measure has been discussed by several authors, see Borell [3], Brascamp and Lieb [4], and Gardner and Zvavitch [18] and so on. As an interesting aspect of convex geometry, Huang, Xi and Zhao [28] posed and solved the Minkowski problem in Gauss measure space. Later, the L_p Minkowski problem for Gauss measure has been established by Liu [31]. Following classical construction of surface area measure introduced by Minkowski, the notions of surface area measure in Gauss measure space introduced by Huang, Xi and Zhao [28] is based on the following beautiful variational formula:

$$\lim_{t\to 0}\frac{\gamma_n(K+tL)-\gamma_n(K)}{t}=\int_{\mathbb{S}^n} h_L dG_K(\xi), \tag{4}$$

where K and L are two convex bodies containing the origin O at their interiors, h_L is the support function of L, $\gamma_n(K)$ is the Gauss measure of K (see Bogachev [2]) and G_K is called the surface area measure of the convex body K, that is,

$$\gamma_n(K)=\int_K \frac{1}{(2\pi)^{\frac{n+1}{2}}}e^{-\frac{|x|^2}{2}}dx, \tag{5}$$

and

$$G_K(E)=\frac{1}{(2\pi)^{\frac{n+1}{2}}}\int_{\nu_K^{-1}(E)} e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1} \tag{6}$$

for any Borel set $E\subseteq \mathbb{S}^n$, where ν_k^{-1} is the generalized inverse of Gauss mapping of K. In particular, if K is smooth and strictly convex, we have

$$G_K(E)=\frac{1}{(2\pi)^{\frac{n+1}{2}}}\int_E \frac{e^{-\frac{\rho_K^2}{2}}}{\sigma_n}d\xi \tag{7}$$

for any Borel set $E\subseteq \mathbb{S}^n$, where ρ_K is the radial function of K. Moreover, motivated by some works on Kähler geometry and by using the powerful optimal transport theory, Cordero-Erausquina and Klartag [11] constructed a moment measure whose density is $e^{-f(|x|^2)}$ provided f is convex and $e^{-f(|x|^2)}\in L^1(\mathbb{R}^{n+1})$, see also Santambrogio [41]. The main results of [3, 4, 11, 18, 28] can be seen as interesting generalizations of classical results in integral geometry and more results have been discussed by Gel'fand et al. [19] or Santalo [40].

It is worth mentioning that the works of Cordero-Erausquina and Klartag [11] and Huang, Xi and Zhao [28] focus on the Gauss curvature of hypersurface in Gauss measure space. Among the Gauss curvature, mean curvature is also a class of important curvature in the differential geometry. In particular, a hypersurface whose mean curvature vanishes is called the minimal hypersurface. The minimal hypersurface problem is associated to the Plateau problem which is a great motivation of modern variational methods and geometric measure theory, see Almgren [1], Courant [12], Pitts [38], Giusti [21], Osserman [36], Struwe [44] or Dierkes, Hildebrandt and Sauvigny [15] and so on. More and more variants of classical Plateau problem have been posed and solved, see for example Morrey [34], Harvey and Lawson [27], Jost [29], Trudinger and Wang [46], Petrache and Rivière [37] and so on. Moreover, by using Gauss-Bonnet theorem and the theory of minimal hypersurface, Schoen and Yau [43] established the positive mass theorem in general relativity. More comments on the theory of minimal surfaces can be referred to Osserman [36].

In the present paper, motivated by the works of Huang, Xi and Zhao [28] and Cordero-Erausquina and Klartag [11], for any f satisfying $e^{-f(|x|^2)} \in L^1(\mathbb{R}^{n+1})$, we may consider the measure

$$F(E) = \int_E e^{-f(|x|^2)} dx \tag{8}$$

for any Borel set $E \subseteq \mathbb{S}^n$ and focus on prescribing mean curvature problem for the measure $e^{-f(|x|^2)}dx$ which may be an attempt on some differential geometric analysis in measure metric space, such as Gauss-Bonnet-Chern theorem and its application on positive mass theorem, the Steiner-Weyl type formula, the Plateau problem and so on. Motivated by (3) and (7), we may call the function

$$e^{-f(|x|^2)}\sigma_1\sqrt{\det g} \tag{9}$$

the mean curvature function for the measure $e^{-f(|\xi|^2)}dx$. In the present paper, we focus on the existence of hypersurface provided the mean curvature function is $\frac{u^p}{n\rho^\beta}\psi(\xi)$ for the measure $e^{-f(|x|^2)}dx$, that is

$$e^{-f(\rho^2)}\sigma_1\sqrt{\det g} = \frac{u^p}{n\rho^\beta}\psi(\xi), \tag{10}$$

where u and ρ are the support function and radial function of the hypersurface M, respectively, that is,

$$u(y) = \max\{x \cdot y : \forall x \in M\} \ \forall y \in \mathbb{R}^{n+1} \tag{11}$$

and

$$\rho(y) = \max\{\lambda \geq 0 : \lambda y \in M\} \ \forall y \in \mathbb{R}^{n+1}. \tag{12}$$

In polar coordinate system, if the metric of hypersurface is $ds^2 = d\rho^2 + \rho^2 d\xi^2$, we can see that

$$\begin{aligned} \sigma_1 &= \frac{1}{n}\sum_{i,j} g^{ij}b_{ji} \\ &= \frac{1}{n}\frac{1}{\rho\sqrt{\rho^2+|\nabla\rho|^2}}(\delta_{ij} - \frac{\rho_i\rho_j}{\rho^2+|\nabla\rho|^2})(-\rho_{ij} + \frac{2}{\rho}\rho_i\rho_j + \rho\delta_{ij}), \end{aligned} \tag{13}$$

and

$$\sqrt{\det g} = \rho^{n-1}\sqrt{\rho^2+|\nabla\rho|^2}, \ u = \frac{\rho^2}{\sqrt{\rho^2+|\nabla\rho|^2}}, \tag{14}$$

(see Lemma 4.1 in Section 4. Appendix). Therefore, we focus on the existence of smooth solutions to the following equation on the unit sphere $\mathbb{S}^n$:

$$\sum_{i,j}(\delta_{ij} - \frac{\rho_i\rho_j}{\rho^2 + |\nabla\rho|^2})(-\rho_{ji} + \frac{2}{\rho}\rho_j\rho_i + \rho\delta_{ji}) = \psi\frac{\rho^{2p+2-n-\beta}e^{f(\rho^2)}}{(\rho^2 + |\nabla\rho|^2)^{\frac{p}{2}}}. \tag{15}$$

Before stating the main result of the present paper, we assume the following conditions hold.

(A.1) $f \in C^1(\mathbb{R})$ and $0 < \psi \in C^1(\mathbb{S}^n)$ satisfying

$$|f|_{C^1(\mathbb{R})} + |\psi|_{C^1(\mathbb{S}^n)} < \infty, \tag{16}$$

$$\lim_{t\to\infty} e^{f(t^2)}t^{p+1-n-\beta} < \frac{n-1}{\max_{\xi\in\mathbb{S}^n}\psi(\xi)}, \ \lim_{t\to 0} e^{f(t^2)}t^{p+1-n-\beta} > \frac{n-1}{\min_{\xi\in\mathbb{S}^n}\psi(\xi)}. \tag{17}$$

(A.2)

$$\begin{cases} p-\beta < n & \text{if } 2p-\beta \geq n-2, \\ p > -2 & \text{if } 2p-\beta < n-2. \end{cases} \tag{18}$$

The main result of the present paper can be stated as follows:

Theorem 1.1. *For any fixed* $n \geq 1$*, suppose the conditions* (A.1) $\sim$ (A.2) *hold. Then there exists a solution* $\rho \in C^2(\mathbb{S}^n)$ *to equation* (15) *satisfying*

$$\|\rho\|_{C^1(\mathbb{S}^n)} \leq c, \tag{19}$$

where c *depends only on* p, β, f *and* ψ*.*

Remark 1.2. If $f \equiv 0$, $p = \beta = 0$, Theorem 1.1 has been proved by Treibergs and Wei [45]. Similar topics can be referred to Yau [47].

Remark 1.3. This can be seen as an embedded result in Gauss measure space and such result will be needed in our forthcoming papers on the differential geometric analysis on some generalizations of some classical theorems in theories of curve and surfaces to Gauss measure space, such as Gauss-Bonnet-Chern theorem and its application on positive mass theorem, (see do Carmo [16] and Schoen and Yau [43]), the Steiner-Weyl type formula, (see Burago and Zalgaller [5], Schneider [42]). In our project, we also focus on formulation of the Plateau problem in the Gauss measure space via the variational method.

The proof of Theorem 1.1 is based on well-known continuous method and one of cores is the a priori bounds of solutions to the following problem on the unit sphere $\mathbb{S}^n$:

$$\sum_{i,j} a_{ij}\rho_{ij} = ((\rho^2 + |\nabla\rho|^2)\delta_{ij} - \rho_i\rho_j)\rho_{ij} = b(\xi, \rho, |\nabla\rho|^2). \tag{20}$$

The rest of paper is organized as follows: In Section 2, we get gradient estimate of solutions. In Section 3, we prove Theorem 1.1.

2. Gradient estimate

In this section, we consider the a priori bounds of solutions to the following equation on the unit sphere $\mathbb{S}^n$:

$$\sum_{i,j}(\delta_{ij}-\frac{\rho_i\rho_j}{\rho^2+|\nabla\rho|^2})(-\rho_{ji}+\frac{2}{\rho}\rho_j\rho_i+\rho\delta_{ji})=\psi\frac{\rho^{2p+2-n-\beta}e^{f(\rho^2)}}{(\rho^2+|\nabla\rho|^2)^{\frac{p}{2}}}\tag{21}$$

which is equivalent to the following equation,

$$\sum_{i,j}((\rho^2+|\nabla\rho|^2)\delta_{ij}-\rho_i\rho_j)(-\rho_{ji}+\frac{2}{\rho}\rho_j\rho_i+\rho\delta_{ji})=\psi\frac{\rho^{2p+2-n-\beta}e^{f(\rho^2)}}{(\rho^2+|\nabla\rho|^2)^{\frac{p-2}{2}}}.\tag{22}$$

We let $a_{ij}=(\rho^2+|\nabla\rho|^2)\delta_{ij}-\rho_i\rho_j$. Then

$$\sum_{i,j}\frac{2a_{ij}}{\rho}\rho_j\rho_i=\frac{2}{\rho}((\rho^2+|\nabla\rho|^2)\Sigma_i\rho_i^2-\sum_{i,j}\rho_i^2\rho_j^2)=2\rho|\nabla\rho|^2\tag{23}$$

and

$$\sum_{i,j}a_{ij}\rho\delta_{ji}=\rho a_{ij}\delta_{ji}=\rho(n(\rho^2+|\nabla\rho|^2)-|\nabla\rho|^2).\tag{24}$$

Putting (24) into (22), we have

$$\begin{aligned}\sum_{i,j}a_{ij}\rho_{ji}&=a_{ij}(\frac{2}{\rho}\rho_j\rho_i+\rho\delta_{ji})-\psi(\xi)\frac{e^{f(\rho^2)}\rho^{2p+2-n-\beta}}{(\rho^2+|\nabla\rho|^2)^{\frac{p-2}{2}}}\\&=\rho(n\rho^2+(n-1)|\nabla\rho|^2)-\psi(\xi)\frac{e^{f(\rho^2)}\rho^{2p+2-n-\beta}}{(\rho^2+|\nabla\rho|^2)^{\frac{p-2}{2}}}.\end{aligned}\tag{25}$$

In divergence form, equation (25) can be rewritten as follows:

$$\begin{aligned}&\operatorname{div}(\frac{\nabla\rho}{\sqrt{\rho^2+|\nabla\rho|^2}})\\&=\rho(n\rho^2+(n-1)|\nabla\rho|^2)-\psi(\xi)\frac{e^{f(\rho^2)}\rho^{2p+2-n-\beta}}{(\rho^2+|\nabla\rho|^2)^{\frac{p-2}{2}}}-\frac{\rho|\nabla\rho|^2}{(\rho^2+|\nabla\rho|^2)^{\frac{3}{2}}}.\end{aligned}\tag{26}$$

Therefore, we focus on the a priori bounds of solution to the following problem:

$$\sum_{i,j}a_{ij}\rho_{ij}=b(\xi,\rho,|\nabla\rho|^2),\tag{27}$$

where $b=b(x,s,w)$ satisfies the following condition;

$(B.0)$ if $2p-\beta\geq n-2$,

$$|b|\leq A_0w^{\frac{\max\{p+4-n-\beta,2\}}{2}},|b_x|\leq A_1w^{\frac{p+4-n-\beta}{2}},$$
$$b_s\geq -A_2w^{\frac{\max\{p+3-n-\beta,2\}}{2}},b_w\geq -A_3w^{\frac{\max\{p+2-n-\beta,-3\}}{2}}$$

for large w; if $2p-\beta<n-2$,

$$|b|\leq A_0w^{\frac{\max\{2-p,2\}}{2}},|b_x|\leq A_1w^{\frac{\max\{2-p,2\}}{2}},b_s\geq -A_2w^{-\frac{p}{2}},b_w\geq -A_3w^{\frac{\max\{-p,-3\}}{2}}$$

for large w.

This main result of this section can be stated as follows:

Theorem 2.1. *For any fixed $n \geq 1$, we let $\rho \in C^1(\mathbb{S}^n)$ be a solution to (27). Suppose that f and ψ are continuous on $\mathbb{S}^n$ satisfying*

$$|f|_{C^1(\mathbb{R})} + |\psi|_{C^1(\mathbb{S}^n)} < \infty, \tag{28}$$

p and β satisfies

$$\begin{cases} p - \beta < n & \text{if } 2p - \beta \geq n - 2, \\ p > -2 & \text{if } 2p - \beta < n - 2, \end{cases} \tag{29}$$

and

$$0 < m = \min_{\xi \in \mathbb{S}^n} \rho(\xi) \leq M = \max_{\xi \in \mathbb{S}^n} \rho(\xi) < \infty. \tag{30}$$

Then there exists a positive constant c, depends only on f, ψ, A_0, A_1, A_2, A_3, m and M, such that

$$\sup_{\xi \in \mathbb{S}^n} |\nabla \rho(\xi)| \leq c < \infty. \tag{31}$$

Proof. The proof is based on Maximum Principle. We let $G = e^{2\rho} v = e^{2\rho}|\nabla \rho|^2$. Suppose that $\sup G$ is achieved at the point $\xi = \xi_0 \in \mathbb{S}^n$. Then, at $\xi = \xi_0$,

$$0 = G_i = 2e^{2\rho}(v\rho_i + \Sigma_l \rho_l \rho_{li}) \tag{32}$$

for any fixed $i \in \{1, 2, \ldots, n\}$ and and $(G_{ij})_{n\times n}$ is non-positive. Direct calculation deduces that

$$G_{ij} = 2e^{2\rho}(v\rho_{ij} + 2\Sigma_\alpha \rho_\alpha \rho_{\alpha j}\rho_i + \Sigma_l \rho_{lj}\rho_{li} + \Sigma_l \rho_l \rho_{lij}) + 4e^{2\rho}\rho_j(vu_i + \Sigma_t \rho_t \rho_{ti}).$$

Therefore, since $(a_{ij})_{n\times n}$ is positive, we have

$$\begin{aligned} 0 \geq & \sum_{i,j} a_{ij}(v\rho_{ij} + 2\Sigma_\alpha \rho_\alpha \rho_{\alpha j}\rho_i + \Sigma_l \rho_{lj}\rho_{li} + \Sigma_l \rho_l \rho_{lij}) \\ & + 2\sum_{i,j} a_{ij}\rho_j(v\rho_i + \Sigma_t \rho_t \rho_{ti}) = \Sigma_{i=1}^5 I_i \end{aligned} \tag{33}$$

at the point $\xi = \xi_0$ where

$$I_1 = v\sum_{i,j} a_{ij}\rho_{ij}, \; I_2 = 2\Sigma_{ij\alpha} a_{ij}\rho_\alpha \rho_{\alpha j}\rho_i, \; I_3 = \Sigma_{ijl} a_{ij}\rho_{lj}\rho_{li}, \tag{34}$$

and

$$I_4 = \Sigma_{ijl} a_{ij} \Sigma_l \rho_l \rho_{lij}, \; I_5 = 2\sum_{i,j} a_{ij}\rho_j(v\rho_i + \Sigma_t \rho_t \rho_{ti}). \tag{35}$$

Without loss of generalization, we may assume that $v(\xi_0) = |\nabla \rho(\xi_0)|^2 \gg 1$. Otherwise, inequality (31) is trivial.

By choosing suitable coordinate, we may assume that

$$\rho_i = \delta_{i1}\sqrt{v} \tag{36}$$

at the point $\xi = \xi_0$. This means that $(a_{ij})_{n\times n}$ is diagonal at the point $\xi = \xi_0$. Moreover, it follows from (32) and (36) that

$$\rho_{1i} = -v\delta_{i1},\ \Sigma_i a_{ii} = (n-1)v + n\rho^2 \tag{37}$$

and

$$\Sigma_i \rho_{ii} = \frac{b - v^2}{\rho^2 + v} \tag{38}$$

at the point $\xi = \xi_0$.

We now get the bounds of $\Sigma_{i=1}^5 I_i$. At the first step, we first analyze the term $I_1 = v\sum_{i,j} a_{ij}\rho_{ij}$. It is easy to see that

$$\begin{aligned} I_1 &= v\sum_{i,j} a_{ij}\rho_{ij} \\ &= v\Sigma_i a_{ii}\rho_{ii} \\ &= vb \geq \begin{cases} -A_0 v^{\frac{\max\{p+6-n-\beta,4\}}{2}} & \text{if } 2p-\beta \geq n-2, \\ -A_0 v^{\frac{\max\{4-p,4\}}{2}} & \text{if } 2p-\beta < n-2, \end{cases} \end{aligned} \tag{39}$$

at the point $\xi = \xi_0$.

We next estimate the term $I_2 = 2\Sigma_{ij\alpha} a_{ij}\rho_\alpha\rho_{\alpha j}\rho_i$.

$$I_2 = 2\Sigma_{ij\alpha} a_{ij}\rho_\alpha\rho_{\alpha j}\rho_i = 2a_{11}\rho_1^2\rho_{11} = -2\rho^2 v^2 \tag{40}$$

at the point $\xi = \xi_0$.

We next estimate the term $I_3 = \Sigma_{ijl} a_{ij}\rho_{lj}\rho_{li}$.

Since $a_{ij} = (\rho^2 + v)\delta_{ij} - \rho_i\rho_j$, we have

$$a_{ii} \geq \rho^2 \tag{41}$$

for any fixed $i \in \{1, 2, \ldots, n\}$ and therefore

$$I_3 = \Sigma_{li} a_{ii}\rho_{li}^2 \geq \rho^2\Sigma_{li}\rho_{li}^2 \geq 0 \tag{42}$$

at the point $\xi = \xi_0$.

We next estimate the term $I_4 = \Sigma_{ijl}\rho_l a_{ij}\rho_{lij}$.

Since $\sum_{i,j} a_{ij}\rho_{ij} = b$ and $a_{ij} = (\rho^2 + v)\delta_{ij} - \rho_i\rho_j$, we have

$$\sum_{i,j} a_{ij}\rho_{ijt} + \sum_{i,j}(2(\rho\rho_t + \Sigma_k\rho_k\rho_{kt})\delta_{ij} - (\rho_{it}\rho_j + \rho_i\rho_{jt}))\rho_{ij} = b_t \tag{43}$$

for any fixed $t \in \{1, 2, \ldots, n\}$. Multiplying ρ_t on both sides of (43) and taking sum for the index t, we get

$$\begin{aligned} &\Sigma_{ijt} a_{ij}\rho_t\rho_{ijt} + \Sigma_{ijt} 2(\rho\rho_t^2 + \Sigma_k\rho_t\rho_k\rho_{kt})\delta_{ij}\rho_{ij} \\ &- \Sigma_{ijt}(\rho_{it}\rho_j + \rho_i\rho_{jt})\rho_{ij}\rho_t = \Sigma_t b_t\rho_t. \end{aligned} \tag{44}$$

At the point $\xi = \xi_0$, we have,

$$\Sigma_i a_{ii}\rho_1\rho_{ii1} = b_1\rho_1 - 2\rho_1^2(\rho + \rho_{11})\Sigma_i\rho_{ii} + \Sigma_i\rho_1^2(\rho_{i1}^2 + \rho_{1i}\rho_{i1}). \tag{45}$$

By Ricci identity,

$$\Sigma_{ti}\rho_t(\rho_{tii} - \rho_{iit}) = n\Sigma_t\rho_t^2 = nv \tag{46}$$

at the point $\xi = \xi_0$. Combining (44), (45) and (46), we have

$$\begin{aligned}\Sigma_i a_{ii}\rho_1\rho_{1ii} &= nv\Sigma_i a_{ii} - 2\rho_1^2(\rho + \rho_{11})\Sigma_i\rho_{ii} + b_1\rho_1 + \Sigma_i\rho_1^2(\rho_{i1}^2 + \rho_{1i}\rho_{i1}) \\ &= I_{41} + I_{42} + I_{43},\end{aligned} \tag{47}$$

where

$$I_{41} = nv\Sigma_i a_{ii} - 2\rho_1^2(\rho + \rho_{11})\Sigma_i\rho_{ii}, I_{42} = b_1\rho_1, I_{43} = \Sigma_i\rho_1^2(\rho_{i1}^2 + \rho_{1i}\rho_{i1}). \tag{48}$$

We first estimate the term $I_{41} = nv\Sigma_i a_{ii} - 2\rho_1^2(\rho + \rho_{11})\Sigma_i\rho_{ii}$.
It is easy to see that

$$\begin{aligned}I_{41} &= nv\Sigma_i a_{ii} - 2\rho_1^2(\rho + \rho_{11})\Sigma_i\rho_{ii} \\ &= nv(nv + n\rho^2) - \frac{2v(\rho - v)(b - v^2)}{\rho^2 + v} \\ &\geq (n^2 - 3)v^2 - c\end{aligned} \tag{49}$$

for sufficiently large $v(\xi_0)$.

We next estimate the term $I_{42} = b_1\rho_1$. Since

$$b_1\rho_1 = \frac{\partial b}{\partial x_1}\rho_1 + \frac{\partial b}{\partial\rho}\rho_1^2 + 2\Sigma_j\frac{\partial b}{\partial v}\rho_j\rho_{j1}\rho_1, \tag{50}$$

it follows that

$$|\frac{\partial b}{\partial x_1}\rho_1| \leq |\nabla b|\sqrt{v} \tag{51}$$

which means that

$$\frac{\partial b}{\partial x_1}\rho_1 \geq \begin{cases} -A_1 v^{\frac{p+5-n-\beta}{2}} & \text{if } 2p - \beta \geq n - 2, \\ -A_1 v^{\frac{\max\{3-p,3\}}{2}} & \text{if } 2p - \beta < n - 2. \end{cases} \tag{52}$$

It is easy to see that

$$\frac{\partial b}{\partial\rho}v \geq \begin{cases} -A_2 v^{\frac{\max\{p+5-n-\beta,4\}}{2}} & \text{if } 2p - \beta \geq n - 2, \\ -A_2 v^{\frac{2-p,}{2}} & \text{if } 2p - \beta < n - 2. \end{cases} \tag{53}$$

Since $u_j = \delta_{j1}$, we have

$$\begin{aligned}\Sigma_j\frac{\partial b}{\partial v}\rho_j\rho_{j1}u_1 &= \frac{\partial b}{\partial v}\rho_{11}\rho_1^2 \\ &= -v^2\frac{\partial b}{\partial v} \geq \begin{cases} -A_3 v^{\frac{\max\{p+6-n-\beta,1\}}{2}} & \text{if } 2p - \beta \geq n - 2, \\ -A_3 v^{\frac{\max\{4-p,1\}}{2}} & \text{if } 2p - \beta < n - 2 \end{cases}\end{aligned} \tag{54}$$

for sufficiently large $v(\xi_0)$. Putting (52), (53) and (54) into (50), we have

$$I_{42} = b_1\rho_1 \geq \begin{cases} -(A_1 + A_2 + A_3)v^{\frac{\max\{p+6-n-\beta,4\}}{2}} & \text{if } 2p - \beta \geq n - 2, \\ -(A_1 + A_2 + A_3)v^{\frac{\max\{4-p,3\}}{2}} & \text{if } 2p - \beta < n - 2 \end{cases} \tag{55}$$

for sufficiently large $v(\xi_0)$.

We next deal with the term $I_{43} = \Sigma_i \rho_1^2(\rho_{i1}^2 + \rho_{1i}\rho_{i1})$.

From (37), we see

$$I_{43} = \rho_1^2(\Sigma_i \rho_{i1}^2 + \rho_{11}^2) \geq 2\rho_1^2\rho_{11}^2 = 2v^3 \tag{56}$$

at the point $\xi = \xi_0$.

Therefore, putting (49), (55) and (56) into (47), we get

$$I_4 \geq \frac{3}{2}v^3 - (A_1 + A_2 + A_3)v^{\tau_0} - c \tag{57}$$

for sufficiently large $v(\xi_0)$ where

$$\tau_0 = \begin{cases} \frac{\max\{p+6-n-\beta,4\}}{2} & \text{if } 2p-\beta \geq n-2, \\ \frac{\max\{4-p,4\}}{2} & \text{if } 2p-\beta < n-2. \end{cases} \tag{58}$$

It follows from (32) that

$$I_5 = 2\sum_{i,j} a_{ij}\rho_j(v\rho_i + \Sigma_t \rho_t \rho_{ti}) = 0 \tag{59}$$

at the point $\xi = \xi_0$.

Therefore, it follows from (39), (40), (42), (57), (59) and (33) that

$$0 \geq \Sigma_{i=1}^5 I_i \geq \frac{3}{2}v^3 - (A_0 + A_1 + A_2 + A_3)v^{\tau_0} - c \tag{60}$$

at the point $\xi = \xi_0$.

Since

$$\begin{cases} p - \beta < n & \text{if } 2p-\beta \geq n-2, \\ p > -2 & \text{if } 2p-\beta < n-2, \end{cases} \tag{61}$$

we have

$$\tau_0 < 3. \tag{62}$$

Therefore, there exists a constant c, depends only on n,f, ψ, A_0, A_1, A_2, A_3, m and M, such that

$$v^3 \leq c \tag{63}$$

for sufficiently large $v(\xi_0)$. Multiplying $e^{6\rho}$ on both sides of (63) and we have,

$$G^3(x_0) \leq ce^{3\rho} \leq c \tag{64}$$

due to the assumption $M = \max_{\xi} \rho(\xi) < \infty$. (64) implies that

$$e^{2\rho}v \leq c \tag{65}$$

at the point $\xi = \xi_0$. Since $m = \min_{\xi \in \mathbb{S}^n} \rho(\xi) > 0$, we can get the conclusion of Theorem 2.1. □

3. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Motivated by Treibergs and Wei [45], we consider the following auxiliary problem with a parameter $t \in [0,1]$ on the unit sphere $\mathbb{S}^n$,

$$L_t[\rho_t](u) = div(\frac{\nabla u_t}{\sqrt{\rho_t^2 + |\nabla \rho_t|^2}}) - u_t = t(g(\xi) - \rho_t), \tag{66}$$

where $g(\xi) = \rho_t((n-1)\rho_t^2 + n|\nabla \rho_t|^2) - \psi(\xi)\frac{e^{f(\rho_t^2)}\rho_t^{2p+2-n-\beta}}{(\rho_t^2+|\nabla\rho_t|^2)^{\frac{p-2}{2}}} - \frac{\rho_t|\nabla\rho_t|^2}{(\rho_t^2+|\nabla\rho_t|^2)^{\frac{3}{2}}}$.

Lemma 3.1. *For any fixed $n \geq 1$, and $t \in [0,1]$, we let $\rho_t \in C^2(\mathbb{S}^n)$ be a solution to*

$$L_t[\rho_t]\rho_t = 0$$

for any $t \in [0,1]$. Suppose the condition (A.1) *holds. Then there exists a constant c, independent on t, such that*

$$0 < c^{-1} \leq |\rho_t|_{C^0(\mathbb{S}^n)} \leq c$$

for any $t \in [0,1]$.

Proof. We consider the following extremal problem,

$$R_t = \max_{\xi \in \mathbb{S}^n} \rho_t(\xi) \tag{67}$$

for any fixed $t \in [0,1]$. It follows from the compactness of $\mathbb{S}^n$ and the continuity of ρ_t that there exists $\xi_1 \in \mathbb{S}^n$ such that

$$R_t = \rho_t(\xi_1) \tag{68}$$

for any fixed $t \in [0,1]$. Suppose that there exists a sequence $\{t_j\}_{j=1}^{\infty} \subseteq [0,1]$ such that

$$R_{t_j} \to +\infty \tag{69}$$

as $j \to \infty$. It follows from (66) and the condition (A.1) that at the point $\xi = \xi_1$,

$$\begin{aligned} L[\rho_{t_j}]\rho_{t_j} &= t_j(g(\xi_1) - \rho_{t_j}) \\ &= tR_{t_j}^3\psi(\xi_1)(\frac{n-1}{\psi(\xi_1)} - e^{f(R_{t_j}^2)}R_{t_j}^{p+1-n-\beta}) - tR_{t_j} \\ &> -t_jR_{t_j} = L[R_{t_j}]R_{t_j} \end{aligned} \tag{70}$$

as $j \to \infty$. However, there exists a contradiction from (70). This implies there exists a positive constant $c > 0$, independent of t, such that

$$R_t \leq c < \infty \tag{71}$$

for any fixed $t \in [0,1]$. We next consider the following extremal problem,

$$r_t = \min_{\xi \in \mathbb{S}^{n-1}} \rho_t(\xi) \tag{72}$$

for any fixed $t \in [0,1]$. By a similar argument mentioned above, there exists a positive constant $c > 0$, independent of t such that

$$r_t \geq \frac{1}{c} > 0 \tag{73}$$

for any fixed $t \in [0,1]$. (71) and (73) yield the desired conclusion of Lemma 3.1. □

As a corollary of Lemma 3.1 and some arguments of Theorem 2.1, we have:

Corollary 3.2. *For any fixed $n \geq 1$ and $t \in [0,1]$, we let $\rho_t \in C^2(\mathbb{S}^n)$ be a solution to*

$$L_t[\rho_t]\rho_t = 0$$

for any $t \in [0,1]$. Suppose the conditions (A.1) *and* (A.2) *hold. Then there exists a constant c, independent on t, such that*

$$0 < c^{-1} \leq |\rho_t|_{C^1(\mathbb{S}^n)} \leq c$$

for any $t \in [0,1]$.

Now, we are in a position to prove Theorem 1.1.

Proof. The final proof of Theorem 1.1 follows from a fixed point argument. Following some arguments of Treibergs and Wei [45], we let $C(\mathbb{S}^n)$, $C^1(\mathbb{S}^n)$ and $C^{1,\alpha}(\mathbb{S}^n)$ be some Banach spaces on $\mathbb{S}^n$ with the standard norm. Define

$$B_r = \{\rho \in C^{1,\alpha}(\mathbb{S}^n) : |\rho|_{C^{1,\alpha}(\mathbb{S}^n)} \leq r\}. \tag{74}$$

We let

$$L_t[\rho_t]u = div(\frac{\nabla u}{\sqrt{\rho_t^2 + |\nabla \rho_t|^2}}) - u. \tag{75}$$

Suppose that $L_t[\rho_t]$ has an inverse in $C^{1,\alpha}(\mathbb{S}^n)$ and H_t be the inverse of $L_t[\rho_t]$ in $C^{1,\alpha}(\mathbb{S}^n)$. Then

$$L_t[\rho_t]u = t(g(\xi) - \rho_t) \Leftrightarrow u = tH_t(g(\xi) - \rho_t) = K_t(\rho_t). \tag{76}$$

Suppose that ρ is a solution to problem (15), we have

$$\rho = K_1(\rho) \tag{77}$$

which means that a fixed point of K_1 is a solution to (15). Therefore, it suffices to prove the existence of a fixed point of K_1 in $C^{1,\alpha}(\mathbb{S}^n)$. To achieve this goal, we prove that there exists a fixed point of K_t in $C^{1,\alpha}(\mathbb{S}^n)$ for any fixed $t \in [0,1]$ provided $L_t[\rho_t]$ has an inverse in $C^{1,\alpha}(\mathbb{S}^n)$. Now we divide the proof of Theorem 1.1 into two steps.

Step 1. We prove that $L_t[\rho_t]$ has an inverse in $C^{1,\alpha}(\mathbb{S}^n)$.

To get this goal, we claim the following argument,

Claim. *For any fixed $t \in [0,1]$, we let $L_t[\rho_t]$ be the operator defined in* (75). *Then*

(i) *$L_t[\rho_t]$ is self-adjoint*;

(ii) *The kernel of $L_t[\rho_t]$ is trivial.*

Indeed, part (i) follows from that definition of $L_t[\rho_t]$. Therefore, it suffices to show part (ii). Suppose that $L_t[\rho_t]u = 0$ for any $u \in C^{1,\alpha}(\mathbb{S}^n)$. It follows from the divergence theorem that

$$0 = -\int_{\mathbb{S}^n} L_t[\rho_t]uu = \int_{\mathbb{S}^n} (\frac{|\nabla u|^2}{\sqrt{\rho_t^2 + |\nabla \rho_t|^2}} + u^2)d\xi \geq \int_{\mathbb{S}^n} u^2 d\xi \tag{78}$$

which means that $u \equiv 0$ for any fixed $t \in [0,1]$. By the arbitrariness of u, we see that

$$\{u \in C^2(\mathbb{S}^n) : L_t[\rho_t]u = 0\} = \{0\} \tag{79}$$

which is the argument of part (ii).

Therefore, it follows from the standard solvability of linear elliptic equation of second order that $L_t[\rho_t]$ has an inverse, see Gilbarg and Trudinger [20].

Step 2. We prove that K_t has a fixed point in $C^{1,\alpha}(\mathbb{S}^n)$.

It follows from Lemma 3.1 and Corollary 3.2 that there exists R_0 such that

$$\|u\|_{C^1(\mathbb{S}^n)} \leq R_0 \tag{80}$$

for any $u \in C^{1,\alpha}(\mathbb{S}^n)$ satisfying $K_t(u) = u$ for any fixed $t \in [0,1]$.

It follows from the standard elliptic regularity theory of linear elliptic equation of second order that $K_t : C^{1,\alpha}(\mathbb{S}^n) \to C^{1,\alpha}(\mathbb{S}^n)$ is compact, see Gilbarg and Trudinger [20]. Therefore, it follows from Leray-Schauder fixed point theorem that there exists $u_t \in B_{R_0}$ such that

$$K_t(u_t) = u_t \tag{81}$$

for any $t \in [0,1]$. Taking $t = 1$, we get the desired conclusion of Theorem 1.1. □

4. Appendix

In this section, we list some basic geometric quantities which are used in the present paper and can be referred to Guan, Li and Li [23].

Lemma 4.1. *Suppose M is a hypersurface in $\mathbb{R}^{n+1}$ with the metric $ds^2 = d\rho^2 + \rho^2 d\xi^2$ and with zero sectional curvature. Then the following statements hold.*

(a) *The components of the metric g and its inverse g^{-1} can be expressed as follows*:

$$g_{ij} = \rho^2\delta_{ij} + \rho_i\rho_j, g^{ij} = \frac{1}{\rho^2}(\delta_{ij} - \frac{\rho_i\rho_j}{\rho^2 + |\nabla\rho|^2}) \tag{82}$$

respectively and therefore $\det g = \rho^{2n-2}(\rho^2 + |\nabla\rho|^2)$.

(b) *The coefficients of second fundamental form b_{ij} are given by*:

$$b_{ij} = \frac{\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}}(-\rho_{ij} + \frac{2}{\rho}\rho_i\rho_j + \rho\delta_{ij}). \tag{83}$$

(c) *The support function u of hypersurface M is given by*:

$$u = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla\rho|^2}}. \tag{84}$$

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