J. Korean Math. Soc. **60** (2023), No. 5, pp. 987–998
https://doi.org/10.4134/JKMS.j220359
pISSN: 0304-9914 / eISSN: 2234-3008

CONTINUUM-WISE EXPANSIVENESS FOR C^1 GENERIC VECTOR FIELDS

Manseob Lee

Abstract. It is shown that every continuum-wise expansive C^1 generic vector field X on a compact connected smooth manifold M satisfies Axiom A and has no cycles, and every continuum-wise expansive homoclinic class of a C^1 generic vector field X on a compact connected smooth manifold M is hyperbolic. Moreover, every continuum-wise expansive C^1 generic divergence-free vector field X on a compact connected smooth manifold M is Anosov.

1. Introduction

Utz [37] introduced a concept in dynamical systems that is called expansiveness. Roughly speaking, expansiveness implies that two orbits cannot stay close to each other under the iteration of the system. Later, Kato [13] introduced a notion generalizing the usual concept of expansiveness that is called continuum-wise expansiveness. These concepts have been studied in the context of hyperbolic systems. Indeed, for a diffeomorphism f on a compact smooth manifold M, Mañé [26] proved that if f belongs to the C^1-interior of the set of expansive diffeomorphisms, then f is quasi-Anosov, that is, if there is a nonzero vector in the tangent space, then the tangent space is unbounded. Sakai [32] proved that if a diffeomorphism f belongs to the C^1-interior of the set of continuum-wise expansive diffeomorphisms, then f is quasi-Anosov. Arbieto [1] proved that every expansive C^1 generic diffeomorphism f on a compact connected smooth manifold M satisfies Axiom A and has no cycles. Lee [20] proved that every continuum-wise expansive C^1 generic diffeomorphism f on a compact connected smooth manifold M satisfies Axiom A and has no cycles. If a diffeomorphism f satisfies Axiom A and has cycles, then by [36], the nonwandering set $\Omega(f)$ admits a disjoint union of transitive invariant closed sets, where these sets are homoclinic classes. To determine whether such classes are

Received July 15, 2022; Accepted December 1, 2022.
2010 *Mathematics Subject Classification.* 37C10, 37C20, 34D10, 37C27.
Key words and phrases. Expansive, continuum-wise expansive, Axiom A, homoclinic class, hyperbolic, generic.

hyperbolic under some dynamical assumption is an important problem. Indeed, in several studies [9,15,17,19,21–25,30,31,33,34,39], it was shown that if a homoclinic class is C^1 expansive in various senses (e.g., robustly, persistently, stably, or generic), then it is hyperbolic. In particular, Das, Lee, and Lee [9] proved that if a homoclinic class is C^1 robustly continuum-wise expansive and satisfies the chain condition, then it is hyperbolic. Yang and Gan [39] proved that every expansive homoclinic class of C^1 generic diffeomorphisms is hyperbolic. Lee [19] proved that every continuum-wise expansive homoclinic class of C^1 generic diffeomorphisms is hyperbolic.

In this study, the previously mentioned results on continuum-wise expansive diffeomorphisms are extended to vector fields.

The paper is organized as follows. In Section 2, basic notions as well as Theorems A and B are introduced. In Section 3, Theorem A is proved. In the proof, we use a notion that is called ϵ-periodic curve and a result by Arbieto, Cordeiro, and Pacifico [2]. In Section 4, Theorem B is proved. In the proof, we do not use local maximality, and we show that every continuum-wise expansive C^1 generic vector field X on a compact connected smooth manifold M is hyperbolic. Finally, in Section 5, divergence-free vector fields are studied.

2. Basic notions and theorems

2.1. Continuum-wise expansive vector fields

Throughout this paper, we assume that M is a compact connected $n(\geq 3)$-dimensional manifold without boundary, and let $X \in \mathfrak{X}(M)$. The flow of X will be denoted by X^t, $t \in \mathbb{R}$. A point $x \in M$ is a *singular* point of X if $X^t(x) = x$ for all $t \in \mathbb{R}$. $Sing(X)$ denotes the set of all singular points of X. A point $x \in M$ is *regular* if $x \notin Sing(X)$. A point $p \in M$ is *periodic* if there is $\pi(p) > 0$ such that $X^{\pi(p)}(p) = p$, where $\pi(p)$ is the prime period of p. Let $Per(X)$ be the set of all closed orbits of X. Let $Crit(X) = Sing(X) \cup Per(X)$. It is clear that $Crit(X) \subset \Omega(X)$, where $\Omega(X)$ is the set of all nonwandering points of X. Following Utz [37], Bowen and Walters [8] introduced a concept of expansiveness for vector fields.

A closed X^t-invariant set $\Lambda \subset M$ is *expansive* for $X \in \mathfrak{X}(M)$ if for every $\epsilon > 0$ there are $\delta > 0$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that for any $x, y \in \Lambda$, if $d(X^t(x), X^{h(t)}(y)) \leq \delta$ for all $t \in \mathbb{R}$, then $y \in X^{(-\epsilon,\epsilon)}(x)$. If $\Lambda = M$, then the vector field X is called *expansive*. Bowen and Walters [8] proved that if a vector field X is expansive, then every $\sigma \in Sing(X)$ is isolated. Oka [29, Lemma 2] showed that if a vector field X is expansive, then $Sing(X) = \emptyset$.

A point $\sigma \in Sing(X)$ is *hyperbolic* if the eigenvalues of the derivative $(DX(\sigma))$ of the vector field at σ have a real part differs from zero.

A closed orbit $\gamma = Orb(p)$ is *hyperbolic* if the eigenvalues of the derivative $(DX^{\pi(p)}(p))$ of diffeomorphism $X^{\pi(p)}$ are all different from one, where $\pi(p)$ is the period of p.

A compact invariant set Λ of X is called *hyperbolic* if there exist constants $C > 0$, $\lambda > 0$ and a splitting $T_xM = E^s_x \oplus F(x) \oplus E^u_x$ such that

(a) the tangent flow DX^t is the invariant continuous splitting,
(b) (C, λ)-contracting, that is, for all $t > 0$ and $x \in \Lambda$,

$$\|DX^t|_{E^s_x}\| \leq Ce^{-\lambda t} \text{ and}$$

(c) (C, λ)-expanding, that is, for all $t > 0$ and $x \in \Lambda$,

$$m(DX^{-t}|_{E^u_x}) \leq Ce^{-\lambda t},$$

where $F(x)$ is the subspace generated by $X(x)$ and $m(O) = \inf_{\|v\|=1} \|O(v)\|$ is the *minimum norm* of a linear operator O.

A point $x \in M$ is a *non-wandering* point of X if for every $t > 0$ and every neighborhood U of x, there is $t_1 > t$ such that $X^{t_1}(U) \cap U \neq \emptyset$. A vector field X satisfies *Axiom A* if the nonwandering set $\Omega(X)$ is hyperbolic and is the closure of $Crit(X)$.

For an Axiom A vector field $X \in \mathfrak{X}(M)$, X satisfies the *quasi-transversality condition* if $T_xW^s(x) \cap T_xW^u(x) = \overrightarrow{O_x}$ for $x \in M$, where $W^s(x)$ is the stable set of x and $W^u(x)$ is the unstable set of x. Moriyasu, Sakai, and Sun [28] proved that if a vector field X belongs to the C^1 interior of the set of all expansive vector fields, then X satisfies Axiom A and the quasi-transversality condition. This is an extension of a result by Mañé [26]. Lee [18] proved that if a vector field X belongs to the C^1 interior of the set of all continuum-wise expansive vector fields, then X satisfies Axiom A and the quasi-transversality condition. This is a generalization of a result by Moriyasu, Sakai, and Sun [28].

An increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ is called a *reparametrization.* Hom$(\mathbb{R})$ denotes the set of all homeomorphisms of $\mathbb{R}$. Let $Rep(\mathbb{R})$ $= \{h \in \text{Hom}(\mathbb{R}) : h \text{ is a reparametrization}\}$. If A is a subset of M, $C^0(A, \mathbb{R})$ denotes the set of real continuous maps defined on A. Let $\mathcal{H}(A) = \{h : A \to Rep(\mathbb{R}) : \text{there is } x_h \in A \text{ with } h(x_h) = id \text{ and } h(\cdot)(t) \in C^0(A, \mathbb{R}) \text{ for all } t \in \mathbb{R}\}$, and if $t \in \mathbb{R}$ and $h \in \mathcal{H}(A)$, let

$$\mathcal{X}^t_h(A) = \{X^{h(x)(t)}(x) : x \in A\}.$$

For convenience, we set $h(x)(t) = h_x(t)$ for all $x \in A$ and $t \in \mathbb{R}$. Let Λ be a closed subset of M. A set A is called *nondegenerate* if it is not reduced to a single point. $A \subset M$ is called a *continuum* if it is a compact connected nondegenerate subset of M. Recently, Arbieto *et al.* [2] introduced the following definition, which is a vector field analogue of continuum-wise expansiveness.

Definition 2.1. Let $X \in \mathfrak{X}(M)$. X is called *continuum-wise expansive* if for any $\epsilon > 0$ there is $\delta > 0$ such that if $A \subset M$ is a continuum and $h \in \mathcal{H}(A)$ satisfies

$$\text{diam}(\mathcal{X}^t_h(A)) < \delta \text{ for all } t \in \mathbb{R},$$

then $A \subset X^{(-\epsilon,\epsilon)}(x)$ for some $x \in A$.

A subset $\mathcal{G} \subset \mathfrak{X}(M)$ is called *residual* if it contains a countable intersection of open and dense subsets of $\mathfrak{X}(M)$. A dynamic property is called C^1 *generic* if it holds in a residual subset of $\mathfrak{X}(M)$.

In the paper, we write "for C^1 generic $X \in \mathfrak{X}(M)$" means that there is an open and dense subset $\mathcal{R} \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{R}, \ldots$.

$X \in \mathfrak{X}(M)$ is called *star* if there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that for any $Y \in \mathcal{U}(X)$, every $\gamma \in Crit(X)$ is hyperbolic. $\mathcal{G}^*(M)$ denotes the set of all star vector fields. Senos [35] proved that for C^1 generic $X \in \mathfrak{X}(M)$, if X is expansive, then $X \in \mathcal{G}^*(M)$, and thus it satisfies Axiom A and has no cycles. In the present study, the following extension will be proved.

Theorem A. *For C^1 generic $X \in \mathfrak{X}(M)$, if X is continuum-wise expansive, then it is Axiom A without cycles.*

To prove Theorem A, we will show that $Sing(X) = \emptyset$, which was proved by Arbieto, Cordeiro, and Pacifico [2]. Using the concept of ϵ-periodic curves, we will show that X is weak star, that is, there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that every $\gamma \in Per(Y)$ is hyperbolic for any $Y \in \mathcal{U}(X)$. The proof follows using the result by Gan and Wen [11].

2.2. Continuum-wise expansive homocinic classes vector fields

For any hyperbolic $\eta, \gamma \in Per(X)$, the notation $\eta \sim \gamma$ implies that

$$W^s(\eta) \pitchfork W^u(\gamma) \neq \emptyset \text{ and } W^u(\eta) \pitchfork W^s(\gamma) \neq \emptyset.$$

Let $H(\gamma, X) = \overline{\{\eta \in Per(X) : \eta \sim \gamma\}}$. It is known that $H(\gamma, X)$ is a compact, X^t-invariant, and transitive subset of X. In [3], Bautista showed that the geometric Lorenz attractor is a homoclinic class. In [14], Komuro proved that the geometric Lorenz attractor is $\mathcal{K}^*$-expansive.

For any $x, y \in M$ and $\delta > 0$, a sequence $\{(x_i, t_i) : t_i \geq 1, i = 1, \ldots, n\} \subset M$ is called a *δ-chain from x to y* if $d(X^{t_{n-1}}(x_{n-1}), x_n) < \delta$, $x_0 = x$, and $x_n = y$. Then an equivalence relation can be defined on the set $\mathcal{CR}(X)$, where $\mathcal{CR}(X)$ is the chain recurrence set of X. It is clear that $\Omega(X) \subset \mathcal{CR}(X)$. Let γ be a hyperbolic periodic orbit of X, and let $C(\gamma, X) = \{x \in M :$ the δ-chain from x to γ and γ to $x\}$. According to the definition, it is easy to see that $C(\gamma, X)$ is closed and X^t-invariant, and $H(\gamma, X) \subset C(\gamma, X)$.

A closed X^t-invariant set Λ is called *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{t \in \mathbb{R}} X^t(U)$. Lee and Park [25] proved that for C^1 generic $X \in \mathfrak{X}(M)$, if a locally maximal homoclinic class $H(\gamma, X)$ is expansive, then it is hyperbolic. Lee and Oh [24] proved that for C^1 generic $X \in \mathfrak{X}(M)$, if a locally maximal homoclinic class $H(\gamma, X)$ is measure expansive, then it is hyperbolic. Recently, Lee [21] proved that for C^1 generic $X \in \mathfrak{X}(M)$, if a homoclinic class $H(\gamma, X)$ is measure expansive, then it is hyperbolic.

In the present study, the following analogue is proved.

Theorem B. *For C^1 generic $X \in \mathfrak{X}(M)$, if for some hyperbolic closed orbit γ, the homoclinic class $H(\gamma, X)$ is continuum-wise expansive, then it is hyperbolic.*

To prove Theorem B, we show that if a homoclinic class $H(\gamma, X)$ is continuum-wise expansive, then $H(\gamma, X)$ does not contain any singular points. We introduce the notion of weak hyperbolic closed orbits, which is related to results by Yang and Gan [39]. According to Bonatti and Crovisier [7], for a C^1 generic vector field X, a homoclinic class $H(\gamma, X)$ is a chain recurrence class $C(\gamma, X)$. Moreover, by the result in [38], if a homoclinic class $H(\gamma, X)$ is continuum-wise expansive, then we show that the class is hyperbolic.

3. Proof of Theorem A

Let M be as before, and let $X \in \mathfrak{X}(M)$.

Lemma 3.1 ([2, Lemma 2.1]). *If X is continuum-wise expansive, then $Sing(X) = \emptyset$.*

Let $T_xM(r)$ be the r-ball $\{v \in T_xM : \|v\| \le r\}$ in T_xM. Let $\mathcal{N}_{x,r} = \exp_x(T_xM(r))$. If x is a regular point of X, $N_x = \{v \in T_xM : v \cdot X(x) = 0\}$ denotes the orthogonal complement of $X(x)$ in T_xM, and $\Phi_x : T_xM \to N_x$ the orthogonal projection of T_xM onto N_x. For all $t \in \mathbb{R}$, we define the linear Poincaré flow of X by

$$P_x^t : N_x \to N_{X^t(x)} \text{ by } P_x^t = \Phi_{X^t(x)} \circ DX^t(x).$$

For any regular point x of X and $t \in \mathbb{R}$, there are a constant $r > 0$ and a C^1 map $\tau : \mathcal{N}_{x,r} \to \mathbb{R}$ such that $\tau(x) = t$ and $X_{\tau(y)}(y) \in \mathcal{N}_{X^t(x),1}$ for any $y \in \mathcal{N}_{x,r}$. Then the *Poincaré map* $f_{x,t} : \mathcal{N}_{x,r_0} \to \mathcal{N}_{X^t(x),1}$ is given by

$$f_{x,t}(y) = X^{\tau(y)}(y) \text{ for all } y \in \mathcal{N}_{x,r_0}.$$

If $X^t(x) \neq x$ for $0 < t \le T$ and r_0 is sufficiently small, then $(t, y) \mapsto X^t(y)$ C^1 embeds $\{(t, y) \in \mathbb{R} \times \mathcal{N}_{x,r} : 0 \le t \le \tau(y)\}$ for $0 < r \le r_0$. The image $\{X^t(y) : y \in \mathcal{N}_{x,r} \text{ and } 0 \le t \le \tau(y)\}$ is denoted by $F_x(X^t, r, T)$ and is called the *flow box* of x. The following is a vector field analogue of Franks' lemma (see [27]).

Lemma 3.2. *Let $p \in \gamma \in Per(X)$ with period $\pi(p) > 0$ and let $f : \mathcal{N}_{p,r_1} \to \mathcal{N}_p$ be the Poincaré map of X for some $r_1 > 0$. Let $\mathcal{U}(X) \subset \mathfrak{X}(M)$ be a C^1 neighborhood of X, and let $0 < r \le r_1$ be given. Then there are $\delta > 0$ and $0 < \epsilon_0 < r/2$ such that for a linear map $L : N_p \to N_p$ with $\|L - D_pf\| < \delta$, there exists $Y \in \mathcal{U}(X)$ having the following properties*:

(a) $Y(x) = X(x)$ *if* $x \notin F_p(X^t, r, \pi(p))$;
(b) $p \in \gamma \in Per(Y)$;
(c) $g(x) = \begin{cases} \exp_p \circ L \circ \exp_p^{-1}(x), & \text{if } x \in B_{\epsilon_0/4}(p) \cap \mathcal{N}_{p,r}, \\ f(x), & \text{if } x \notin B_{\epsilon_0}(p) \cap \mathcal{N}_{p,r}, \end{cases}$

where $B_{\epsilon_0}(x)$ is an ϵ_0 neighborhood of $x \in M$, and $g : \mathcal{N}_{p,r} \to \mathcal{N}_p$ is the Poincaré map of Y.

Lemma 3.3. *Let $Sing(X) = \emptyset$. If X is not star, then there is Y that is C^1 close to X such that the Poincaré map f of Y has a closed small arc $\mathcal{J}$ whose endpoints are hyperbolic.*

Proof. As X is not star, we may assume that there is Y that is C^1 close to X such that Y has a non-hyperbolic closed orbit η. Let $q \in \eta$ and let $f : \mathcal{N}_{q,r} \to \mathcal{N}_q$ (for some $r > 0$) be the Poincaré map of Y. As q is not hyperbolic, there is an eigenvalue λ of $D_q f$ such that $|\lambda| = 1$. As $|\lambda| = 1$, we may assume that $\lambda = 1$ (the other case is similar). Then by Lemma 3.2, there are $\delta > 0$ and $0 < \epsilon_0 < r/2$ such that for a linear map $L : N_q \to N_q$ with $\|L - D_q f\| < \delta$, there exists Z that is C^1 closed to Y satisfying

(a) $Z(x) = Y(x)$ if $x \notin F_q(X^t, r, \pi(q))$,
(b) $q \in \eta \in Per(Z)$, and
(c) $g(x) = \begin{cases} \exp_q \circ L \circ \exp_q^{-1}(x), & \text{if } x \in B_{\epsilon_0/4}(q) \cap \mathcal{N}_{q,r}, \\ f(x), & \text{if } x \notin B_{\epsilon_0}(q) \cap \mathcal{N}_{q,r}, \end{cases}$

where $B_{\epsilon_0}(x)$ is an ϵ_0 neighborhood of $x \in M$, and $g : \mathcal{N}_{q,r} \to \mathcal{N}_q$ is the Poincaré map of Y.

Let $u \in T_q M$ be a nonzero vector such that u is associated with the eigenvalue λ and $\|u\| = \epsilon_0/4$. Then we have

$$g(\exp_q(u)) = \exp_q \circ L \circ \exp_q^{-1}(\exp_q(u)) = \exp_q(u).$$

Let $J = \{tu : -\epsilon_0/4 \le t \le \epsilon_0/4\}$ and let $\mathcal{J} = \exp_q(J)$. Then $g|_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$ is the identity. Let $r_1, r_2 \in \mathcal{J}$ be the endpoints of $\mathcal{J}$. Again, using Lemma 3.2, there is W that is C^1 close to Z (also, C^1 close to X) such that r_1' and $r_2' \in Per(h)$ are hyperbolic, and $h|_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$ is the identity map, where h is the Poincaré map of W. □

Let $p \in \gamma \in Per(X)$ and let $f : \mathcal{N}_{p,r} \to \mathcal{N}_p$ be the Pioincaré map of $X \in \mathfrak{X}(M)$. For any $\epsilon > 0$, a closed small arc $\mathcal{I}$ is called an *ϵ-periodic curve* if

(a) $f(\mathcal{I}) = \mathcal{I}$,
(b) the length of $f(\mathcal{I})$ is less than or equal to ϵ and the endpoints are hyperbolic, and
(c) $\mathcal{I}$ is normally hyperbolic (see [12]).

Remark 3.4. Let $X \in \mathfrak{X}(M)$ and $\mathcal{I}$ be a normally hyperbolic manifold. Then, there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that for any $Y \in \mathcal{U}(X)$, there is a closed small curve $\mathcal{J}$ close to $\mathcal{I}$ such that $\mathcal{J}$ is also normally hyperbolic (see [4]).

The following is a vector field analogue of the result on diffeomorphisms in [20].

Proposition 3.5. *There is a residual set $\mathcal{R} \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{R}$, either* (a) *X is star or* (b) *X has an ϵ-periodic closed arc $\mathcal{I}$.*

Proof. As M is compact, there is a countable basis $\{U_n\}_{n \in \mathbb{N}}$ for the topology of M. For any $\epsilon > 0$, let $\mathcal{H}_n(\epsilon) = \{Y \in \mathfrak{X}(M) : Y$ has an ϵ-periodic closed curve

$\mathcal{I}$ and the endpoints are hyperbolic}. By Remark 3.4 and the hyperbolicity of closed orbits, $\mathcal{H}_n(\epsilon)$ is open in $\mathfrak{X}(M)$. Let $\mathcal{N}_n(\epsilon) = \mathfrak{X}(M) - \overline{\mathcal{H}_n(\epsilon)}$. Then $\mathcal{H}_n(\epsilon) \cup \mathcal{N}_n(\epsilon)$ is an open dense subset of $\mathfrak{X}(M)$. Let

$$\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{H}_n(\epsilon) \cup \mathcal{N}_n(\epsilon).$$

Then $\mathcal{R}$ is a residual subset of $\mathfrak{X}(M)$. Let $X \in \mathcal{R}$, and we assume that X is not star. By Lemma 3.3, $X \in \overline{\mathcal{H}_n(\epsilon)}$ for any $n \in \mathbb{N}$. Then $X \notin \mathcal{N}_n(\epsilon)$, and therefore $X \in \mathcal{H}_n(\epsilon)$. Thus, X has an ϵ-periodic closed arc $\mathcal{I}$ and the endpoints are hyperbolic. □

Proposition 3.6. *For C^1 generic $X \in \mathfrak{X}(M)$, if X is continuum-wise expansive, then $X \in \mathcal{G}^*(M)$.*

Proof. Let $X \in \mathcal{R}$ be continuum-wise expansive. We assume toward a contradiction that $X \notin \mathcal{G}^*(M)$. As X is continuum-wise expansive, by Lemma 3.1, $Sing(X) = \emptyset$. As $X \in \mathcal{R}$, by Proposition 3.5, X has an ϵ-periodic closed arc $\mathcal{I}$. It is clear that $\mathcal{I} \subset M$ is a nontrivial continuum. Let $\delta = \epsilon$ be the constant of continuum-wise expansiveness for X. As $\mathcal{I}$ is an ϵ-periodic curve,

$$\text{diam}(f^i(\mathcal{I})) = \text{diam}(\mathcal{I}) \leq \delta$$

for all $i \in \mathbb{Z}$, where f is the Poincaré map of X. As $\mathcal{I}$ is not a singleton, f is not continuum-wise expansive. By [2, Theorem 3.2], X is not continuum-wise expansive, which is a contradiction. □

Proof of Theorem A. Let $X \in \mathcal{R}$ be continuum-wise expansive. By Proposition 3.6, $X \in \mathcal{G}^*(M)$. By the result in [11], if $Sing(X) = \emptyset$ and $X \in \mathcal{G}^*(M)$, then X satisfies Axiom A and has no cycles. □

4. Proof of Theorem B

In this section, it is proved that for a C^1 generic vector field X, if a homoclinic class $H(\gamma, X)$ is continuum-wise expansive, then $H(\gamma, X)$ has no singular points and is hyperbolic. The following provides information on singularities in homoclinic classes.

Lemma 4.1. *Let $\Lambda \subset M$ be a closed X^t-invariant subset of X. If Λ is continuum-wise expansive, then $\Lambda \cap Sing(X)$ is totally disconnected.*

Proof. We assume toward a contradiction that $\Lambda \cap Sing(X)$ is not totally disconnected. Then for any $\eta > 0$, there is a closed set $A \subset \Lambda \cap Sing(X)$ such that A is a continuum with $\text{dim}A < \eta$. Let $\alpha : A \to \text{Hom}(\mathbb{R}, 0)$ such that $\alpha(x)(t) = \alpha_x(t) = id$ for all $x \in A$ and all $t \in \mathbb{R}$. As $A \subset \Lambda \cap Sing(X)$, we have $X^t(A) = A$ for all $t \in \mathbb{R}$. Let $\delta = 2\eta$ and let $\mathcal{X}^t_\alpha(A) = \{X^{\alpha_x(t)}(x) : \forall x \in A, \alpha \in \mathcal{H}(A), \forall t \in \mathbb{R}\}$. Then

$$\text{diam}(\mathcal{X}^t_\alpha(A)) = \text{diam}(X^t(A)) = \text{diam}(A) < \delta.$$

As A is not an orbit, this is a contradiction. Thus, if Λ is continuum-wise expansive, then $\Lambda \cap Sing(X)$ is totally disconnected. $\square$

In general, a homoclinic class $H(\gamma, X)$ has a singular point (see [3]). The following lemma shows that if a homoclinic class $H(\gamma, X)$ is continuum-wise expansive, then it does not contain singular points.

Lemma 4.2. *Let $\gamma \in Per(X)$ be hyperbolic. If $H(\gamma, X)$ is continuum-wise expansive, then $H(\gamma, X) \cap Sing(X) = \emptyset$.*

Proof. We assume toward a contradiction that $H(\gamma, X) \cap Sing(X) \neq \emptyset$. As $H(\gamma, X)$ is continuum-wise expansive, by Lemma 4.1, $H(\gamma, X) \cap Sing(X)$ is totally disconnected. This is a contradiction because M is connected. Thus, $H(\gamma, X) \cap Sing(X) = \emptyset$. $\square$

A closed orbit γ is *weak hyperbolic* if for any $p \in \gamma$ and any $\delta > 0$, $D_p f$ has an eigenvalue μ such that $(1-\delta) \leq |\mu| \leq (1+\delta)$, where f is the Poincaré map of X.

The proof of the following lemma is similar to that in [21], which is still unpublished. For convenience, a proof is provided here.

Lemma 4.3. *Let $\eta \in H(\gamma, X) \cap Per(X)$ with $\eta \sim \gamma$. If η is a weak hyperbolic closed orbit, then there is Y that is C^1 close to X such that g has a small arc $\mathcal{J}$ whose endpoints are homoclinically related to $p \in \gamma_Y$, where g is the Poincaré map of Y, and γ_Y is the continuation of γ.*

Proof. Let $\eta \in H(\gamma, X) \cap Per(X)$ with $\eta \sim \gamma$. We assume that η is a weak hyperbolic closed orbit. Let $q \in \eta$ and let $f : \mathcal{N}_{q,r} \to \mathcal{N}_q$ (for some $r > 0$) be the Poincaré map of X. As η is a weak hyperbolic closed orbit of X, by Lemma 3.2, there are $\delta > 0$ and $0 < \epsilon_0 < r/2$ such that for a linear map $L : N_q \to N_q$ with $\|L - D_q f\| < \delta$, there is Y that is C^1 closed to X such that

(a) $Y(x) = X(x)$ if $x \notin F_q(X^t, r, \pi(q))$,
(b) $q \in \eta \in Per(Y)$,
(c) $\eta \sim \gamma_Y$, and
(d) $g(x) = \begin{cases} \exp_q \circ L \circ \exp_q^{-1}(x), & \text{if } x \in B_{\epsilon_0/4}(q) \cap \mathcal{N}_{q,r}, \\ f(x), & \text{if } x \notin B_{\epsilon_0}(q) \cap \mathcal{N}_{q,r}, \end{cases}$

where $B_{\epsilon_0}(x)$ is an ϵ_0 neighborhood of $x \in M$, and $g : \mathcal{N}_{q,r} \to \mathcal{N}_q$ is the Poincaré map of Y. Then $D_q g$ has an eigenvalue μ such that $|\mu| = 1$. As $|\mu| = 1$, we may assume that $\mu = 1$ (the other case similar). Let u be a nonzero vector that is associated with the eigenvalue μ and $\|u\| = \epsilon_0/4$. Then

$$g(\exp_q(u)) = \exp_q \circ L \circ \exp_q^{-1}(\exp_q(u)) = \exp_q(u).$$

Let $J = \{tu : -\epsilon_0/4 \leq t \leq \epsilon_0/4\}$ and $\mathcal{J} = \exp_q(J)$. Then $g|_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$ is the identity. Let q_1 and q_2 be the endpoints of $\mathcal{J}$. Then, as in the proof in [33, Proposition 3], there is h that is C^1 close to g (also C^1 close to f) such that q_1, q_2, and q are the only periodic points of h, $h|_{\mathcal{J}_1}$ is the identity, and $q_1 \sim p_h \in \gamma_Z$ and $q_2 \sim p_h \in \gamma_Z$, where γ_Z is the continuation of γ, and the

Poincaré map h is associated with a vector field Z that is C^1 close to Y. Thus, if η is a weak hyperbolic closed orbit, then there is Z that C^1 close to X such that h has a small arc $\mathcal{J}_1$ whose endpoints q_1 and q_2 are homoclinically related to $p_h \in \gamma_Z$, where h is the Poincaré map of Z. This proves Lemma 4.3. □

Yang and Gan [39] introduced the following vector field analogue. Let $H(\gamma, X)$ be a homoclinic class. For any $\epsilon > 0$, a closed small curve $\mathcal{I}$ is called *ϵ-periodically simple* if

(a) $f(\mathcal{I}) = \mathcal{I}$,
(b) the length of $f(\mathcal{I})$ is less than or equal to ϵ and the endpoints are homoclinically related to $p \in \gamma$, and
(c) $\mathcal{I}$ is normally hyperbolic (see [12]), where f is the Poincaré map of X.

This slightly differs from an ϵ-simple curve in (b). It should be noted that if $\gamma \in Per(X)$ is a 2-weak hyperbolic closed orbit of X, then for any $\delta > 0$, there is an eigenvalue μ of D_pf such that $(1 - 2\delta) \leq |\mu| \leq (1 + 2\delta)$ in the notion above.

Lemma 4.4. *There is a residual set $\mathcal{G}_1 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}_1$ and any hyperbolic closed orbit γ of X, we have*:

(a) *For any $\epsilon > 0$, if for any C^1 neighborhood $\mathcal{U}(X)$ of X, there is $Y \in \mathcal{U}(X)$ that has an ϵ-periodically simple curve $\mathcal{J}$ such that the two endpoints of $\mathcal{J}$ are homoclinically related to γ_Y, then X has a 2ϵ-periodically simple curve $\mathcal{L}$ such that the two endpoints of $\mathcal{L}$ are homoclinically related to γ* (*see* [39, Lemma 2.1]).
(b) *For any $\delta > 0$, if for any C^1 neighborhood $\mathcal{U}(X)$ of X, there is $Y \in \mathcal{U}(X)$ that has a periodic orbit $\eta \sim \gamma_Y$ with weak hyperbolic orbit, then X has a periodic orbit $\eta \sim \gamma$ with 2-weak hyperbolic orbit* (*see* [39, Lemma 2.1]).
(c) *For any $\delta > 0$, if X has a periodic closed orbit $\eta \sim \gamma$ with weak periodic orbit, then X has a periodic orbit $\tau \sim \gamma$ with weak periodic orbit whose eigenvalues are all real* (*see* [39, Lemma 2.3]).

Lemma 4.5. *There is a residual set $\mathcal{G}_2 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}_2$, we have*:

(a) *X is Kupka–Smale, that is, every $\sigma \in Crit(X)$ is hyperbolic and its stable and unstable manifolds intersect transversally* (*see* [16]).
(b) *$H(\gamma, X) = C(\gamma, X)$ for some hyperbolic closed orbit γ* (*see* [7]).

Lemma 4.6. *There is a residual set $\mathcal{G}_3 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}_3$, if $H(\gamma, X)$ is continuum-wise expansive, then every $\eta \in H(\gamma, X) \cap Per(X)$ with $\eta \sim \gamma$ is not a weak hyperbolic closed orbit of X.*

Proof. Let $X \in \mathcal{G}_3 = \mathcal{G}_1 \cap \mathcal{G}_2$, and let $H(\gamma, X)$ be continuum-wise expansive. We assume that there is $\eta \in H(\gamma, X) \cap Per(X)$ with $\eta \sim \gamma$ such that η is a weak hyperbolic closed orbit. By Lemma 4.3, for any $\epsilon > 0$, there is Y that is C^1 close to X such that g has a small arc $\mathcal{J}$ whose endpoints are homoclinically related to $p_Y \in \gamma_Y$, and is an ϵ-periodically simple curve, where g is the Poincaré map of Y and γ_Y is the continuation of γ. By Lemmas 4.4 and 4.5, f has a small arc $\mathcal{L} \subset H(\gamma, X) = C(\gamma, X)$ whose endpoints are homoclinically related to $p \in \gamma$, and is a 2ϵ-periodically simple curve. Let $\delta = 2\epsilon$ be the constant of continuum-wise expansiveness. As $\mathcal{L} \subset H(\gamma, X)$ is a 2ϵ-periodically simple curve, $f^i(\mathcal{L}) = \mathcal{L}$ for all $i \in \mathbb{Z}$. Thus, $\mathrm{diam}(f^i(\mathcal{L})) = \mathrm{diam}\mathcal{L} \leq \delta$ for all $i \in \mathbb{Z}$. It is clear that $\mathcal{L}$ is not a singleton. Thus, by [2, Theorem 3.2], $H(\gamma, X)$ is not continuum-wise expansive, which is a contradiction. □

The following is a vector field analogue of the result in [38]. It provides information on nonhyperbolic homoclinic classes if a homoclinic class $H(\gamma, X)$ does not contain any singular points.

Lemma 4.7 ([38])**.** *There is a residual set $\mathcal{G}_4 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}_4$, if a homoclinic class $H(\gamma, X)$ does not contain any singular points and $H(\gamma, X)$ is not hyperbolic, then there is $\eta \in H(\gamma, X) \cap Per(X)$ with $\eta \sim \gamma$ such that η is a weak hyperbolic periodic orbit of X.*

Proof of Theorem B. Let $X \in \mathcal{G}_3 \cap \mathcal{G}_4$ and $H(\gamma, X)$ be continuum-wise expansive. We assume that $H(\gamma, X)$ is not hyperbolic. As $H(\gamma, X)$ is continuum-wise expansive $X \in \mathcal{G}_4$, by Lemma 4.7, there is $\eta \in H(\gamma, X) \cap Per(X)$ with $\eta \sim \gamma$ such that η is a weak hyperbolic closed orbit of X. As $H(\gamma, X)$ is continuum-wise expansive, by Lemma 4.6, X has no weak hyperbolic closed orbits. This is a contradiction. Thus, C^1 generically, if $H(\gamma, X)$ is continuum-wise expansive, then $H(\gamma, X)$ is hyperbolic. □

5. Divergence-free vector fields

Let M be a compact, connected, and smooth $n(\geq 3)$-dimensional Riemannian manifold endowed with a volume form with respect to Lebesgue measure μ. Let $\mathfrak{X}_\mu(M)$ denote the space of C^1 divergence-free vector fields, and we consider the usual C^1 Whitney topology on this space.

A vector field X is called *divergenc-free* if its divergence is equal to zero. It is known that by Liouville's formula, a flow X^t is volume preserving if and only if the corresponding $X \in \mathfrak{X}_\mu(M)$ is divergence-free. Ferreira [10] proved that if a divergence-free vector field X belongs to the C^1 interior of the set of all expansive divergence-free vector fields, then it is Anosov. Lee [19] proved that if a divergence-free vector field X belongs to the C^1 interior of the set of all continuum-wise expansive divergence-free vector fields, then it is Anosov. Bessa, Lee, and Wen [6] proved that C^1 generically, if a divergence-free vector field X is expansive, then it is Anosov. Here, the following analogue is proved.

Theorem C. *For C^1 generic $X \in \mathfrak{X}_\mu(M)$, if a vector field X is continuum-wise expansive, then it is Anosov.*

Proof. By the result in [5], for a C^1 generic vector field $X \in \mathfrak{X}_\mu(M)$, $M = H(\gamma, X)$. As in the proof of Theorem B, we have that $H(\gamma, X)$ is hyperbolic. Thus, C^1 generically, if X is continuum-wise expansive, then X is Anosov. □

References

[1] A. Arbieto, *Peirodic orbits and expansiveness*, Math. Z. **269** (2011), 801–807.

[2] A. Arbieto, W. Cordeiro and M. J. Pacifico, *Continuum-wise expansivity and entropy for flows*, to appear in Ergodic Theory Dynam. Systems.

[3] S. Bautista, *The geometric Lorenz attractor is a homoclinic class*, Bol. Mat. **11** (2004), 69–78.

[4] P. Bergera and A. Bounemourab, *A geometrical proof of the persistence of normally hyperbolic submanifolds*, Dynam. Syst. **28**(2013), 567–581.

[5] M. Bessa, *A generic incompressible flow is topological mixing*, C. R. Math. Acad. Sci. Paris **346** (2008), 1169–1174.

[6] M. Bessa, M. Lee and X. Wen, *Shadowing, expansiveness and specification for C^1-conservative systems*, Acta Math. Sci. **35** (2015), 583–600.

[7] C. Bonatti and S. Crovisier, *Récurrence et généricité*, Invent. Math. **158** (2004), 180–193.

[8] R. Bowen and P. Walters, *Expansive one-parameter flows*, J. Diff. Eqns. **12** (1972), 180–193.

[9] T. Das, K. Lee and M. Lee, *C^1-persistently continuum-wise expansive homoclinic classes and recurrent sets*, Topol. Its Appl. **160** (2013), 350–359.

[10] C. Ferreira, *Shadowing, expansiveness and stability of divergence-free vector fields*, Bull. Korean Math. Soc. **51** (2014), no. 1, 67–76. `https://doi.org/10.4134/BKMS.2014.51.1.067`

[11] S. Gan and L. Wen, *Nonsingular star flows satisfy Axiom A and the no-cycle condition*, Invent. Math. **164** (2006), no. 2, 279–315. `https://doi.org/10.1007/s00222-005-0479-3`

[12] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin, 1977.

[13] H. Kato, *Continuum-wise expansive homeomorphisms*, Canad. J. Math. **45** (1993), no. 3, 576–598. `https://doi.org/10.4153/CJM-1993-030-4`

[14] M. Komuro, *Expansive properties of Lorenz attractors*, in The theory of dynamical systems and its applications to nonlinear problems (Kyoto, 1984), 4–26, World Sci. Publishing, Singapore, 1984.

[15] N. Koo, K. Lee, and M. Lee, *Generic diffeomorphisms with measure-expansive homoclinic classes*, J. Difference Equ. Appl. **20** (2014), no. 2, 228–236. `https://doi.org/10.1080/10236198.2013.829053`

[16] I. Kupka, *Contribution à la théorie des champs génériques*, Contributions to Differential Equations **2** (1963), 457–484.

[17] M. Lee, *Measure expansive homoclinic classes for generic diffeomorphisms*, Appl. Math. Sci. **73** (2015), 3623–3628.

[18] M. Lee, *Continuum-wise expansiveness for non-conservative or conservative systems*, Chaos Solitons & Fractals **87** (2016), 314–318.

[19] M. Lee, *Continuum-wise expansive homoclinic classes for generic diffeomorphisms*, Publ. Math. Debrecen **88** (2016), no. 1-2, 193–200.

[20] M. Lee, *Continuum-wise expansiveness for generic diffeomorphisms*, Nonlinearity **31** (2018), no. 6, 2982–2988. `https://doi.org/10.1088/1361-6544/aaba38`

[21] M. Lee, *Measure expansive homoclinic classes for C^1 generic vector fields*, Mathematics **8** (2020), 1232.

[22] K. Lee and M. Lee, *Hyperbolicity of C^1-stably expansive homoclinic classes*, Discrete Contin. Dyn. Syst. **27** (2010), no. 3, 1133–1145. `https://doi.org/10.3934/dcds.2010.27.1133`

[23] K. Lee and M. Lee, *Measure-expansive homoclinic classes*, Osaka J. Math. **53** (2016), no. 4, 873–887. `http://projecteuclid.org/euclid.ojm/1475601821`

[24] M. Lee and J. Oh, *Measure expansive flows for the generic view point*, J. Difference Equ. Appl. **22** (2016), no. 7, 1005–1018. https://doi.org/10.1080/10236198.2016.1172573
[25] S. Lee and J. Park, *Expansive homoclinic classes of generic C^1-vector fields*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 12, 1451–1458. https://doi.org/10.1007/s10114-016-5207-y
[26] R. Mañé, *Expansive diffeomorphisms*, in Dynamical systems—Warwick 1974 (Proc. Sympos. Appl. Topology and Dynamical Systems, Univ. Warwick, Coventry, 1973/1974; presented to E. C. Zeeman on his fiftieth birthday), 162–174, Lecture Notes in Math., Vol. 468, Springer, Berlin, 1975.
[27] K. Moriyasu, K. Sakai, and N. Sumi, *Vector fields with topological stability*, Trans. Amer. Math. Soc. **353** (2001), no. 8, 3391–3408. https://doi.org/10.1090/S0002-9947-01-02748-9
[28] K. Moriyasu, K. Sakai, and W. Sun, *C^1-stably expansive flows*, J. Differential Equations **213** (2005), no. 2, 352–367. https://doi.org/10.1016/j.jde.2004.08.003
[29] M. Oka, *Expansiveness of real flows*, Tsukuba J. Math. **14** (1990), no. 1, 1–8. https://doi.org/10.21099/tkbjm/1496161314
[30] M. J. Pacifico, E. R. Pujals, M. Sambarino, and J. L. Vieitez, *Robustly expansive codimension-one homoclinic classes are hyperbolic*, Ergodic Theory Dynam. Systems **29** (2009), no. 1, 179–200. https://doi.org/10.1017/S0143385708000175
[31] M. J. Pacifico, E. R. Pujals, and J. L. Vieitez, *Robustly expansive homoclinic classes*, Ergodic Theory Dynam. Systems **25** (2005), no. 1, 271–300. https://doi.org/10.1017/S0143385704000203
[32] K. Sakai, *Continuum-wise expansive diffeomorphisms*, Publ. Mat. **41** (1997), no. 2, 375–382. https://doi.org/10.5565/PUBLMAT_41297_04
[33] M. Sambarino and J. L. Vieitez, *On C^1-persistently expansive homoclinic classes*, Discrete Contin. Dyn. Syst. **14** (2006), no. 3, 465–481. https://doi.org/10.3934/dcds.2006.14.465
[34] M. Sambarino and J. L. Vieitez, *Robustly expansive homoclinic classes are generically hyperbolic*, Discrete Contin. Dyn. Syst. **24** (2009), no. 4, 1325–1333. https://doi.org/10.3934/dcds.2009.24.1325
[35] L. Senos, *Generic Bowen-expansive flows*, Bull. Braz. Math. Soc. (N.S.) **43** (2012), no. 1, 59–71. https://doi.org/10.1007/s00574-012-0005-3
[36] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. https://doi.org/10.1090/S0002-9904-1967-11798-1
[37] W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc. **1** (1950), 769–774. https://doi.org/10.2307/2031982
[38] X. Wang, *Hyperbolicity versus weak periodic orbits inside homoclinic classes*, Ergodic Theory Dynam. Systems **38** (2018), no. 6, 2345–2400. https://doi.org/10.1017/etds.2016.122
[39] D. Yang and S. Gan, *Expansive homoclinic classes*, Nonlinearity **22** (2009), no. 4, 729–733. https://doi.org/10.1088/0951-7715/22/4/002

Manseob Lee
Department of Marketing Big Data
Mokwon University
Daejeon 35349, Korea
Email address: lmsds@mokwon.ac.kr