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WEAKLY BERWALD SPACE WITH A SPECIAL (α, β)-METRIC

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Abstract. As a generalization of Berwald spaces, we have the ideas of Douglas spaces and Landsberg spaces. S. Bacso defined a weakly-Berwald space as another generalization of Berwald spaces. In 1972, Matsumoto proposed the (α, β) metric, which is a Finsler metric derived from a Riemannian metric α and a differential 1-form β. In this paper, we investigated an important class of (α, β)-metrics of the form $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\beta^2}{\alpha}$, which is recognized as a special form of the first approximate Matsumoto metric on an n-dimensional manifold, and we obtain the criteria for such metrics to be weakly-Berwald metrics. A Finsler space with a special (α, β)-metric is a weakly Berwald space if and only if B^m_m is a 1-form. We have shown that under certain geometric and algebraic circumstances, it transforms into a weakly Berwald space.

1. Introduction

In 1972 [13], Matsumoto introduced the concept of a (α, β)-metric on a Finsler space $F^n = (M^n, F)$ and it has been studied by numerous authors [1, 4, 6, 7, 9, 11, 14, 17]. The study of several well-known metrics, such as the Randers metric and the Kropina metric, has significantly contributed to the expansion of Finsler geometry and its applications to relativity theory. A Finsler metric $F(x, y)$ is known as (α, β)-metric, if F is a positively homogeneous function of α and β of degree one, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and 1-form $\beta = b_i(x)y^i$ on M^n.

Let $F^n = (M^n, F)$ be an n-dimensional Finsler space, where M^n be an n-dimensional differential manifold and fundamental function F. Let the fundamental tensor $g_{ij} = \dot{\partial}_i\dot{\partial}_j\dfrac{F^2}{2}$, where $\dot{\partial}_i$ represents $\dfrac{\partial}{\partial y^i}$ and we define G_i as follows

$$G_i = \frac{1}{4}\left(y^r(\partial_r\dot{\partial}_iF^2) - \partial_iF^2\right),$$

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and $G^i = g^{ij} G_j$. Here ∂_i means $\dfrac{\partial}{\partial x^i}$ and g^{ij} is inverse of g_{ij} fundamental tensor. The coefficients of (G^i_{jk}, G^i_j) of the Berwald connection $B\Gamma$ are determined as $G^i_{jk} = \dot{\partial}_k G^i_j$ and $G^i_j = \dot{\partial}_j G^i$. A Berwald space is a Finsler space that satisfy the criterion $G^h_{ijk} = 0$, which means that the Berwald connection coefficients G^h_{ij} are functions of the position (x^i) alone. Thus the equation $y_r G^r_{ijk} = 0$ holds, so $2G^i = G^i_{rs} y^r y^s$ are homogeneous polynomials of degree two in (y^i), so $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials of degree three in (y^i). Then, as two extensions of Berwald spaces, we can study the concepts of Landsberg spaces and Douglas spaces. The third extension of Berwald spaces is the concept of weakly-Berwald spaces. As a result, if a Finsler space satisfies the criterion $G_{ij} = 0$, it is referred to as a weakly-Berwald space.

Berwald space is a Finsler space, if G^i_{jk} are the functions of position alone, that is, Berwald connection $B\Gamma$ is linear. If the (hv)-Ricci curvature tensor $G_{jk} = 0$, a Finsler space is said to be a weakly Berwald space. The spray functions G^i of a Finsler space with an (α,β)-metric are given by $2G^i = \gamma^i_{00} + 2B^i$, where γ^i_{jk} represents the Christoffel symbols in the associated Riemannian space (M^n, α). Then we have $G^i_{jk} = B^i_{jk} + \gamma^i_{jk}$ and $G^i_j = B^i_j + \gamma^i_{0j}$, where $\dot{\partial}_k B^i_j = B^i_{jk}$ and $\dot{\partial}_j B^i = B^i_j$. A Finsler space with an (α,β)-metric is a weakly Berwald space if and only if $B^m_m = \dfrac{\partial B^m}{\partial y^m}$ is an one-form.

In [3], Bacso and Szilagyi proposed the concept of weakly-Berwald space as another extension of Berwald spaces as well as a necessary condition for the existence of a weakly Berwald Finsler space of Kropina type. L. Lee and M. Lee have investigated weakly Berwald spaces with special (α,β)-metric in [12]. In 2004, Yoshikawa et al. [19] developed the conditions for generalised Kropina and Matsumoto spaces to be weakly-Berwald and Berwald spaces, respectively. In [18], Tayebi obtained a new class of weakly Berwald Finsler metric. Shanker and Choudhary has obtained the conditions for Finsler space with a second approximate Matsumoto metric to be weakly Berwald space in [16]. In [15], Narasimhamurthy has proved that under some conditions, a Finsler space with special (α,β)-metric becomes a weakly-Berwald space. Recently, Khoshdani and Abazari [5] have discussed the characteristics of weakly Berwald space for fourth-root (α,β)-metric. Pradeep and Ajaykumar [10] have examined the weakly Berwald space with special (α,β)-metric.

In this paper, we extend the study on weakly Berwald spaces with a special form of the first approximate Matsumoto metric. We proposed a special (α,β)-metric

$$F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha},$$

where μ_1, μ_2 and μ_3 are constants in [8], which is a special form of the first approximate Matsumoto metric. Firstly, we gave a brief introduction to Berwald and Weakly-Berwald space in section one. We have discussed the basic notations and conditions for a Finsler space F^n with an (α, β)-metric to be a weakly Berwald space in section two. Finally, we obtained the conditions for Finsler space to be weakly Berwald space with a special form of the first approximate Matsumoto metric F.

2. Weakly-Berwald space with respect to (α, β)-metric

This section discusses the conditions for a Finsler space with a (α, β)-metric to be a weakly-Berwald space.

Let $F^n = (M^n, F)$ be a Finsler space defined on n-dimensional differential manifold M equipped with (α, β)-metric $F(\alpha, \beta)$, where Riemannian metric $\alpha^2 = a_{ij}(x)y^i y^j$ and one-form $\beta = b_i(x)y^i$. The symbol (;) in this paper stands for h-covariant derivation in the space (M, α) with regard to the Riemannian connection, while γ^i_{jk} stands for Christoffel symbols in the space (M, α). The notations are as follows [3]:

$$
\begin{aligned}
&i. \quad b^2 = a^{rs} b_r b_s, \quad b^i = a^{ir} b_r,\\
&ii. \quad 2r_{ij} = b_{j;i} + b_{i;j}, \quad 2s_{ij} = b_{i;j} - b_{j;i},\\
&iii. \quad r^i_j = a^{ir} r_{rj}, \quad s_i = b_r s^r_i, \quad s^i_j = s_{rj}, \quad r_i = b_r r^r_i.
\end{aligned}
$$

Now, we consider the function $G^i(x, y)$ of F^n with an (α, β)-metric. According to [13], they are being written in the form

$$
\begin{aligned}
2G^m &= 2B^m + \gamma^m_{00},\\
B^m &= \frac{\alpha F_\beta}{F_\alpha} s^m_0 + \frac{E^*}{\alpha} y^m - \frac{\alpha F_{\alpha\alpha}}{F_\alpha}\left(\frac{1}{\alpha} y^m - \frac{\alpha}{\beta} b^m\right) C^*,
\end{aligned}
\tag{1}
$$

where

$$
\begin{aligned}
C^* &= \frac{\alpha\beta\left(r_{00} F_\alpha - 2\alpha s_0 F_\beta\right)}{2\left(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}\right)},\\
\gamma^2 &= b^2\alpha^2 - \beta^2, \quad E^* = \left(\frac{\beta F_\beta}{F}\right) C^*
\end{aligned}
\tag{2}
$$

and

$$
F_\alpha = \frac{\partial F}{\partial \alpha}, \quad F_\beta = \frac{\partial F}{\partial \beta}, \quad F_{\alpha\alpha} = \frac{\partial^2 F}{\partial \alpha^2}, \quad F_{\alpha\beta} = \frac{\partial^2 F}{\partial\alpha\partial\beta}, \quad F_{\alpha\alpha\alpha} = \frac{\partial^3 F}{\partial \alpha^3}.
\tag{3}
$$

Since, $\gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k$ are homogeneous polynomial in $\left(y^i\right)$ of degree two, it is well-known that a Finsler space with an $(\alpha, \beta)-$ metric is a Berwald space, if and only if B^m are homogeneous polynomial in $\left(y^i\right)$ of degree two and Berwald connection $B\Gamma$ is linear.

Differentiating equation (1) by y^n and contracting m and n in the obtained equation, we get

(4)
$$\begin{aligned} B^m_m =& \left\{\dot{\partial}_m\left(\frac{\beta F_\beta}{\alpha F}\right)y^m + \frac{n\beta F_\beta}{\alpha F} - \dot{\partial}_m\left(\frac{\alpha F_{\alpha\alpha}}{F_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right)\right\}C^* \\ &- \frac{\alpha F_{\alpha\alpha}}{F_\alpha}\left\{\dot{\partial}_m\left(\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \dot{\partial}_m\left(\frac{\alpha}{\beta}\right)b^m\right\}C^* + \dot{\partial}_m\left(\frac{\alpha F_\beta}{F_\alpha}\right)s^m_0 \\ &+ \left(\frac{\beta F_\alpha F_\beta - \alpha F F_{\alpha\alpha}}{\alpha F F_\alpha}\right)\left(\dot{\partial}_m C^*\right)y^m + \left(\frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha}\right)\left(\dot{\partial}_m C^*\right)b^m. \end{aligned}$$

Since $F = F(\alpha, \beta)$ is a positively homogeneous function of α and β of degree one, we have

$$\begin{gathered} F_\alpha \alpha + F_\beta \beta = F, \quad F_{\alpha\alpha}\alpha + F_{\alpha\beta}\beta = 0, \\ F_{\beta\alpha}\alpha + F_{\beta\beta}\beta = 0, \quad F_{\alpha\alpha\alpha}\alpha + F_{\alpha\alpha\beta}\beta = -F_{\alpha\alpha}. \end{gathered}$$

Using the above inequalities and the homogeneity of $\left(y^i\right)$, we obtain the following

$$\dot{\partial}_m\left(\frac{\beta F_\beta}{\alpha F}\right)y^m = -\frac{\beta F_\beta}{\alpha F}, \tag{5}$$

$$\begin{aligned} \dot{\partial}_m\left(\frac{\alpha F_{\alpha\alpha}}{F_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right) =& \frac{\gamma^2}{\left(\beta F_\alpha\right)^2}\left\{F_\alpha F_{\alpha\alpha} + \alpha F_\alpha F_{\alpha\alpha\alpha}\right. \\ & \left. - \alpha\left(F_{\alpha\alpha}\right)^2\right\}, \end{aligned} \tag{6}$$

$$\left\{\dot{\partial}_m\left(\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \dot{\partial}_m\left(\frac{\alpha}{\beta}\right)b^m\right\} = \frac{1}{\alpha\beta^2}\left\{\gamma^2 + (n-1)\beta^2\right\}, \tag{7}$$

$$\left(\dot{\partial}_m C^*\right)y^m = 2C^*, \tag{8}$$

$$\begin{aligned} \left(\dot{\partial}_m C^*\right)b^m =& \frac{1}{2\alpha\beta\Omega^2}\left[\Omega\{\beta\left(\gamma^2 + 2\beta^2\right)W + 2\alpha^2\beta^2 F_\alpha r_0 - \alpha\beta\gamma^2 F_{\alpha\alpha}\right. \\ & r_{00} - 2\alpha\left(\beta^3 F_\beta + \alpha^2\gamma^2 F_{\alpha\alpha}\right)s_0\} - \alpha^2\beta W\{2b^2\beta^2 F_\alpha \\ & \left. - \gamma^4 F_{\alpha\alpha\alpha} - b^2\alpha\gamma^2 F_{\alpha\alpha}\}\right], \end{aligned} \tag{9}$$

$$\dot{\partial}_m\left(\frac{\alpha F_\beta}{F_\alpha}\right)s^m_0 = \frac{\alpha^2 F F_{\alpha\alpha} s_0}{\left(\beta F_\alpha\right)^2}, \tag{10}$$

where

$$\begin{aligned} & W = \left(r_{00}F_\alpha - 2\alpha s_0 F_\beta\right), \\ & \Omega = \left(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}\right), \quad \text{provided that } (\Omega \neq 0) \\ & Y_i = a_{ir}y^r, \quad s_{00} = 0, \quad b^r s_r = 0, \quad a^{ij}s_{ij} = 0. \end{aligned} \tag{11}$$

Substituting (2)-(3) and (5)-(10) into (4), we get

$$B^m_m = \frac{1}{2\alpha F\left(\beta F_\alpha\right)^2\Omega^2}\left[2\Omega^2 AC^* + 2\alpha F\Omega^2 Bs_0 + \alpha^2 FF_\alpha F_{\alpha\alpha}\right.$$
$$\left.(Cr_{00} + Ds_0 + Er_0)\right], \tag{12}$$

where

$$A = (t+1)\beta^2 F_\alpha\left(\beta F_\alpha F_\beta - \alpha FF_{\alpha\alpha}\right) + \alpha\gamma^2 F\left\{\alpha\left(F_{\alpha\alpha}\right)^2 - 2F_\alpha F_{\alpha\alpha}\right.$$
$$\left. - \alpha F_\alpha F_{\alpha\alpha\alpha}\right\},$$
$$B = \alpha^2 FF_{\alpha\alpha},$$
$$C = \beta\gamma^2\left\{-\beta^2\left(F_\alpha\right)^2 + 2b^2\alpha^3 F_\alpha F_{\alpha\alpha} - \alpha^2\gamma^2\left(F_{\alpha\alpha}\right)^2 + \alpha^2\gamma^2 F_\alpha F_{\alpha\alpha\alpha}\right\},$$
$$D = 2\alpha\left\{\beta^3\left(\gamma^2 - \beta^2\right)F_\alpha F_\beta - \alpha^2\beta^2\gamma^2 F_\alpha F_{\alpha\alpha} - 2\alpha\beta\gamma^2\left(\gamma^2 + 2\beta^2\right)\right.$$
$$\left.F_\beta F_{\alpha\alpha} - \alpha^3\gamma^4\left(F_{\alpha\alpha}\right)^2 - \alpha^2\beta\gamma^4 F_\beta F_{\alpha\alpha\alpha}\right\},$$
$$E = 2\alpha^2\beta^2 F_\alpha\Omega.$$

Summarizing the above, we have

Theorem 2.1. *A Finsler space F^n with an (α,β)-metric is a weakly-Berwald space if $G^m_m = B^m_m + \gamma^m_{0m}$ is a homogeneous polynomial in (y^m) of degree one, where B^m_m is given by equations (11) and (12), provided that $\Omega \neq 0$.*

Lemma 2.2. [4] *If $\alpha^2 \equiv 0(\mathrm{mod}\,\beta)$, that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case we have 1 -form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

3. Finsler space with a special (α,β)-metric

In this section, we investigated the Finsler space with the generalized (α,β)-metric, which is a weakly Berwald space.

Let us consider $F^n = (M^n, F)$ be a Finsler space with generalized (α,β)-metric

$$F(\alpha,\beta) = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}, \tag{13}$$

where μ_1, μ_2 and μ_3 are constants. We now establish the conditions for F^n with the metric (13) being a weakly Berwald space. For F^n with metric (13), we have

$$F_\alpha = \mu_1 - \mu_3\frac{\beta^2}{\alpha^2},\quad F_\beta = \mu_2 + 2\mu_3\frac{\beta}{\alpha},$$
$$F_{\alpha\alpha} = 2\mu_3\frac{\beta^2}{\alpha^3},\quad F_{\alpha\alpha\alpha} = -6\mu_3\frac{\beta^2}{\alpha^4}. \tag{14}$$

Substituting (14) into (1), we have

$$
\begin{aligned}
B^m =& \frac{\alpha\left(\mu_2\alpha+2\mu_3\beta\right)\left\{r_{00}\left(\mu_1\alpha^2-\mu_3\beta^2\right)-2\alpha^2 s_0\left(\mu_2\alpha+2\mu_3\beta\right)\right\}}{2\alpha\left(\mu_1\alpha^2+\mu_2\alpha\beta+\mu_3\beta^2\right)\left[(\mu_1+2b^2\mu_3)\alpha^2-3\mu_3\beta^2\right]}y^m \\
&-\frac{\mu_3\beta\{r_{00}\left(\mu_1\alpha^2-\mu_3\beta^2\right)-2\alpha^2 s_0\left(\mu_2\alpha+2\mu_3\beta\right)\}}{\left(\mu_1\alpha^2-\mu_3\beta^2\right)\left[(\mu_1+2b^2\mu_3)\alpha^2-3\mu_3\beta^2\right]}y^m \\
&+\frac{\mu_3\alpha^2\left[r_{00}\left(\mu_1\alpha^2-\mu_3\beta^2\right)-2\alpha^2 s_0\left(\mu_2\alpha+2\mu_3\beta\right)\right]}{\left(\mu_1\alpha^2-\mu_3\beta^2\right)\left[(\mu_1+2b^2\mu_3)\alpha^2-3\mu_3\beta^2\right]}b^m \\
&+\frac{\alpha^2\left(\mu_2\alpha+2\mu_3\beta\right)}{\left(\mu_1\alpha^2-\mu_3\beta^2\right)}s_0^m.
\end{aligned}
\tag{15}
$$

Again substituting (14) into (2), (4) and (12) in respective quantities, we get

$$
\begin{aligned}
A =& \frac{(t+1)\beta^3}{\alpha^5}\left(\mu_1\alpha^2-\mu_3\beta^2\right)\left[\left(\mu_1\alpha^2-\mu_3\beta^2\right)\left(\mu_1\alpha+2\mu_3\beta\right)\right. \\
&\left.-2\mu_3\beta(\mu_1\alpha^2+\mu_2\alpha\beta+\mu_3\beta^2)\right]+\frac{2\gamma^2\beta^2}{\alpha^5} \\
&\left(\mu_1\alpha^2+\mu_2\alpha\beta+\mu_3\beta^2\right)\left(\mu_1\mu_3\alpha^2+\mu_3\beta^2\right), \\
B =& \frac{2\mu_3\beta^2\left(\mu_1\alpha^2+\mu_2\alpha\beta+\mu_3\beta^2\right)}{\alpha^2}, \\
C =& \frac{\beta^3\gamma^2}{\alpha^4}\left[4\mu_3 b^2\alpha^2-\left(\mu_1\alpha^2-\mu_3\beta^2\right)^2-2\gamma^2\left(3\mu_1\mu_3\alpha^2-\mu_3^2\beta^2\right)\right], \\
D =& \frac{2\beta^3}{\alpha^2}\left[\left(\gamma^2-\beta^2\right)\left(\mu_1\alpha^2-\mu_3\beta^2\right)\left(\mu_2\alpha+2\mu_3\beta\right)\right. \\
&-2\mu_3\beta\gamma^2\left(\mu_1\alpha^2-\mu_3\beta^2\right)-4\mu_3\gamma^2\left(\gamma^2+2\beta^2\right)\left(\mu_2\alpha+2\mu_3\beta\right) \\
&\left.+2\gamma^4\left(3\mu_2\mu_3\alpha+4\mu_3^2\beta\right)\right], \\
E =& \frac{2\beta^4}{\alpha^2}\left(\mu_1\alpha^2-\mu_3\beta^2\right)\left[(\mu_1+2b^2\mu_3)\alpha^2-3\mu_3\beta^2\right], \\
\Omega =& \frac{\beta^2}{\alpha^2}\left[(\mu_1+2b^2\mu_3)\alpha^2-3\mu_3\beta^2\right], \\
W =& \frac{1}{\alpha^2}\left[r_{00}\left(\mu_1\alpha^2-\mu_3\beta^2\right)-2\alpha^2 s_0\left(\mu_2\alpha+2\mu_3\beta\right)\right], \\
C^* =& \frac{\alpha\left[r_{00}\left(\mu_1\alpha^2-\mu_3\beta^2\right)-2\alpha^2 s_0\left(\mu_2\alpha+2\mu_3\beta\right)\right]}{2\beta\left[(\mu_1+2b^2\mu_3)\alpha^2-3\mu_3\beta^2\right]}, \\
E^* =& \frac{\alpha\left(\mu_2\alpha+2\mu_3\beta\right)\left[r_{00}\left(\mu_1\alpha^2-\mu_3\beta^2\right)-2\alpha^2 s_0\left(\mu_2\alpha+2\mu_3\beta\right)\right]}{2\left(\mu_1\alpha^2+\mu_2\alpha\beta+\mu_3\beta^2\right)\left[(\mu_1+2b^2\mu_3)\alpha^2-3\mu_3\beta^2\right]}.
\end{aligned}
\tag{16}
$$

Substituting (16) into (11), we get

$$
\frac{B^m_m}{\alpha^8}\left[2\beta^6\left(\mu_1\alpha^2-\mu_3\beta^2\right)^2\left(\mu_1\alpha^2+2\mu_3 b^2\alpha^2-3\mu_3\beta^2\right)^2\left(\mu_1\alpha^2\mu_2\alpha\beta+\mu_3\beta^2\right)\right]
$$

$$-\frac{1}{\alpha^8}\left[\beta^3\{\beta^3(t+1)\left(\mu_1\alpha^2-\mu_3\beta^2\right)\left(4\mu_3\beta^3+3\mu_2\mu_3\alpha\beta^2-\mu_1\mu_2\alpha^3\right)+2\mu_3\beta^2\right.$$
$$\left(\mu_1\alpha^2+\mu_3\beta^3\right)\left(\beta^2-b^2\alpha^2\right)\left(\mu_1\alpha^2\mu_2\alpha\beta+\mu_3\beta^2\right)\}\left(\mu_1\alpha^2+2\mu_3b^2\alpha^2-3\mu_3\beta^2\right)$$
$$\{r_{00}\left(\mu_1\alpha^2-\mu_3\beta^2\right)-2\alpha^2s_0\left(\mu_2\alpha+2\mu_3\beta\right)\}\Big]-\frac{2s_0}{\alpha^8}\left[2\mu_3\alpha^2\beta^6(2\mu_3b^2\alpha^2+\mu_1\right.$$
$$\left.\alpha^2-3\mu_3\beta^2)^2(\mu_1\alpha^2+\mu_2\alpha\beta+\mu_3\beta^2)^2\right]-\frac{r_{00}}{\alpha^8}\left[2\mu_3\beta^5(2\mu_1\mu_3b^2\alpha^4+2\mu_3^2b^2\alpha^2\right.$$
$$\beta^2+\mu_1^2\alpha^4-8\mu_1\mu_3\alpha^2\beta^2+3\mu_3^2\beta^4)\left(\beta^2-b^2\alpha^2\right)\left(\mu_1\alpha^2-\mu_3\beta^2\right)(\mu_1\alpha^2+\mu_2\alpha$$
$$\left.\beta+\mu_3\beta^2)\right]-\frac{s_0}{\alpha^8}\left[4\mu_3\beta^5\left(\mu_1\alpha^2-\mu_3\beta^2\right)\left(\mu_1\alpha^2+\mu_2\alpha\beta+\mu_3\beta^2\right)(2\mu_2\mu_3b^4\alpha^5\right.$$
$$-16\mu_3^2b^2\alpha^2\beta^3-13\mu_2\mu_3b^2\alpha^3\beta^2+\mu_1\mu_2b^2\alpha^5+18\mu_3^2\beta^5-2\mu_1\mu_3\alpha^2\beta^3+12\mu_2$$
$$\left.\mu_3\alpha\beta^4-2\mu_1\mu_2\alpha^3\beta^2)\right]-\frac{r_0}{\alpha^8}\Big[4\mu_3\beta^6\left(\mu_1\alpha^2-\mu_3\beta^2\right)^2(\mu_1\alpha^2+2\mu_3b^2\alpha^2-3$$
$$\mu_3\beta^2)\left(\mu_1\alpha^2+\mu_2\alpha\beta+\mu_3\beta^2\right)\Big]=0.$$

Above equation can be re-written as

$$\begin{aligned}
&2B^m_m\left[a_1\alpha^{10}\beta+a_2\alpha^9\beta^2-a_3\alpha^8\beta^3-a_4\alpha^7\beta^4+a_5\alpha^6\beta^5+a_6\alpha^5\beta^6\right.\\
&\left.+a_7\alpha^4\beta^7-a_8\alpha^3\beta^8-a_9\alpha^2\beta^9+9\mu_2\mu_3^4\alpha\beta^{10}+9\mu_3^5\beta^{11}\right]-r_{00}\,[\\
&a_{10}\alpha^9\beta+a_{11}\alpha^8\beta^2-a_{12}\alpha^7\beta^3+a_{13}\alpha^6\beta^4+a_{14}\alpha^5\beta^5+a_{15}\alpha^4\beta^6\\
&+a_{16}\alpha^3\beta^7+a_{17}\alpha^2\beta^8+3t\mu_2\mu_3^4\alpha\beta^9+3(1+4t)\mu_3^5\beta^{10}]-2s_0\,[\\
&a_{18}\alpha^{10}\beta-a_{19}\alpha^9\beta^2+a_{20}\alpha^8\beta^3+a_{21}\alpha^7\beta^4+a_{22}\alpha^6\beta^5+a_{23}\alpha^5\beta^6\\
&+a_{24}\alpha^4\beta^7+(30t-12)\mu_2\mu_3^4\alpha^3\beta^8+(24t-t)\mu_3^5\alpha^2\beta^9]-2r_0\,[\\
&a_{25}\alpha^{10}\beta+a_{26}\alpha^9\beta^2-a_{27}\alpha^8\beta^3-a_{28}\alpha^7\beta^4+a_{29}\alpha^6\beta^5+a_{30}\alpha^5\beta^6\\
&+a_{31}\alpha^4\beta^7-6\mu_2\mu_3^4\alpha^3\beta^8-6\mu_3^5\alpha^2\beta^9]=0,
\end{aligned}\tag{17}$$

where
$a_1=\mu_1^5+4\mu_1^4\mu_3b^2+4\mu_1^3\mu_3^2b^4,$
$a_2=\mu_1^4\mu_2+4\mu_1^3\mu_2\mu_3b^2+4\mu_1^2\mu_2\mu_3^2b^4,$
$a_3=7\mu_1^4\mu_3+16\mu_1^3\mu_3^2b^2+4\mu_1^2\mu_3^3b^4,$
$a_4=8\mu_1^3\mu_2\mu_3+8\mu_1\mu_2\mu_3b^4+20\mu_1^2\mu_2\mu_3^2b^2,$
$a_5=14\mu_1^3\mu_3^2-4\mu_1\mu_3^4b^4+8\mu_1^2\mu_3^2b^2,$
$a_6=22\mu_1^2\mu_2\mu_3^2+4\mu_2\mu_3^4b^4+28\mu_1\mu_2\mu_3^3b^2,$
$a_7=4\mu_3^5b^4+16\mu_1\mu_3^4b^2-2\mu_1^2\mu_3^2,$
$a_8=12\mu_2\mu_3^4b^2+24\mu_1\mu_2\mu_3^3,$
$a_9=15\mu_1\mu_3^4+12\mu_3^5b^2,$
$a_{10}=(1+t)\mu_1^4\mu_2+(2+t)\mu_1^3\mu_2\mu_3,$
$a_{11}=4\mu_1^2\mu_3^3b^4+8\mu_1^3\mu_3^2b^2,$

$a_{12} = \mu_1^3\mu_2\mu_3 + 10t\mu_1^2\mu_2\mu_3^2 b^4 + 2\mu_1^2\mu_2\mu_3^2 b^2,$

$a_{13} = 4\mu_1^2\mu_3^3 b^2(5b^2 + 2t) - (16 + 4t)\mu_1^3\mu_3^2,$

$a_{14} = 32\mu_1^2\mu_2\mu_3^2 + \mu_1\mu_2\mu_3^2(12b^4 - 8b^2),$

$a_{15} = 4\mu_1\mu_3^4 b^2(1 + 4t) + 4\mu_1^2\mu_3^3(5t + 8),$

$a_{17} = 4\mu_3^5 b^2(1 - 2t) - 4\mu_1\mu_3^4(4 + 7t),$

$a_{18} = 2\mu_1^4\mu_3 + 4\mu_1^3\mu_3^2 b^2 + 4\mu_1\mu_2^2\mu_3^2 b^4 - (1 + t)\mu_1^3\mu_2^2 - 2(1 + t)\mu_1^2\mu_2^2\mu_3 b^2,$

$a_{19} = 2t\mu_1^3\mu_2\mu_3 + (4t + 22)\mu_1^2\mu_2\mu_3^2 b^2,$

$a_{20} = -12\mu_1^3\mu_3^2 - 28\mu_1^2\mu_3^3 b^2 + 7(t + 1)\mu_1^2\mu_2^2\mu_3 + (60 + 8t)\mu_1\mu_2^2\mu_3^2,$

$a_{21} = (24t + 60)\mu_1\mu_2\mu_3^3 b^2 + (18t + 20)\mu_1^2\mu_2\mu_3^2,$

$a_{22} = (10 - 6t)\mu_1^2\mu_3^3 b^2 + (36 + 8t)\mu_1^2\mu_3^3 - 15t\mu_1\mu_2^2\mu_3^2 + (16t + 12)\mu_1\mu_3^4 b^2,$

$a_{23} = (12 - 48t)\mu_1\mu_2\mu_3^3 - 20\mu_2\mu_3^4 b^2,$

$a_{24} = (12 - 16t)\mu_3^5 b^2 - (24 + 32t)\mu_1\mu_3^4 + (9t - 3)\mu_2^2\mu_3^3,$

$a_{25} = 2\mu_1^4\mu_3 + 4\mu_1^3\mu_3^2 b^2,$

$a_{26} = 2\mu_1^3\mu_2\mu_3 + 4\mu_1^2\mu_2\mu_3^2 b^2,$

$a_{27} = 8\mu_1^3\mu_3^2 + 4\mu_1^2\mu_3^3 b^2,$

$a_{28} = 10\mu_1^2\mu_2\mu_3^2 + 8\mu_1\mu_2\mu_3^3 b^2,$

$a_{29} = 4\mu_1^2\mu_3^2 - 4\mu_1\mu_3^4 b^2,$

$a_{30} = 4\mu_2\mu_3^4 b^2 + 14\mu_1\mu_2\mu_3^3,$

$a_{31} = 4\mu_3^5 + 8\mu_1\mu_3^4.$

Now, we can assume that F^n is a weakly Berwald space, then B^m_m is $hp(1)$. Since, α is irrational in (y^i), the equation (17) is divided into two equations as follows

$$K_1 B^m_m + \beta L_1 r_{00} + \alpha^2 M_1 s_0 + \alpha^2 N_1 r_0 = 0, \tag{18}$$

$$\beta K_2 B^m_m + L_2 r_{00} + \alpha^2 \beta M_2 s_0 + \alpha^2 \beta N_2 r_0 = 0, \tag{19}$$

where

$K_1 = 2a_1\alpha^{10} - 2a_3\alpha^8\beta^2 + 2a_5\alpha^6\beta^4 + 2a_7\alpha^4\beta^6 - 2a_9\alpha^2\beta^8 + 18\mu_3^5\beta^{10},$

$K_2 = 2a_2\alpha^8 - 2a_4\alpha^6\beta^2 + 2a_6\alpha^4\beta^4 + 2a_{17}\alpha^2\beta^6 + 18\mu_2\mu_3^4\beta^8,$

$L_1 = -\{a_{11}\alpha^8 + a_{13}\alpha^6\beta^2 + a_{15}\alpha^4\beta^4 + a_{17}\alpha^2\beta^6 + 3(1 + 4t)\mu_3^5\beta^8\},$

$L_2 = -\{a_{10}\alpha^8 - a_{12}\alpha^6\beta^2 + a_{14}\alpha^4\beta^4 + a_{16}\alpha^2\beta^6 + 3t\mu_2\mu_3^4\beta^8\},$

$M_1 = -2\{a_{18}\alpha^8 + a_{20}\alpha^6\beta^2 + a_{22}\alpha^4\beta^4 + a_{24}\alpha^2\beta^6 + 3(8t - 2)\mu_3^5\beta^8\},$

$M_2 = -2\{-a_{19}\alpha^6 + a_{21}\alpha^4\beta^2 + a_{24}\alpha^2\beta^4 + 3(10t - 4)\mu_2\mu_3^4\beta^6\},$

$$N_1 = -2\{a_{25}\alpha^8 - a_{27}\alpha^6\beta^2 + a_{29}\alpha^4\beta^4 + a_{31}\alpha^2\beta^6 - 6\mu_3^5\beta^8\},$$
$$N_2 = -2\{a_{26}\alpha^6 - a_{28}\alpha^4\beta^2 + a_{30}\alpha^2\beta^4 - 6\mu_2\mu_3^4\beta^6\}\ .$$

Eliminating B^m_m from equations (18) and (19), we get

$$Fr_{00} + \alpha^2\beta G s_0 + \alpha^2\beta H r_0 = 0, \tag{20}$$

where

$$F = \beta^2 K_2 L_1 - K_1 L_2,$$
$$G = K_2 M_1 - K_1 M_2,$$
$$H = K_2 N_1 - K_1 N_2.$$

Equation (20) re-written as

$$\left(\frac{F}{\alpha^2\beta}\right) r_{00} + G s_0 + H r_0 = 0. \tag{21}$$

Since, only the term $\epsilon_1\alpha^{16}$ of Gs_0 in (21) does not contain β, we must have $hp(16)V_{16}$ such that

$$\alpha^{16} s_0 = \beta V_{16}, \tag{22}$$

where $\epsilon_1 = -4\left(a_2 a_{18} - 2a_1 a_{19}\right)$.

First consider that $\alpha^2 \not\equiv 0 \pmod{\beta}$ and $b^2 \neq 0$. Equation (22) shows the existence of a function $q(x)$ satisfy $V_{16} = q\alpha^{16}$ and hence, $s_0 = q\beta$. Then equation (21) reduces to

$$\left(\frac{F}{\alpha^2\beta}\right) r_{00} + Gq\beta + H r_0 = 0,$$

which implies that

$$Fr_{00} + Gq\alpha^2\beta^2 + \alpha^2\beta H r_0 = 0.$$

Only the term $2a_1 a_{10}\alpha^{18} r_{00}$ of the above relation does not contain β. Thus there exist $hp(19)U_{19}$ satisfying $2a_1 a_{10}\alpha^{18} r_{00} = \beta U_{19}$. It is a contradiction, which implies that $q = 0$. Hence, we obtain $s_0 = 0$, $s_j = 0$. Then equation (20) becomes

$$Fr_{00} + \alpha^2\beta H r_0 = 0. \tag{23}$$

Only the term $54(1+3t)\mu_2\mu_3^9\beta^{18} r_{00}$ of (23) seemingly does not contain α^2 and hence, we must have $hp(18)V_{18}$ such that $\beta^{18} r_{00} = \alpha^2 V_{18}$. From $\alpha^2 \not\equiv 0 \pmod{\beta}$ there exist a function $g(x)$ such that

$$r_{00} = \alpha^2 g(x); \quad r_{ij} = a_{ij} g(x). \tag{24}$$

Transvecting above equation by $b^i y^j$, we have

$$r_0 = \beta g(x); \quad r_j = b_j g(x). \tag{25}$$

Plugging (24) and (25) into (23), we get

$$g(x)\left(F+\beta^2 H\right)=0. \tag{26}$$

Assuming that $g(x)\neq 0$, we can deduce from equation (26) that

$$F+\beta^2 H=0.$$

The term $2a_1a_{10}\alpha^{18}r_{00}$ of above relation does not contain β. Then there exist $hp(17)V_{17}$ satisfying $2a_1a_{10}\alpha^{18}=\beta V_{17}$, where V_{17} is $hp(17)$ this implies $V_{17}=0$, provided that $b^2\neq 0$. Hence, $g(x)=0$ must hold and we get

$$r_{00}=0,\quad r_{ij}=0\quad and\quad r_0=0;\quad r_j=0.$$

Conversely, substituting $r_{00}=0,\ \ s_0=0,\ \ and\ \ r_0=0$ into equation (17), we get $B^m_m=0$. That is, the Finsler space with (13) is a Weakly Berwald space.

Consequently, we assume that the Finsler space with (13) is a Berwald space. As a result of the preceding discussion, we have $r_{00},\ \ s_0=0\ \ and\ \ r_0=0$, indicating that the space is Weakly Berwald space. When we plug the above into (15), we get $B^m_m=0$, noting that the Finsler space with (13) is a Berwald space. Hence s_{ij} is holds good.

Now, consider $\alpha^2\equiv 0(\text{mod }\beta)$, Lemma(2.2) shows that $t=2$, $b^2=0$ and $\alpha^2=\beta\delta$, $\delta=d_i(x)y^i$. From these conditions (20) is rewritten in the form below

$$F^{'}r_{00}+\beta\delta G^{'}s_0=0, \tag{27}$$

where

$$\begin{aligned}F^{'}=&-2a_1a_{10}\delta^9+\beta\delta^8\left(a_1a_{12}+2a_3a_{10}+2a_2a_{11}\right)+\beta^2\delta^7\left(2a_2a_{13}-2a_4a_{11}\right.\\&\left.-2a_1a_{14}-2a_3a_{12}-2a_5a_{10}\right)+\beta^3\delta^6\left(2a_2a_{15}-2a_4a_{13}+2a_6a_{11}-2a_1\right.\\&\left.a_{16}+2a_3a_{14}+2a_5a_{12}-2a_7a_{10}\right)+\beta^4\delta^5\left(2a_2a_{17}-2a_4a_{15}+2a_6a_{13}-\right.\\&\left.2a_2a_{11}+2a_3a_{16}+2a_5a_{14}+2a_7a_{12}+2a_9a_{10}-6\mu_2\mu_3^4a_1\right)+\beta^5\delta^4\left(2a_6\right.\\&a_{13}-2a_4a_{17}-2a_8a_{13}-2a_5a_{16}-2a_9a_{12}+30\mu_3^5a_2+18\mu_2\mu_3^4a_{11}+6\\&\left.\mu_2\mu_3^4\right)+\beta^6\delta^3\left(2a_6a_{17}-2a_8a_{15}-2a_7a_{16}+2a_9a_{12}-30\mu_3^5a_4+18\mu_2\right.\\&\left.\mu_3^4a_{13}-6\mu_2\mu_3^4+18\mu_3^5a_{12}\right)+\beta^7\delta^2\left(2a_9a_{16}-2a_2a_{17}+30\mu_3^5a_6+18\mu_2\right.\\&\left.\mu_3^4a_{15}-6\mu_2\mu_3^4-18\mu_3^5a_{12}\right)+\beta^8\delta\left(30\mu_3^5a_8+18\mu_2\mu_3^4a_{17}+6\mu_2\mu_3^4-18\right.\\&\left.\mu_3^5a_{16}\right)-216\mu_2\mu_3^9\beta^9,\\G^{'}=&4\delta^9\left(a_1a_{19}-a_2a_{18}\right)+4\beta\delta^7\left(a_4a_{18}-a_2a_{20}-a_1a_{21}-a_3a_{19}\right)+4\beta^2\delta^6\left(\right.\\&a_4a_{20}-a_2a_{22}-a_6a_{18}-a_6a_{18}-a_1a_{24}+a_3a_{21}+a_5a_{19}\left.\right)+4\beta^3\delta^5\left(a_4\right.\\&\left.a_{22}-a_2a_{24}-a_6a_{20}+a_8a_{18}+a_3a_{24}-a_5a_{21}+a_7a_{19}-18\mu_2\mu_3^4a_1\right)+\\&4\beta^4\delta^4(a_4a_{24}-a_6a_{22}+a_8a_{20}-a_5a_{24}-a_7a_{21}-a_9a_{19}+18\mu_2\mu_3^4a_3-\end{aligned}$$

$$9\mu_2\mu_3^4a_{18} - a8\mu_3^5a_2\big) + 4\beta^5\delta^3\left(a_8a_{22} - a_6a_{24} - a_7a_{24} + a_9a_{21} + 18\mu_3^5\right.$$
$$a_4 - 18\mu_2\mu_3^4a_5 - 9\mu_2\mu_3^4a_{20} + 9\mu_3^5a_{19}\big) + 4\beta^6\delta^2\left(a_8a_{24} + 9a_9a_{24} - 18\right.$$
$$\mu_3^5a_6 - 18\mu_2\mu_3^4a_7 - 9\mu_2\mu_3^4 - 9\mu_3^5a_{12}\big) + 4\beta^7\delta\left(18\mu_3^5a_8 + 18\mu_2\mu_3^4a_9 - \right.$$
$$9\mu_2\mu_3^4a_{24} - 9\mu_3^5a_{24}\big) + 972\mu_2\mu_3^5\beta^8.$$

Since, only the term $216\mu_2\mu_3^9\beta^9r_{00}$ of $F^{'}r_{00} + \beta\delta G^{'}s_0$ in (27) seemingly does not contain δ. We must have $hp(1)V_1$ such that $r_{00} = \delta V_1$. We have $s_0 = 0$, $s_j = 0$, now (27) becomes

$$F^{'}r_{00} = 0,$$

which implies

$$r_{00} = 0, \quad r_{ij} = 0 \quad and \quad r_0 = 0; \quad r_j = 0.$$

Consequently from $r_{00} = 0, \quad r_0 = 0 \quad and \quad s_0 = 0$, we have $B^m_m = 0$. Thus the space with (13) is weakly Berwald space. Hence we state the following

Theorem 3.1. *Let F be a Finsler space with (α, β)-metric* (13) *is weakly Berwald space if and only if the following properties satisfies*

i. $\alpha^2 \not\equiv 0(\text{mod } \beta)$ *implies* $r_{ij} = 0$ *and* $s_j = 0$,
ii. $\alpha^2 \equiv 0(\text{mod } \beta)$ *implies* $t = 2$, $b^2 = 0$ *and* $r_{ij} = 0$, $s_j = 0$ *are satisfied, where* $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$.

4. Conclusion

In this article, we look at a Finsler space where the (hv)-Ricci tensor G_{ij} vanishes but the (hv)-curvature tensor G^h_{ijk} does not always equal to zero. The primary goal of this research is to present an example of a weakly Berwald Finsler space and to show a required condition for the existence of a weakly Berwald Finsler space of the (α, β)-metric

$$F(\alpha, \beta) = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}.$$

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