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ON THE SPHERICAL INDICATRIX CURVES OF THE SPACELIKE SALKOWSKI CURVE WITH TIMELIKE PRINCIPAL NORMAL IN LORENTZIAN 3-SPACE

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Abstract. In this paper, we calculate Frenet frames, Frenet derivative formulas, curvatures, arc lengths, geodesic curvatures according to the Lorentzian 3-space $\mathbb{R}_1^3$, Lorentzian sphere $\mathbb{S}_1^2$ and hyperbolic sphere $\mathbb{H}_0^2$ of the spherical indicatrix curves of the spacelike Salkowski curve with the timelike principal normal in $\mathbb{R}_1^3$ and draw the graphs of these indicatrix curves on the spheres.

1. Introduction

Lorentz-Minkowski space is an enjoyable and up-to-date field where those interested in theoretical physics and geometry work extensively. In this space, which became more interesting and had a chance to develop after Einstein's special and general relativity theories in the 20^{th} century, differential geometry studies also have an important place in addition to physics. General concepts about Lorentz-Minkowski space are available in the sources [2, 6, 7, 15, 32, 34, 43, 45, 46].

One of the most interesting topics in the differential geometry is the theory of curves. The system $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$ of a differentiable curve $(\overrightarrow{\alpha})$ in $\mathbb{R}_1^3$, is the Frenet frame of the curve, [22, 33, 36, 41]. The functions curvature κ and torsion τ achieved from these vectors of a regular curve play an important part in appointing the characterization of the curve. For example, curves with a constant ratio $\frac{\tau}{\kappa}$ are helices, with nonzero curvature and torsion. The general helix curve, which was expressed by Lancret in 1802 and is the generalization of the helix curve, is defined as the curve whose tangent vector makes a fixed angle with a fixed axis. Examples of general helices are the sequence of molecules in the DNA structure, carbon nanotubes, helical ladders, the way a bean string is wound on a rod, and screw movements. Some studies on curves or surfaces

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in Lorentz space are available in [1, 8, 9, 17, 16, 19, 20, 24, 25, 26, 27, 28, 29, 30, 42, 44, 47].

Besides, slant helices are curves whose principal normal vector makes a constant angle with a fixed direction, [4, 5, 23]. The Salkowski curve, a well-known example of slant helices, was described in 1909 by Ernst Leopold Salkowski, [35]. Similarly, the family of curves with constant torsion and non-constant curvature is known as anti-Salkowski curve. Since then, many authors have studied these curves in both Euclidean and Lorentz 3 space, [3, 18, 21, 31, 37, 38, 39, 40]. The unit tangent vectors along a regular curve $\overrightarrow{\alpha}(t)$ in $\mathbb{R}_1^3$ create a curve $\left(\overrightarrow{T}\right)$ on the unit hyperbolic sphere $\mathbb{H}_0^2$ (or unit Lorentzian sphere $\mathbb{S}_1^2$). The curve $(\overrightarrow{T})$ is called tangent indicatrix curve of $(\overrightarrow{\alpha})$. Similarly $\left(\overrightarrow{N}\right)$ and $\left(\overrightarrow{B}\right)$ are called the principal normal indicatrix curve and the binormal indicatrix curve. Let t_T, t_N, t_B be the parameters for the curves $\left(\overrightarrow{T}\right)$, $\left(\overrightarrow{N}\right)$, $\left(\overrightarrow{B}\right)$, respectively. The parametric equations for these curves are as follows:

$$\left(\overrightarrow{T}\right) = \overrightarrow{\alpha}\,(t_T) = \overrightarrow{T}\,(t)\,,\ \left(\overrightarrow{N}\right) = \overrightarrow{\alpha}\,(t_N) = \overrightarrow{N}\,(t)\,,\ \left(\overrightarrow{B}\right) = \overrightarrow{\alpha}\,(t_B) = \overrightarrow{B}\,(t)\,,$$

[43]. Some studies on spherical indicatrix curves of various curves are [10, 11, 12, 13, 14].

In this paper, we calculate Frenet frames, Frenet derivative formulas, curvatures, Darboux vectors, pole vectors, arc lengths, geodesic curvatures of spherical indicatrix curves of the spacelike Salkowski curve with the timelike principal normal in $\mathbb{R}_1^3$ and draw the graphs of these indicatrix curves.

2. Preliminaries

For the vectors $\overrightarrow{G} = (g_1, g_2, g_3)$ and $\overrightarrow{P} = (p_1, p_2, p_3) \in \mathbb{R}^3$, the Lorentzian inner product is identified by

$$\langle,\rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\,,\ \left\langle \overrightarrow{G}, \overrightarrow{P} \right\rangle = g_1p_1 + g_2p_2 - g_3p_3,$$

and the Lorentzian vector product is identified by

$$(1)\ \wedge:\ \ \mathbb{R}_1^3 \times \mathbb{R}_1^3 \to \mathbb{R}_1^3\,,\quad \overrightarrow{G} \wedge \overrightarrow{P} = (g_3p_2 - g_2p_3, g_1p_3 - g_3p_1, g_1p_2 - g_2p_1)\,.$$

Here, the function $\langle,\rangle$ is Lorentzian metric and the space $\mathbb{R}^3$ together with this metric is called the Lorentzian 3-space and is denoted by $\mathbb{R}_1^3$. For $\overrightarrow{G} \in \mathbb{R}_1^3$, if $\left\langle \overrightarrow{G}, \overrightarrow{G} \right\rangle > 0$ or $\overrightarrow{G} = 0$, $\overrightarrow{G}$ is spacelike (sl) vector, if $\left\langle \overrightarrow{G}, \overrightarrow{G} \right\rangle < 0$, $\overrightarrow{G}$ is timelike (tl) vector, if $\left\langle \overrightarrow{G}, \overrightarrow{G} \right\rangle = 0$ and $\overrightarrow{G} \neq 0$, $\overrightarrow{G}$ is lightlike (null) vector. For a timelike vector $\overrightarrow{G}$, if $\left\langle \overrightarrow{G}, \overrightarrow{E} \right\rangle < 0$, it is future pointing timelike (fptl) vector, if $\left\langle \overrightarrow{G}, \overrightarrow{E} \right\rangle > 0$, it is past pointing timelike (pptl) vector, where $\overrightarrow{E} = (0, 0, 1)$.

The norm of $\overrightarrow{G} \in \mathbb{R}^3_1$ is $\left\|\overrightarrow{G}\right\| = \sqrt{\left|\left\langle \overrightarrow{G}, \overrightarrow{G}\right\rangle\right|}$ and if $\left\|\overrightarrow{G}\right\| = 1$, $\overrightarrow{G}$ is called a unit vector. The Lorentzian unit sphere, future pointing hyperbolic unit sphere, past pointing hyperbolic unit sphere and light cone are

$$\begin{aligned}
\mathbb{S}^2_1 &= \left\{\overrightarrow{G} \in \mathbb{R}^3_1 \;\middle|\; \left\langle \overrightarrow{G}, \overrightarrow{G}\right\rangle = 1\right\}, \\
\mathbb{H}^{2+}_0 &= \left\{\overrightarrow{G} \in \mathbb{R}^3_1 \;\middle|\; \left\langle \overrightarrow{G}, \overrightarrow{G}\right\rangle = -1\,,\;\; \left\langle \overrightarrow{G}, \overrightarrow{E}\right\rangle < 0\right\}, \\
\mathbb{H}^{2-}_0 &= \left\{\overrightarrow{G} \in \mathbb{R}^3_1 \;\middle|\; \left\langle \overrightarrow{G}, \overrightarrow{G}\right\rangle = -1\,,\;\; \left\langle \overrightarrow{G}, \overrightarrow{E}\right\rangle > 0\right\}, \\
\Lambda &= \left\{\overrightarrow{G} \in \mathbb{R}^3_1 \;\middle|\; \left\langle \overrightarrow{G}, \overrightarrow{G}\right\rangle = 0\right\},
\end{aligned}$$

respectively. For the vectors $\overrightarrow{G}$ and $\overrightarrow{P}$ in $\mathbb{R}^3_1$, if $\left\langle \overrightarrow{G}, \overrightarrow{P}\right\rangle = 0$, $\overrightarrow{G}$ and $\overrightarrow{P}$ are Lorentz orthogonal vectors. Let $\overrightarrow{G}$ and $\overrightarrow{P}$ be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$, if $\overrightarrow{G}$ is timelike, then $\overrightarrow{P}$ is spacelike, [34]. Let $\overrightarrow{G}$ and $\overrightarrow{P}$ be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$, if $\overrightarrow{G}$ is timelike, then $\overrightarrow{P}$ is spacelike, [34]. Now, let's mention the angle in between two vectors in Lorentzian 3-space:

i) The hyperbolic angle between of the future pointing timelike vector $\overrightarrow{G}$ and the past pointing timelike vector $\overrightarrow{P}$ in $\mathbb{R}^3_1$ is stated as follows [34]:

$$\left\langle \overrightarrow{G}, \overrightarrow{P}\right\rangle = \left\|\overrightarrow{G}\right\| \left\|\overrightarrow{P}\right\| \cosh\varphi. \tag{2}$$

If the timelike vectors $\overrightarrow{G}$ and $\overrightarrow{P}$ are simultaneously future (or past) pointing, in the case, the hyperbolic angle between these vectors is [32]

$$\left\langle \overrightarrow{G}, \overrightarrow{P}\right\rangle = -\left\|\overrightarrow{G}\right\| \left\|\overrightarrow{P}\right\| \cosh\varphi. \tag{3}$$

ii) The hyperbolic angle between of the spacelike vectors $\overrightarrow{G}$ and $\overrightarrow{P}$ in $\mathbb{R}^3_1$ that span a timelike vector subspace is stated as follows [34]:

$$\left|\left\langle \overrightarrow{G}, \overrightarrow{P}\right\rangle\right| = \left\|\overrightarrow{G}\right\| \left\|\overrightarrow{P}\right\| \cosh\varphi.$$

iii) The hyperbolic angle between of the spacelike vector $\overrightarrow{G}$ and the timelike vector $\overrightarrow{P}$ in $\mathbb{R}^3_1$ is stated as follows [34]:

$$\left|\left\langle \overrightarrow{G}, \overrightarrow{P}\right\rangle\right| = \left\|\overrightarrow{G}\right\| \left\|\overrightarrow{P}\right\| \sinh\varphi, \tag{4}$$

If all of the velocity vector of a curve $(\overrightarrow{\alpha})$ in $\mathbb{R}^3_1$ are spacelike, timelike or lightlike, it is called spacelike, timelike or lightlike curve, respectively. Let the Frenet frame, curvature, torsion, Darboux vector and pole vector be $\left\{\overrightarrow{T}(t), \overrightarrow{N}(t), \overrightarrow{B}(t)\right\}$, $\kappa(t)$, $\tau(t)$, $\overrightarrow{F}(t)$ and $\overrightarrow{C}(t)$ at the point t of differentiable curve $(\overrightarrow{\alpha})$ in $\mathbb{R}^3_1$, respectively. Also, there are two cases depending on causal character of $(\overrightarrow{\alpha})$:

i) If $(\overrightarrow{\alpha})$ is any timelike space curve, its tangent vector $\overrightarrow{T}(t)$ is timelike, its normal and binormal vectors $\overrightarrow{N}(t)$ and $\overrightarrow{B}(t)$ are spacelike. Then,

$$\overrightarrow{T}(t)\wedge\overrightarrow{N}(t)=-\overrightarrow{B}(t),\qquad \overrightarrow{N}(t)\wedge\overrightarrow{B}(t)=\overrightarrow{T}(t),\qquad \overrightarrow{B}(t)\wedge\overrightarrow{T}(t)=-\overrightarrow{N}(t). \tag{5}$$

Let $\varphi(t)$ be the hyperbolic angle between binormal vector $\overrightarrow{B}(t)$ and Darboux vector $\overrightarrow{F}(t)$. The pole vector $\overrightarrow{C}(t)$ of a timelike curve is classified as follows depending on the causal character of the Darboux vector, [9, 43]:

a) If $|\kappa(t)|>|\tau(t)|$, $\overrightarrow{F}(t)$ is a spacelike vector. Then,

$$\overrightarrow{C}(t)=\sinh\varphi(t)\overrightarrow{T}(t)-\cosh\varphi(t)\overrightarrow{B}(t).$$

b) If $|\kappa(t)|<|\tau(t)|$, then $\overrightarrow{F}(t)$ is a timelike vector. Then,

$$\overrightarrow{C}(t)=\cosh\varphi(t)\overrightarrow{T}(t)-\sinh(t)\varphi\overrightarrow{B}(t). \tag{6}$$

ii) Let $(\overrightarrow{\alpha})$ be any spacelike space curve. We assume that the tangent vector $\overrightarrow{T}(t)$ and the normal vector $\overrightarrow{N}(t)$ are spacelike, the binormal vector $\overrightarrow{B}(t)$ is timelike. Then,

$$\overrightarrow{T}(t)\wedge\overrightarrow{N}(t)=\overrightarrow{B}(t),\qquad \overrightarrow{N}(t)\wedge\overrightarrow{B}(t)=-\overrightarrow{T}(t),\qquad \overrightarrow{B}(t)\wedge\overrightarrow{T}(t)=-\overrightarrow{N}(t). \tag{7}$$

Let $\varphi(t)$ be the hyperbolic angle between binormal vector $\overrightarrow{B}(t)$ and Darboux vector $\overrightarrow{F}(t)$. The pole vector for any spacelike curve is classified as follows depending on the causal characteristic of the Darboux vector, [9, 43]:

a) If $|\kappa(t)|<|\tau(t)|$, $\overrightarrow{F}(t)$ is a spacelike vector. Then,

$$\overrightarrow{C}(t)=\cosh\varphi(t)\overrightarrow{T}(t)-\sinh\varphi(t)\overrightarrow{B}(t).$$

b) If $|\kappa(t)|>|\tau(t)|$, $\overrightarrow{F}(t)$ is a timelike vector. Then,

$$\overrightarrow{C}(t)=\sinh\varphi(t)\overrightarrow{T}(t)-\cosh\varphi(t)\overrightarrow{B}(t). \tag{8}$$

The arc length between the points $\overrightarrow{\alpha}(a)$ and $\overrightarrow{\alpha}(b)$ of a curve $(\overrightarrow{\alpha})$ in $\mathbb{R}^3_1$ is real number

$$\oint_a^b \left\| \frac{d\overrightarrow{\alpha}(t)}{dt} \right\| dt \tag{9}$$

where $a\le t\le b$, [34]. Besides, the geodesic curvature with regard to $\mathbb{R}^3_1$ is

$$k_g(t)=\left\|\overrightarrow{D_T T}(t)\right\|. \tag{10}$$

D is the connection of $\mathbb{R}^3_1$. The normal vector of the surface for the tangent indicatrix curve $(\overrightarrow{T})$ on $\mathbb{S}^2_1$ or $\mathbb{H}^2_0$ is $\overrightarrow{T}(t)$. Then, there are the following equations:

a) If $\overrightarrow{T}(t)$ is spacelike, the geodesic curvature γ_g with regard to $\mathbb{S}_1^2$ of $(\overrightarrow{T})$ is

$$\gamma_g(t) = \left\| \overrightarrow{\overline{D}_T T(t)} \right\|, \qquad \overrightarrow{D_T T}(t) = \overrightarrow{\overline{D}_T T}(t) + \left\langle \overrightarrow{\overline{S}(T)}(t), \overrightarrow{T}(t) \right\rangle \overrightarrow{T}(t), \tag{11}$$

where, $\overline{D}$ is the connection and $\overline{S}$ is the shape operator of $\mathbb{S}_1^2$.

b) If $\overrightarrow{T}(t)$ is timelike, the geodesic curvature ψ_g with regard to $\mathbb{H}_0^2$ of $(\overrightarrow{T})$ is

$$\psi_g(t) = \left\| \overrightarrow{\overline{\overline{D}}_T T(t)} \right\|, \qquad \overrightarrow{D_T T}(t) = \overrightarrow{\overline{\overline{D}}_T T}(t) - \left\langle \overrightarrow{\overline{\overline{S}}(T)}(t), \overrightarrow{T}(t) \right\rangle \overrightarrow{T}(t), \tag{12}$$

where, $\overline{\overline{D}}$ is the connection and $\overline{\overline{S}}$ is the shape operator of $\mathbb{H}_0^2$, [32]. Similar equations are also written for other indicatrix curves.

3. On The Spherical Indicatrix Curves of the Spacelike Salkowski Curve with Timelike Principal Normal in Lorentzian 3-Space

The spacelike Salkowski curve with the timelike principal normal in $\mathbb{R}_1^3$ is:

$$\begin{aligned}\overrightarrow{\gamma_m}(t) \;=\; \Big(& 2\sin t - \frac{1+n}{1-2n}\sin\left[(1-2n)t\right] - \frac{1-n}{1+2n}\sin\left[(1+2n)t\right], \\ & 2\cos t - \frac{1+n}{1-2n}\cos\left[(1-2n)t\right] - \frac{1-n}{1+2n}\cos\left[(1+2n)t\right], \frac{1}{m}\cos(2nt) \Big),\end{aligned}$$

where $m \in (-\infty, -1)$ or $m \in (1, +\infty)$ and $n = \dfrac{m}{\sqrt{m^2-1}}$, Figure 1.

The Frenet vectors of this curve are

$$\left\{ \begin{aligned} \overrightarrow{T}(t) &= (\cos t \sin(nt) - n \sin t \cos(nt), \\ &\qquad -\sin t \sin(nt) - n\cos t \cos(nt), -\frac{n}{m}\cos(nt)\Big), \\ \overrightarrow{N}(t) &= \frac{n}{m}(\sin t, \cos t, m), \\ \overrightarrow{B}(t) &= (-\cos t \cos(nt) - n \sin t \sin(nt), \\ &\qquad \sin t \cos(nt) - n \cos t \sin(nt), -\frac{n}{m}\sin(nt)\Big). \end{aligned} \right. \tag{13}$$

Also, the curvature is $\kappa(t) = 1$, the torsion is $\tau(t) = -\cot(nt)$ and the arc length is $s_\gamma = -\dfrac{\cos(nt)}{m}$ of $\overrightarrow{\gamma_m}(t)$, [3].

We will now examine some properties of tangent, principal normal and binormal indicatrix curves of $\overrightarrow{\gamma_m}(t)$. The representation of these three indicatrix curves on the spheres in $\mathbb{R}_1^3$ is as in Figure 2.

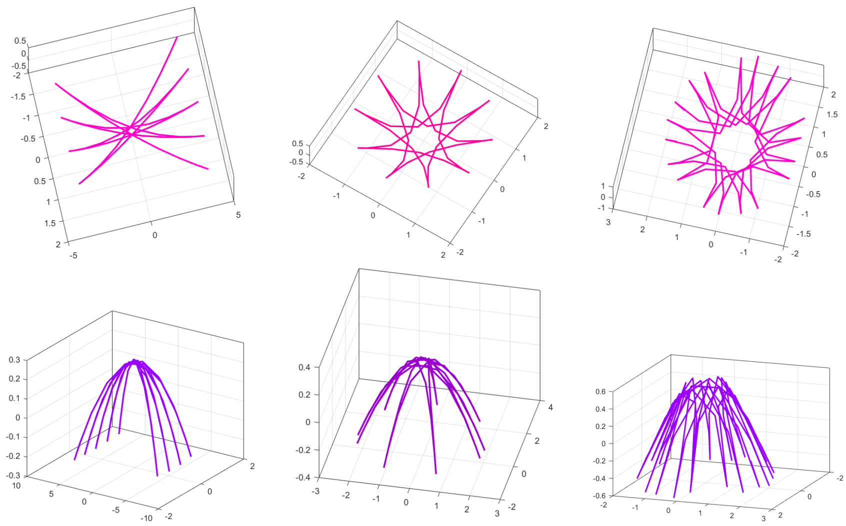

FIGURE 1. Spacelike Salkowski Curves with Timelike Principal Normal for $m = \frac{5}{4}$, 4, $\frac{10}{9}$, -6, -2, -$\frac{10}{9}$ respectively.

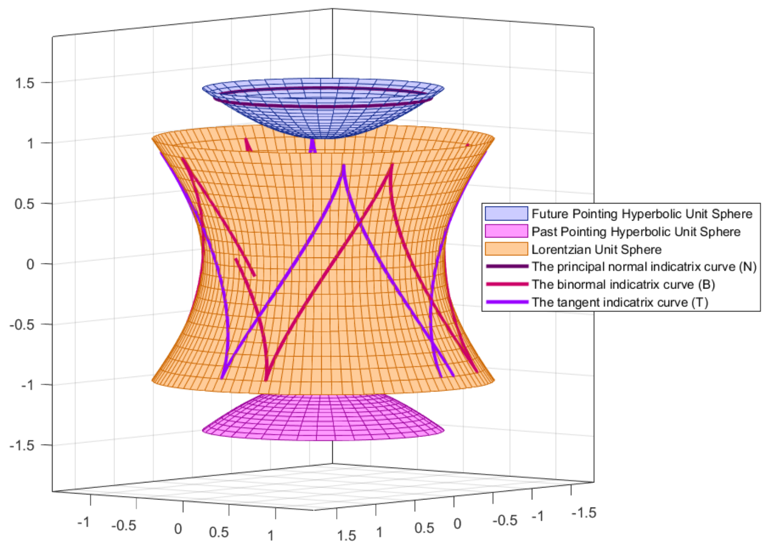

FIGURE 2. The Spherical Indicatrix Curves on $\mathbb{S}_1^2$ and $\mathbb{H}_0^2$

First of all, let's specify the interval of $\overrightarrow{\gamma_m}(t)$. So, if we take into consideration $\left[-\frac{\pi}{2n}, t\right]$, from (9), we indeed obtain

$$s_\gamma = \int_{-\frac{\pi}{2n}}^{t} \left\| \overrightarrow{\gamma_m}'(t) \right\| dt = \int_{-\frac{\pi}{2n}}^{t} \frac{\sin(nt)}{\sqrt{m^2-1}} dt = \frac{1}{\sqrt{m^2-1}} \left[-\frac{\cos(nt)}{n} \Big|_{-\frac{\pi}{2n}}^{t} \right],$$

(14) $$s_\gamma = -\frac{\cos(nt)}{m}.$$

3.1. The Tangent Indicatrix Curve of $\overrightarrow{\gamma_m}(t)$

Theorem 3.1. *The Frenet frame $\left\{\overrightarrow{T_T}(t), \overrightarrow{N_T}(t), \overrightarrow{B_T}(t)\right\}$ of the timelike tangent indicatrix curve $(\overrightarrow{T})$ of $\overrightarrow{\gamma_m}(t)$ is as follows, respectively:*

(15) $$\begin{cases} \overrightarrow{T_T}(t) = \left(\frac{n}{m}\sin t, \frac{n}{m}\cos t, \mathrm{n}\right) \quad (tl), \\ \overrightarrow{N_T}(t) = (\cos t, -\sin t, 0) \quad (sl), \\ \overrightarrow{B_T}(t) = \left(n\sin t, n\cos t, \frac{n}{m}\right) \quad (sl). \end{cases}$$

Proof. Let the parameter of $(\overrightarrow{T})$ be t_T. If we derivate of $\overrightarrow{T}(t)$ in (13) according to t, we have

(16) $$\overrightarrow{T'}(t) = \left(\frac{n^2}{m^2}\sin t\sin(nt), \frac{n^2}{m^2}\cos t\sin(nt), \frac{n^2}{m}\sin(nt)\right).$$

And, if we take into consideration the equation

(17) $$\overrightarrow{T_T}(t) = \frac{d\overrightarrow{T}(t)}{dt_T} = \frac{d\overrightarrow{T}(t)}{dt}\frac{dt}{dt_T} = \overrightarrow{T'}(t)\frac{dt}{dt_T},$$

from (16) and since $\left\|\overrightarrow{T_T}(t)\right\| = 1$, we achieve

(18) $$v_T = \frac{dt_T}{dt} = \left\|\overrightarrow{T'}(t)\right\| = \frac{n}{m}\sin(nt).$$

From (17) and (18), we write $\overrightarrow{T_T}(t) = \dfrac{\overrightarrow{T'}(t)}{\left\|\overrightarrow{T'}(t)\right\|}$. Here, if we pay regard to (16) and (18), we achieve the timelike vector $\overrightarrow{T_T}(t)$ in (15). Also, if we take the first and second derivatives of $\overrightarrow{T}(t)$ in (15) according to t and pay regard to (1), we obtain

$$\overrightarrow{T'}(t) \wedge \overrightarrow{T''}(t) = \left(-\frac{n^4}{m^3}\sin t\sin^2(nt), -\frac{n^4}{m^3}\cos t\sin^2(nt), -\frac{n^4}{m^4}\sin^2(nt)\right),$$

(19) $$\left\|\overrightarrow{T'}(t) \wedge \overrightarrow{T''}(t)\right\| = \frac{n^3}{m^3}\sin^2(nt).$$

If we take into consideration $\overrightarrow{B_T}(t) = -\dfrac{\overrightarrow{T'}(t) \wedge \overrightarrow{T''}(t)}{\left\|\overrightarrow{T'}(t) \wedge \overrightarrow{T''}(t)\right\|}$, we obtain the spacelike vector $\overrightarrow{B_T}(t)$ in (15). Last, from (5), if we pay regard to $\overrightarrow{N_T}(t) =$

$-\overrightarrow{B_T}(t) \wedge \overrightarrow{T_T}(t)$, we obtain the spacelike vector $\overrightarrow{N_T}(t)$ in (15). Thus, the proof is completed. □

Theorem 3.2. *The curvature $\kappa_T(t)$ and the torsion $\tau_T(t)$ of the timelike tangent indicatrix curve $(\overrightarrow{T})$ of $\overrightarrow{\gamma_m}(t)$ are as follows:*

$$\kappa_T(t) = \frac{1}{\sin(nt)} \quad \text{and} \quad \tau_T(t) = \frac{m}{\sin(nt)}. \tag{20}$$

Proof. If we take into consideration $\kappa_T(t) = \frac{\left\|\overrightarrow{T'}(t) \wedge \overrightarrow{T''}(t)\right\|}{\left\|\overrightarrow{T'}(t)\right\|^3}$, from (18) and (19), we obtain the curvature $\kappa_T(t)$ as (20). And, if we take into consideration the equation $\tau_T(t) = \frac{\left\langle \overrightarrow{T'}(t) \wedge \overrightarrow{T''}(t), \overrightarrow{T'''}(t)\right\rangle}{\left\|\overrightarrow{T'}(t) \wedge \overrightarrow{T''}(t)\right\|^2}$ and pay regard to (19), we obtain the torsion $\tau_T(t)$ in (20). So, the proof is completed. □

Theorem 3.3. *The Frenet formulas of timelike tangent indicatrix curve $(\overrightarrow{T})$ of $\overrightarrow{\gamma_m}(t)$ are as follows:*

$$\begin{cases} \overrightarrow{D_{T_T} T_T}(t) = \dfrac{\overrightarrow{T_T}'(t)}{v_T} = \left(\dfrac{\cos t}{\sin(nt)}, -\dfrac{\sin t}{\sin(nt)}, 0\right) \quad (sl), \\ \overrightarrow{D_{T_T} N_T}(t) = \dfrac{\overrightarrow{N_T}'(t)}{v_T} = \left(-\dfrac{m \sin t}{n \sin(nt)}, -\dfrac{m \cos t}{n \sin(nt)}, 0\right) \quad (sl), \\ \overrightarrow{D_{T_T} B_T}(t) = \dfrac{\overrightarrow{B_T}'(t)}{v_T} = \left(\dfrac{m \cos t}{\sin(nt)}, -\dfrac{m \sin t}{\sin(nt)}, 0\right) \quad (sl). \end{cases} \tag{21}$$

Proof. For the Frenet derivative formulas of $(\overrightarrow{T})$, we write

$$\begin{bmatrix} \overrightarrow{D_{T_T} T_T}(t) \\ \overrightarrow{D_{T_T} N_T}(t) \\ \overrightarrow{D_{T_T} B_T}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_T(t) & 0 \\ \kappa_T(t) & 0 & -\tau_T(t) \\ 0 & \tau_T(t) & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T_T}(t) \\ \overrightarrow{N_T}(t) \\ \overrightarrow{B_T}(t) \end{bmatrix}, \tag{22}$$

[43]. If we substitute (15) and (20) in (22) , we obtain (21). On the other hand, we can write

$$\overrightarrow{D_{T_T} T_T}(t) = \frac{d\overrightarrow{T_T}(t)}{dt_T} = \frac{d\overrightarrow{T_T}}{dt}\frac{dt}{dt_T} = \frac{\overrightarrow{T_T}'(t)}{v_T}. \tag{23}$$

So, if we derivate of $\overrightarrow{T_T}(t)$ in (15) according to t_T and take into consideration (18), we obtain

$$\overrightarrow{D_{T_T}T_T}(t) = \left(\frac{\cos t}{\sin(nt)}, -\frac{\sin t}{\sin(nt)}, 0\right). \tag{24}$$

From (23) and (24), we get $\overrightarrow{D_{T_T}T_T}(t)$ in (21). Similarly, we achieve the vectors $\overrightarrow{D_{T_T}N_T}(t)$ and $\overrightarrow{D_{T_T}B_T}(t)$. □

Theorem 3.4. *There are the following differential equations for the Frenet derivative vectors of the timelike tangent indicatrix curve* $(\overrightarrow{T})$ *of* $\overrightarrow{\gamma_m}(t)$*:*

$$\begin{bmatrix} \overrightarrow{D_{T_T}T_T}(t) & \overrightarrow{D^2{}_{T_T}T_T}(t) & \overrightarrow{D^3{}_{T_T}T_T}(t) \\ \overrightarrow{D_{T_T}N_T}(t) & \overrightarrow{D^2{}_{T_T}N_T}(t) & \overrightarrow{D^3{}_{T_T}N_T}(t) \\ \overrightarrow{D_{T_T}B_T}(t) & \overrightarrow{D^2{}_{T_T}B_T}(t) & \overrightarrow{D^3{}_{T_T}B_T}(t) \end{bmatrix} \begin{bmatrix} \lambda_T \\ \mu_T \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

(25)

where

$$\lambda_T(t) = -\frac{1}{\sin^2(nt)} \quad \text{and} \quad \mu_T(t) = \frac{3m\cos(nt)}{\sin^2(nt)}.$$

Proof. If we take the first, second and third derivatives of $\overrightarrow{T_T}(t)$ in (15) according to t_T:

$$\overrightarrow{D_{T_T}T_T}(t) = \left(\frac{\cos t}{\sin(nt)}, -\frac{\sin t}{\sin(nt)}, 0\right),$$

$$\overrightarrow{D^2{}_{T_T}T_T}(t) = \left(-\frac{m\sin t}{n\sin^2(nt)} - \frac{m\cos t\cos(nt)}{\sin^3(nt)}, -\frac{m\cos t}{n\sin^2(nt)} + \frac{m\sin t\cos(nt)}{\sin^3(nt)}, 0\right),$$

$$\overrightarrow{D^3{}_{T_T}T_T}(t) = \left(\frac{\cos t}{\sin^3(nt)} + \frac{3m^2\sin t\cos(nt)}{n\sin^4(nt)} + \frac{3m^2\cos t\cos^2(nt)}{\sin^5(nt)},\right.$$
$$\left. -\frac{\sin t}{\sin^3(nt)} + \frac{3m^2\cos t\cos(nt)}{n\sin^4(nt)} + \frac{3m^2\sin t\cos^2(nt)}{\sin^5(nt)}, 0\right).$$

If we substitute these vectors in (25) and resolve this differential equation, we obtain the values of $\lambda_T(t)$ and $\mu_T(t)$. Similarly, the values are also achieved for $\overrightarrow{D_{T_T}N_T}(t)$ and $\overrightarrow{D_{T_T}B_T}(t)$. □

Theorem 3.5. *The Darboux vector $\overrightarrow{F_T}(t)$ belong to the Frenet frame of the timelike tangent indicatrix curve $(\overrightarrow{T})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$\overrightarrow{F_T}(t) = (0,0,1) \qquad (tl). \tag{26}$$

Proof. From (1), (15) and (21, we obtain

$$\overrightarrow{F_T}(t) = -\overrightarrow{N_T}(t) \wedge \overrightarrow{N_T}'(t) = (0,0,1).$$

Or from (15), (18) and (20), we obtain again

$$\overrightarrow{F_T}(t) = v_T(t)\left(\tau_T(t)\overrightarrow{T_T}(t) - \kappa_T(t)\overrightarrow{B_T}(t)\right) = (0,0,1),$$

[43]. □

Theorem 3.6. *The hyperbolic angle φ_T between the unit timelike vectors $\overrightarrow{T_T}(t)$ and $\overrightarrow{F_T}(t)$ of the timelike tangent indicatrix curve $(\overrightarrow{T})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$\varphi_T = \operatorname{arctanh}\left(\frac{1}{m}\right) = \frac{1}{2}\ln\left(\frac{m+1}{m-1}\right).$$

Proof. Since both the timelike vectors $\overrightarrow{T_T}(t)$ and $\overrightarrow{F_T}(t)$ are future pointing, Figure 3, from (3), (15) and (26), we obtain

$$\left\langle \overrightarrow{F_T}(t), \overrightarrow{T_T}(t) \right\rangle = -\left\|\overrightarrow{F_T}(t)\right\| \left\|\overrightarrow{T_T}(t)\right\| \cosh\varphi_T = -\cosh\varphi_T. \tag{27}$$

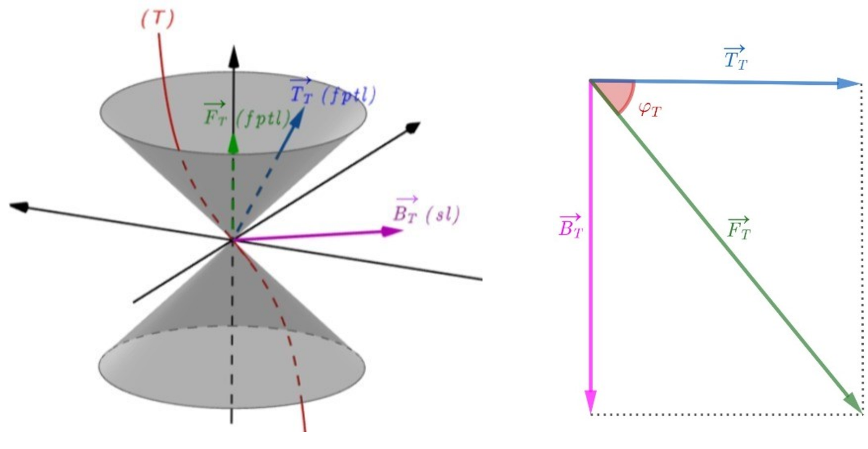

FIGURE 3. The Darboux vector $\overrightarrow{F_T}$ of the tangent indicatrix curve $(\overrightarrow{T})$ on $\mathbb{S}^2_1$

In addition to this, from the inner product of $\overrightarrow{T_T}(t)$ and $\overrightarrow{F_T}(t)$, in (15) and (26), we obtain

$$\left\langle \overrightarrow{F_T}(t), \overrightarrow{T_T}(t) \right\rangle = -n. \tag{28}$$

From the equality of (27) and (28), we achieve

$$\cosh \varphi_T = n. \tag{29}$$

Similarly, from the inner product formula, (4), (15) and (26), we obtain

$$\left| \left\langle \overrightarrow{F_T}(t), \overrightarrow{B_T}(t) \right\rangle \right| = \sinh \varphi_T = \frac{n}{m}. \tag{30}$$

From (29) and (30), we achieve

$$\varphi_T = \operatorname{arctanh}\left(\frac{1}{m}\right). \tag{31}$$

On the other hand, we know

$$\cosh \varphi_T + \sinh \varphi_T = e^{\varphi_T}. \tag{32}$$

Then, from (29), (30) and (32), we obtain

$$\varphi_T = \ln\left(\frac{n(m+1)}{m}\right),$$

Here, if we follow the necessary procedures, we obtain

$$\varphi_T = \frac{1}{2}\ln\left(\frac{m+1}{m-1}\right). \tag{33}$$

So, from the equality of (31) and (33), the proof is completed. □

Corollary 3.7. *The unit vector $\overrightarrow{C_T}(t)$ in the direction of the Darboux vector $\overrightarrow{F_T}(t)$ of the timelike tangent indicatrix curve $(\overrightarrow{T})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$\overrightarrow{C_T}(t) = (0, 0, 1). \tag{34}$$

Proof. We can easily see (34) by using (26) in $\overrightarrow{C_T}(t) = \dfrac{\overrightarrow{F_T}(t)}{\left\| \overrightarrow{F_T}(t) \right\|}$, but we want to show that (6) works. So, indeed, we pay regard to (15), (29) and (30) in the equation

$$\overrightarrow{C_T}(t) = \cosh \varphi_T \overrightarrow{T_T}(t) - \sinh \varphi_T \overrightarrow{B_T}(t),$$

we obtain (34). □

Corollary 3.8. *The timelike indicatrix curve $(\overrightarrow{T})$ of $\overrightarrow{\gamma_m}(t)$ is a helix on $\mathbb{S}^2_1$ and the axis of this helix is $\overrightarrow{C_T}(t)$, Figure 4.*

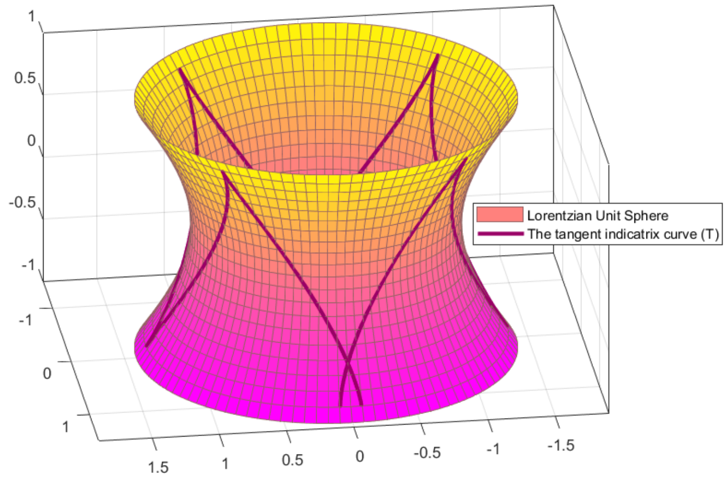

FIGURE 4. The Tangent Indicatrix Curve $(\vec{T})$ on $\mathbb{S}_1^2$

Proof. Let θ_T be the fixed angle between of $\overrightarrow{T_T}(t)$ and the axis $\overrightarrow{U_T}(t)$. From (20), the rate of the curvatures of the timelike curve $(\vec{T})$ is

$$\frac{\tau_T(t)}{\kappa_T(t)} = \coth\theta_T = m = constant. \tag{35}$$

Then, the curve $(\vec{T})$ is a helix, [41]. The axis $\overrightarrow{U_T}(t)$ of this helix is $\vec{C}(t)$. Indeed, from the definition of helix and (35), we write

$$\begin{aligned} \left\langle \overrightarrow{T_T}(t), \overrightarrow{U_T}(t) \right\rangle &= -\cosh\theta_T = -n, \\ \left\langle \overrightarrow{B_T}(t), \overrightarrow{U_T}(t) \right\rangle &= \sinh\theta_T = \frac{n}{m}. \end{aligned} \tag{36}$$

And so, from (15) and (36), we obtain

$$\overrightarrow{U_T}(t) = \cosh\theta_T \overrightarrow{T_T}(t) - \sinh\theta_T \overrightarrow{B_T}(t) = (0,0,1) = \overrightarrow{C_T}(t),$$

where, we can say $\theta_T = \varphi_T$. □

Theorem 3.9. *The arc length s_T of the timelike tangent indicatrix curve $(\vec{T})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$s_T = -\frac{\cos(nt)}{m}. \tag{37}$$

Proof. Considering (14), since $\overrightarrow{\gamma_m}(t)$ is in $\left[-\frac{\pi}{2n}, t\right]$, from (9), (17) and (18), the arc length of $(\overrightarrow{T})$ is

$$s_T = \int_{-\frac{\pi}{2n}}^{t} \left\| \frac{d\overrightarrow{T}(t)}{dt} \right\| dt = \int_{-\frac{\pi}{2n}}^{t} \left\| \overrightarrow{T'}(t) \right\| dt = \frac{n}{m} \int_{-\frac{\pi}{2n}}^{t} \sin(nt) dt = -\frac{\cos(nt)}{m}.$$

From (37), $t = \dfrac{\arccos(-m s_T)}{n}$. Indeed, if t is substituted into $(\overrightarrow{T})$ in (13), $\left\| \dfrac{d\overrightarrow{T}}{ds_T} \right\| = 1$ is achieved. □

Theorem 3.10. *The geodesic curvatures k_T and γ_T with regard to $\mathbb{R}_1^3$ and $\mathbb{S}_1^2$ of the timelike tangent indicatrix curve $(\overrightarrow{T})$ of $\overrightarrow{\gamma_m}(t)$ are as follows:*

$$k_T(t) = \frac{1}{\sin(nt)} \qquad \text{and} \qquad \gamma_T(t) = \cot(nt).$$

Proof. From (10) and (21), the geodesic curvature k_T with regard to $\mathbb{R}_1^3$ of (T) is given as follows:

$$k_T(t) = \sqrt{\left| \left\langle \overrightarrow{D_{T_T} T_T}(t), \overrightarrow{D_{T_T} T_T}(t) \right\rangle \right|} = \frac{1}{|\sin(nt)|}.$$

Also, from (11) and (21), we obtain

$$\begin{aligned} \overrightarrow{\overline{D}_{T_T} T_T}(t) &= \overrightarrow{D_{T_T} T_T}(t) - \overrightarrow{T}(t) \\ &= \Bigg(\frac{\cos t \cos^2(nt) + n \sin t \cos(nt) \sin(nt)}{\sin(nt)}, \\ &\quad -\frac{\sin t \cos^2(nt) + n \cos t \cos(nt) \sin(nt)}{\sin(nt)}, -\frac{n}{m}\cos(nt) \Bigg). \end{aligned}$$

So, the geodesic curvature γ_T with regard to $\mathbb{S}_1^2$ of $(\overrightarrow{T})$ is achieved as follows:

$$\gamma_T(t) = \sqrt{\left| \left\langle \overrightarrow{\overline{D}_{T_T} T_T}(t), \overrightarrow{\overline{D}_{T_T} T_T}(t) \right\rangle \right|} = |\cot(nt)|.$$

□

3.2. The Principal Normal Indicatrix Curve of $\overrightarrow{\gamma_m}(t)$

Theorem 3.11. *The Frenet frame $\left\{ \overrightarrow{T_N}(t), \overrightarrow{N_N}(t), \overrightarrow{B_N}(t) \right\}$ of the spacelike principal normal indicatrix curve $(\overrightarrow{N})$ of $\overrightarrow{\gamma_m}(t)$ is as follows, respectively:*

$$\left\{\begin{array}{ll}\overrightarrow{T_N}(t) = (\cos t, -\sin t, 0) & (sl),\\ \overrightarrow{N_N}(t) = (-\sin t, -\cos t, 0) & (sl),\\ \overrightarrow{B_N}(t) = (0, 0, -1) & (tl).\end{array}\right. \tag{38}$$

Proof. Let the parameter of $(\overrightarrow{N})$ be t_N. If we derivate of $\overrightarrow{N}(t)$ in (13) according to t, we obtain

$$\overrightarrow{N'}(t) = \left(\frac{n}{m}\cos t, -\frac{n}{m}\sin t, 0\right). \tag{39}$$

Also, if we pay regard to the equation

$$\overrightarrow{T_N}(t) = \frac{d\overrightarrow{N}(t)}{dt_N} = \frac{d\overrightarrow{N}(t)}{dt}\frac{dt}{dt_N} = \overrightarrow{N'}(t)\frac{dt}{dt_N}, \tag{40}$$

from (39) and since $\left\|\overrightarrow{T_N}(t)\right\| = 1$, we achieve

$$v_N = \frac{dt_N}{dt} = \left\|\overrightarrow{N'}(t)\right\| = \frac{n}{m}. \tag{41}$$

From (40) and (41), we write $\overrightarrow{T_N}(t) = \dfrac{\overrightarrow{N'}(t)}{\left\|\overrightarrow{N'}(t)\right\|}$. Here, if we take into consideration (39) and (41), we obtain the spacelike vector $\overrightarrow{T_N}(t)$ in (38). Also, if we take the first and second derivatives of $\overrightarrow{N}(t)$ in (15) according to t and pay regard to (1), we obtain

$$\overrightarrow{N'}(t) \wedge \overrightarrow{N''}(t) = \left(0, 0, -\frac{n^2}{m^2}\right).$$

$$\left\|\overrightarrow{N'}(t) \wedge \overrightarrow{N''}(t)\right\| = \frac{n^2}{m^2}. \tag{42}$$

If we take into consideration $\overrightarrow{B_N}(t) = \dfrac{\overrightarrow{N'}(t) \wedge \overrightarrow{N''}(t)}{\left\|\overrightarrow{N'}(t) \wedge \overrightarrow{N''}(t)\right\|}$, we obtain the timelike vector $\overrightarrow{B_N}(t)$ in (38). Last, from (7), here if we pay regard to $\overrightarrow{N_N}(t) = -\overrightarrow{B_N}(t) \wedge \overrightarrow{T_N}(t)$, we obtain the spacelike vector $\overrightarrow{N_N}(t)$ in (38). □

Theorem 3.12. *The curvature $\kappa_N(t)$ and the torsion $\tau_N(t)$ of the spacelike principal normal indicatrix curve $(\overrightarrow{N})$ of $\overrightarrow{\gamma_m}(t)$ are as follows:*

$$\kappa_N(t) = \frac{m}{n} \quad \text{and} \quad \tau_N(t) = 0. \tag{43}$$

Proof. If we take into consideration $\kappa_N(t) = \dfrac{\left\| \overrightarrow{N'}(t) \wedge \overrightarrow{N''}(t) \right\|}{\left\| \overrightarrow{N'}(t) \right\|^3}$, from (41) and (42), we obtain the curvature $\kappa_N(t)$ as (20). And if we take into consideration $\tau_N(t) = \dfrac{\left\langle \overrightarrow{N'}(t) \wedge \overrightarrow{N''}(t), \overrightarrow{N'''}(t) \right\rangle}{\left\| \overrightarrow{N'}(t) \wedge \overrightarrow{N''}(t) \right\|^2}$ and pay regard to (42), we obtain the torsion $\tau_N(t)$ in (20). □

Corollary 3.13. *The spacelike principal normal indicatrix curve* $(\overrightarrow{N})$ *of* $\overrightarrow{\gamma_m}(t)$ *is a planar circle of radius* $\dfrac{n}{m}$ *on* $\mathbb{H}_0^{2+}$*, Figure 5.*

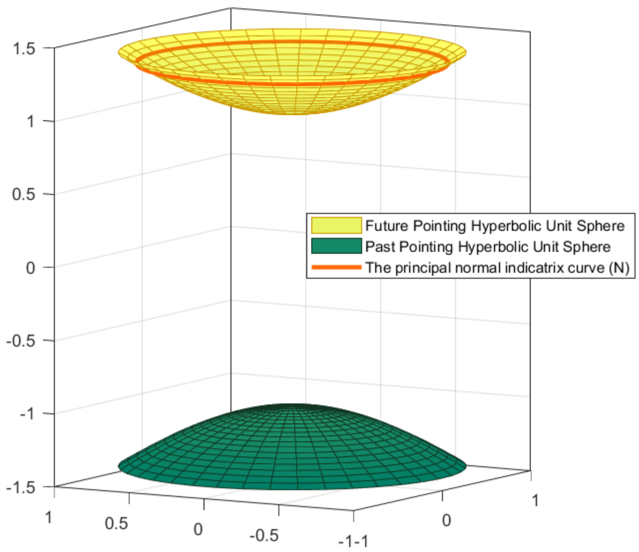

FIGURE 5. The Principal Normal Indicatrix Curve $(\vec{N})$ on $\mathbb{H}_0^{2+}$

Proof. Since the torsion of $(\overrightarrow{N}) = (-\sin t, \cos t, 0)$ is $\tau(t) = 0$, the curve $(\overrightarrow{N})$ is a planar circle on $\mathbb{H}_0^{2+}$ (on timelike plane). Also the radius of the circle is $R_N(t) = \dfrac{1}{\kappa_N(t)} = \dfrac{n}{m}$. □

Theorem 3.14. *The Frenet derivative formulas of the spacelike principal normal indicatrix curve* $(\overrightarrow{N})$ *of* $\overrightarrow{\gamma_m}(t)$ *are as follows:*

$$(44)\quad \begin{cases} \overrightarrow{D_{T_N}T_N}(t) = \dfrac{\overrightarrow{T_N}'(t)}{v_N} = \Big(-\dfrac{m}{n}\sin t, -\dfrac{m}{n}\cos t, 0\Big) \qquad (sl), \\[2ex] \overrightarrow{D_{T_N}N_N}(t) = \dfrac{\overrightarrow{N_N}'(t)}{v_N} = \Big(-\dfrac{m}{n}\cos t, -\dfrac{m}{n}\sin t, 0\Big) \qquad (sl), \\[2ex] \overrightarrow{D_{T_N}B_N}(t) = \dfrac{\overrightarrow{B_N}'(t)}{v_N} = (0,0,0) \qquad (sl). \end{cases}$$

Proof. For the Frenet derivative formulas of $(\overrightarrow{N})$, we write

$$(45)\quad \begin{bmatrix} \overrightarrow{D_{T_N}T_N}(t) \\ \overrightarrow{D_{T_N}N_N}(t) \\ \overrightarrow{D_{T_N}B_N}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_N(t) & 0 \\ -\kappa_N(t) & 0 & \tau_N(t) \\ 0 & \tau_N(t) & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T_N}(t) \\ \overrightarrow{N_N}(t) \\ \overrightarrow{B_N}(t) \end{bmatrix},$$

[43]. If we substitute (38) and (43) in (45), we obtain (44). In addition to this, we can write

$$(46)\quad \overrightarrow{D_{T_N}T_N}(t) = \frac{d\overrightarrow{T_N}(t)}{dt_N} = \frac{d\overrightarrow{T_N}}{dt}\frac{dt}{dt_N} = \frac{\overrightarrow{T_N}'(t)}{v_N}.$$

So, if we derivate of $\overrightarrow{T_N}(t)$ in (38) according to t_N and take into consideration (41), we obtain

$$(47)\quad \overrightarrow{D_{T_N}T_N}(t) = \Big(-\frac{m}{n}\sin t, -\frac{m}{n}\cos t, 0\Big) \qquad (sl).$$

From (46) and (47), we get $\overrightarrow{D_{T_N}T_N}(t)$ in (44). Similarly, we achieve the vectors $\overrightarrow{D_{T_N}N_N}(t)$ and $\overrightarrow{D_{T_N}B_N}(t)$. □

Theorem 3.15. *There are the following differential equations for the Frenet derivative vectors of the spacelike principal normal indicatrix curve $(\overrightarrow{N})$ of $\overrightarrow{\gamma_m}(t)$:*

$$\begin{bmatrix} \overrightarrow{D_{T_N}T_N}(t) & \overrightarrow{D^2{}_{T_N}T_N}(t) & \overrightarrow{D^3{}_{T_N}T_N}(t) \\ \overrightarrow{D_{T_N}N_N}(t) & \overrightarrow{D^2{}_{T_N}N_N}(t) & \overrightarrow{D^3{}_{T_N}N_N}(t) \\ \overrightarrow{D_{T_N}B_N}(t) & \overrightarrow{D^2{}_{T_N}B_N}(t) & \overrightarrow{D^3{}_{T_N}B_N}(t) \end{bmatrix} \begin{bmatrix} \lambda_N \\ \mu_N \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

(48)

where

$$\lambda_N(t) = \frac{m^2}{n^2} \qquad \textit{and} \qquad \mu_N(t) = 0.$$

Proof. If we take the first, second and third derivatives of $\overrightarrow{N_N}(t)$ in (38) according to t_N:

$$\overrightarrow{D_{T_N} N_N}(t) = \left(-\frac{m}{n}\cos t, \frac{m}{n}\sin t, 0\right),$$

$$\overrightarrow{D^2{}_{T_N} N_N}(t) = \left(\frac{m^2}{n^2}\sin t, \frac{m^2}{n^2}\cos t, 0\right),$$

$$\overrightarrow{D^3{}_{T_N} N_N}(t) = \left(\frac{m^3}{n^3}\cos t, -\frac{m^3}{n^3}\sin t, 0\right).$$

If we substitute these vectors in (48) and resolve this differential equation, we obtain the values of $\lambda_N(t)$ and $\mu_N(t)$. Similarly, the values are also achieved for $\overrightarrow{D_{T_N} T_N}(t)$ and $\overrightarrow{D_{T_N} B_N}(t)$. □

Theorem 3.16. *The Darboux vector $\overrightarrow{F_N}(t)$ belong to the Frenet frame of the spacelike principal normal indicatrix curve $(\overrightarrow{N})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$\overrightarrow{F_N}(t) = (0,0,1) \qquad (tl). \tag{49}$$

Proof. From (38) and (44, we obtain

$$\overrightarrow{F_N}(t) = -\overrightarrow{N_N}(t) \wedge \overrightarrow{N_N}'(t) = (0,0,1).$$

Or from (38), (41) and (43), we obtain again

$$\overrightarrow{F_N}(t) = \upsilon_N(t)\left(\tau_N(t)\overrightarrow{T_N}(t) - \kappa_N(t)\overrightarrow{B_N}(t)\right) = (0,0,1),$$

[43]. □

Theorem 3.17. *The hyperbolic angle φ_N between the spacelike vector $\overrightarrow{T_N}(t)$ and timelike vector $\overrightarrow{F_N}(t)$ of the spacelike principal normal indicatrix curve $(\overrightarrow{N})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$\varphi_N = 0. \tag{50}$$

Proof. Since $\overrightarrow{T_N}(t)$ is spacelike and $\overrightarrow{F_N}(t)$ is timelike, Figure 6, from (4), (38) and (49), we obtain

$$\left|\left\langle \overrightarrow{F_N}(t), \overrightarrow{T_N}(t)\right\rangle\right| = \left\|\overrightarrow{F_N}(t)\right\| \left\|\overrightarrow{T_N}(t)\right\| \sinh\varphi_N = \sinh\varphi_N. \tag{51}$$

In addition to this , from the inner product of $\overrightarrow{T_N}(t)$ and $\overrightarrow{F_N}(t)$ in (38) and (49), we obtain

$$\left\langle \overrightarrow{F_N}(t), \overrightarrow{T_N}(t)\right\rangle = 0. \tag{52}$$

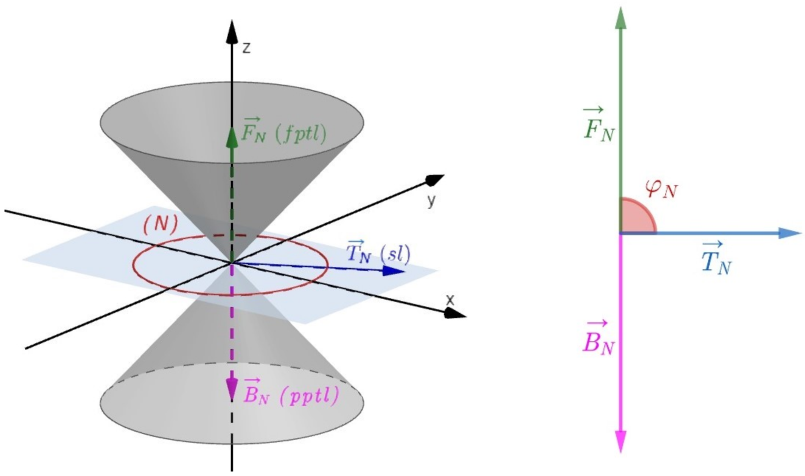

FIGURE 6. The Darboux vector $\overrightarrow{F_N}$ of the principal normal indicatrix curve $(\overrightarrow{N})$ on $\mathbb{H}_0^{2+}$

From the equality of (51) and (52), we achieve

$$\sinh \varphi_N = 0. \tag{53}$$

Similarly, since vector $\overrightarrow{B_N}(t)$ is past pointing timelike vector and $\overrightarrow{F_N}(t)$ is future pointing timelike vector, Figure 6, from (2), (38), (49) and the inner product formula, we obtain

$$\left\langle \overrightarrow{F_N}(t), \overrightarrow{B_N}(t) \right\rangle = \cosh \varphi_N = 1. \tag{54}$$

From (53) and (54), we achieve (50). □

Corollary 3.18. *The unit vector $\overrightarrow{C_N}(t)$ in the direction of the Darboux vector $\overrightarrow{F_N}(t)$ of the spacelike principal normal indicatrix curve $(\overrightarrow{N})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$\overrightarrow{C_N}(t) = (0, 0, 1). \tag{55}$$

Proof. We can easily see (55) by using (49) in $\overrightarrow{C_N}(t) = \dfrac{\overrightarrow{F_N}(t)}{\left\|\overrightarrow{F_N}(t)\right\|}$, but we want to show that (8) works. So, indeed, we pay regard to (38), (53) and (54) in the equation

$$\overrightarrow{C_N}(t) = \sinh \varphi_N \overrightarrow{T_N}(t) - \cosh \varphi_N \overrightarrow{B_N}(t),$$

we obtain (34). □

Theorem 3.19. *The arc length s_N of the spacelike principal normal indicatrix curve $(\overrightarrow{N})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$s_N = \frac{nt}{m} + \frac{\pi}{2m}. \tag{56}$$

Proof. Considering (14), since $\overrightarrow{\gamma_m}(t)$ is in $\left[-\frac{\pi}{2n}, t\right]$, from (9), (40) and (41), the arc length of $(\overrightarrow{N})$ is

$$s_N = \int_{-\frac{\pi}{2n}}^{t} \left\| \frac{d\overrightarrow{N}(t)}{dt} \right\| dt = \int_{-\frac{\pi}{2n}}^{t} \left\| \overrightarrow{N'}(t) \right\| dt = \frac{n}{m} \int_{-\frac{\pi}{2n}}^{t} dt = \frac{nt}{m} + \frac{\pi}{2m}.$$

From (56), $t = \frac{m s_N}{n} - \frac{\pi}{2n}$. Indeed, if this value is substituted into $(\overrightarrow{N})$ in (13), $\left\| \frac{d\overrightarrow{N}}{ds_T} \right\| = 1$ is achieved. □

Theorem 3.20. *The geodesic curvatures k_N and ψ_N with regard to $\mathbb{R}^3_1$ and $\mathbb{H}^2_0$ of the spacelike principal normal indicatrix curve $(\overrightarrow{N})$ of $\overrightarrow{\gamma_m}(t)$ are as follows:*

$$k_N(t) = \frac{m}{n} \qquad \text{and} \qquad \psi_N(t) = |m|.$$

Proof. From (10) and (44), the geodesic curvature k_N with regard to $\mathbb{R}^3_1$ of $(\overrightarrow{N})$ is

$$k_N(t) = \sqrt{\left\langle \overrightarrow{D_{T_N} T_N}(t), \overrightarrow{D_{T_N} T_N}(t) \right\rangle} = \frac{m}{n}.$$

Also, from (12) and (44), we obtain

$$\overrightarrow{\overline{\overline{D}}_{T_N} T_N}(t) = \overrightarrow{D_{T_N} T_N}(t) - \overrightarrow{N}(t) = \left(-\frac{(n^2+m^2)}{nm} \sin t,\ -\frac{(n^2+m^2)}{nm} \cos t, -n \right).$$

So, the geodesic curvature ψ_N with regard to $\mathbb{H}^2_0$ of $(\overrightarrow{N})$ is achieved as follows:

$$\psi_N(t) = \sqrt{\left| \left\langle \overrightarrow{\overline{\overline{D}}_{T_N} T_N}(t), \overrightarrow{\overline{\overline{D}}_{T_N} T_N}(t) \right\rangle \right|} = |m|.$$

□

3.3. The Binormal Indicatrix Curve of $\overrightarrow{\gamma_m}(t)$

Theorem 3.21. *The Frenet frame* $\left\{\overrightarrow{T_B}(t), \overrightarrow{N_B}(t), \overrightarrow{B_B}(t)\right\}$ *of the timelike binormal indicatrix curve* $(\overrightarrow{B})$ *of* $\overrightarrow{\gamma_m}(t)$ *is as follows, respectively:*

$$\begin{cases} \overrightarrow{T_B}(t) = \left(-\frac{n}{m}\sin t, \, -\frac{n}{m}\cos t, \, -n\right) \qquad (tl), \\ \overrightarrow{N_B} = (-\cos t, \, \sin t, 0) \qquad (sl), \\ \overrightarrow{B_B}(t) = \left(n\sin t, n\cos t, \frac{n}{m}\right) \qquad (sl). \end{cases} \tag{57}$$

Proof. Let the parameter of $(\overrightarrow{B})$ be t_B. If we derivate of $\overrightarrow{B}(t)$ in (13) according to t, we obtain

$$\overrightarrow{B'}(t) = \left(-\frac{n^2}{m^2}\sin t\cos(nt), \, -\frac{n^2}{m^2}\cos t\cos(nt), \, -\frac{n^2}{m}\cos(nt)\right). \tag{58}$$

Also, if we pay regard to the equation

$$\overrightarrow{T_B}(t) = \frac{d\overrightarrow{B}(t)}{dt_B} = \frac{d\overrightarrow{B}(t)}{dt}\frac{dt}{dt_B} = \overrightarrow{B'}(t)\frac{dt}{dt_B}, \tag{59}$$

from (58) and since $\left\|\overrightarrow{T_B}(t)\right\| = 1$, we achieve

$$v_B = \frac{dt_B}{dt} = \left\|\overrightarrow{B'}(t)\right\| = \frac{n}{m}\cos(nt). \tag{60}$$

From (59) and (60), we write $\overrightarrow{T_B}(t) = \dfrac{\overrightarrow{B'}(t)}{\left\|\overrightarrow{B'}(t)\right\|}$. Here, if we take into consideration (58) and (60), we obtain the timelike vector $\overrightarrow{T_B}(t)$ in (57). Also, if we take the first and second derivatives of $\overrightarrow{B}(t)$ in (15) according to t and pay regard to (1), we obtain

$$\overrightarrow{B'}(t) \wedge \overrightarrow{B''}(t) = \left(-\frac{n^4}{m^3}\sin t\cos^2(nt), \, -\frac{n^4}{m^3}\cos t\cos^2(nt), \, -\frac{n^4}{m^4}\cos^2(nt)\right),$$

$$\left\|\overrightarrow{B'}(t) \wedge \overrightarrow{B''}(t)\right\| = \frac{n^3}{m^3}\cos^2(nt). \tag{61}$$

If we take into consideration the equation $\overrightarrow{B_T}(t) = -\dfrac{\overrightarrow{B'}(t) \wedge \overrightarrow{B''}(t)}{\left\|\overrightarrow{B'}(t) \wedge \overrightarrow{B''}(t)\right\|}$, we obtain the spacelike vector $\overrightarrow{B_B}(t)$ in (57). Last, from (5), if we pay regard to $\overrightarrow{N_B}(t) = -\overrightarrow{B_B}(t) \wedge \overrightarrow{T_B}(t)$, we obtain the spacelike vector $\overrightarrow{N_B}(t)$ in (57). Thus, the proof is completed. □

Theorem 3.22. *The curvature $\kappa_B(t)$ and the torsion $\tau_B(t)$ of the timelike binormal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$ are as follows:*

$$\kappa_B(t) = \frac{1}{\cos(nt)} \quad \text{and} \quad \tau_B(t) = -\frac{m}{\cos(nt)}. \tag{62}$$

Proof. If we take into consideration the equation $\kappa_B(t) = \frac{\left\|\overrightarrow{B'}(t) \wedge \overrightarrow{B''}(t)\right\|}{\left\|\overrightarrow{B'}(t)\right\|^3}$, from (60) and (61), we obtain the curvature $\kappa_B(t)$ as (62). And if we take into consideration $\tau_B(t) = \frac{\left\langle \overrightarrow{B'}(t) \wedge \overrightarrow{B''}(t), \overrightarrow{B'''}(t)\right\rangle}{\left\|\overrightarrow{B'}(t) \wedge \overrightarrow{B''}(t)\right\|^2}$ and pay regard to (61), we obtain the torsion $\tau_B(t)$ in (62). Thus, the proof is completed. □

Theorem 3.23. *The Frenet derivative formulas of the timelike binormal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$ are as follows:*

$$\begin{cases} \overrightarrow{D_{T_B} T_B}(t) = \frac{\overrightarrow{T_B'}(t)}{v_B} = \left(-\frac{\cos t}{\cos(nt)}, \frac{\sin t}{\cos(nt)}, 0\right) \quad (sl), \\ \overrightarrow{D_{T_B} N_B}(t) = \frac{\overrightarrow{N_B'}(t)}{v_B} = \left(-\frac{m \sin t}{n\cos(nt)}, \frac{m\cos t}{n\cos(nt)}, 0\right) \quad (sl), \\ \overrightarrow{D_{T_B} B_B}(t) = \frac{\overrightarrow{B_B'}(t)}{v_B} = \left(\frac{m\cos t}{\cos(nt)}, -\frac{m \sin t}{\cos(nt)}, 0\right) \quad (sl). \end{cases} \tag{63}$$

Proof. For the Frenet derivative formulas of $(\overrightarrow{B})$, we write

$$\begin{bmatrix} \overrightarrow{D_{T_B} T_B}(t) \\ \overrightarrow{D_{T_B} N_B}(t) \\ \overrightarrow{D_{T_B} B_B}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_B(t) & 0 \\ \kappa_B(t) & 0 & -\tau_B(t) \\ 0 & \tau_B(t) & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T_B}(t) \\ \overrightarrow{N_B}(t) \\ \overrightarrow{B_B}(t) \end{bmatrix}, \tag{64}$$

[43]. If we substitute (57) and (62) in the last equation, we obtain (63). In addition to this, we can write

$$\overrightarrow{D_{T_B} T_B}(t) = \frac{d\overrightarrow{T_B}(t)}{dt_B} = \frac{d\overrightarrow{T_B}}{dt}\frac{dt}{dt_B} = \frac{\overrightarrow{T_B'}(t)}{v_B}. \tag{65}$$

So, if we derivate of $\overrightarrow{T_B}(t)$ in (57) according to t_B and take into consideration (60), we obtain

$$\overrightarrow{D_{T_B} T_B}(t) = \left(-\frac{\cos t}{\cos(nt)}, \frac{\sin t}{\cos(nt)}, 0\right) \quad (sl). \tag{66}$$

From (65) and (66), we get $\overrightarrow{D_{T_B}T_B}(t)$ in (63). Similarly, we achieve the vectors $\overrightarrow{D_{T_B}N_B}(t)$ and $\overrightarrow{D_{T_B}B_B}(t)$. □

Theorem 3.24. *There are the following differential equations for the Frenet derivative vectors of the timelike binormal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$:*

$$\begin{bmatrix} \overrightarrow{D_{T_B}T_B}(t) & \overrightarrow{D^2{}_{T_B}T_B}(t) & \overrightarrow{D^3{}_{T_B}T_B}(t) \\ \overrightarrow{D_{T_B}N_B}(t) & \overrightarrow{D^2{}_{T_B}N_B}(t) & \overrightarrow{D^3{}_{T_B}N_B}(t) \\ \overrightarrow{D_{T_B}B_B}(t) & \overrightarrow{D^2{}_{T_B}B_B}(t) & \overrightarrow{D^3{}_{T_B}B_B}(t) \end{bmatrix} \begin{bmatrix} \lambda_B \\ \mu_B \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

(67)

where

$$\lambda_B(t) = -\frac{1}{\cos^2(nt)} \quad \textit{and} \quad \mu_B(t) = \frac{3m\sin(nt)}{\cos^2(nt)}.$$

Proof. If we take the first, second and third derivatives of $\overrightarrow{T_B}(t)$ in (57) according to t_B:

$$\overrightarrow{D_{T_B}T_B}(t) = \left(-\frac{\cos t}{\cos(nt)}, \frac{\sin t}{\cos(nt)}, 0\right),$$

$$\overrightarrow{D^2{}_{T_B}T_B}(t) = \left(\frac{m\sin t}{n\cos^2(nt)} - \frac{m\cos t\sin(nt)}{\cos^3(nt)}, \frac{m\cos t}{n\cos^2(nt)} + \frac{m\sin t\sin(nt)}{\cos^3(nt)}, 0\right),$$

$$\overrightarrow{D^3{}_{T_B}T_B}(t) = \left(-\frac{\cos t}{\cos^3(nt)} + \frac{3m^2\sin t\sin(nt)}{n\cos^4(nt)} - \frac{3m^2\cos t\sin^2(nt)}{\cos^5(nt)}, \right.$$
$$\left. \frac{\sin t}{\cos^3(nt)} + \frac{3m^2\cos t\sin(nt)}{n\cos^4(nt)} + \frac{3m^2\sin t\sin^2(nt)}{\cos^5(nt)}, 0\right).$$

If we substitute these vectors in (67) and resolve this differential equation, we obtain the values of $\lambda_B(t)$ and $\mu_B(t)$. Similar values are also achieved $\overrightarrow{D_{T_B}N_B}(t)$ and $\overrightarrow{D_{T_B}B_B}(t)$. □

Theorem 3.25. *The Darboux vector $\overrightarrow{F_B}(t)$ belong to the Frenet frame of the timelike binormal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

(68) $$\overrightarrow{F_B}(t) = (0, 0, 1) \quad (tl).$$

Proof. From (1), (57) and (63, we obtain

$$\overrightarrow{F_B}(t) = -\overrightarrow{N_B}(t) \wedge \overrightarrow{N_B'}(t) = (0,0,1).$$

Or from (57), (60) and (62), we obtain again

$$\overrightarrow{F_B}(t) = v_B(t)\left(\tau_B(t)\overrightarrow{T_B}(t) - \kappa_B(t)\overrightarrow{B_B}(t)\right) = (0,0,1),$$

[43]. □

Theorem 3.26. *The hyperbolic angle φ_B between the timelike vectors $\overrightarrow{T_B}(t)$ and $\overrightarrow{F_B}(t)$ of the timelike binormal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$\varphi_B = \operatorname{arctanh}\left(-\frac{1}{m}\right) = \frac{1}{2}\ln\left(\frac{m-1}{m+1}\right).$$

Proof. Since both the timelike vectors $\overrightarrow{T_B}(t)$ and $\overrightarrow{F_B}(t)$ are future pointing, Figure 7, from (3), (57), (68) and the inner product formula, we obtain

$$\left\langle \overrightarrow{F_B}(t), \overrightarrow{T_B}(t)\right\rangle = -\left\|\overrightarrow{F_B}(t)\right\|\left\|\overrightarrow{T_B}(t)\right\|\cosh\varphi_B = -\cosh\varphi_B. \tag{69}$$

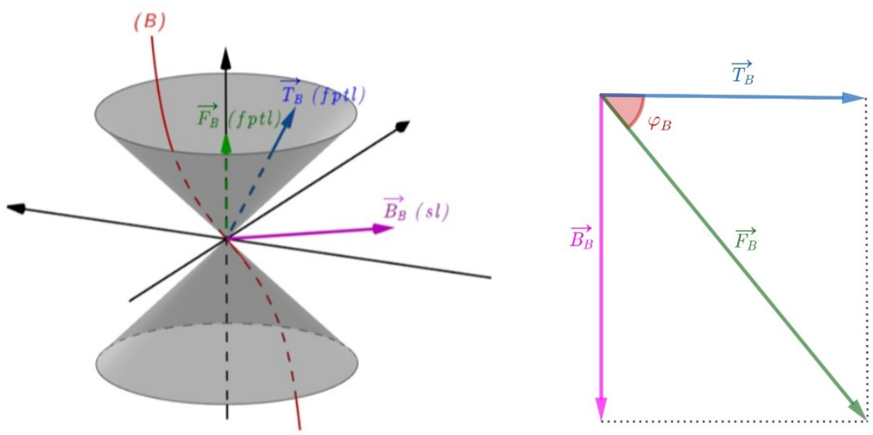

FIGURE 7. The Darboux vector $\overrightarrow{F_B}$ of the binormal indicatrix curve $(\overrightarrow{B})$ on $\mathbb{S}^2_1$

In addition to this, from the inner product of $\overrightarrow{T_B}(t)$ and $\overrightarrow{F_B}(t)$, in (57) and (68), we obtain

$$\left\langle \overrightarrow{F_B}(t), \overrightarrow{T_B}(t)\right\rangle = n. \tag{70}$$

From the equality of (69) and (70), we achieve

$$\cosh\varphi_B = -n. \tag{71}$$

Similarly, from the inner product formula, (4), (57) and (68), we obtain

$$\left|\left\langle \overrightarrow{F_B}(t), \overrightarrow{B_B}(t)\right\rangle\right| = \sinh\varphi_B = \frac{n}{m}. \tag{72}$$

From (71) and (72), we achieve

$$\varphi_B = \operatorname{arctanh}\left(-\frac{1}{m}\right). \tag{73}$$

On the other hand, we know

$$\cosh\varphi_B + \sinh\varphi_B = e^{\varphi_B}. \tag{74}$$

Then, from (71), (72) and (74), we obtain

$$\varphi_B = \ln\left(\frac{n(1-m)}{m}\right),$$

Here, if we follow the necessary procedures, we obtain

$$\varphi_B = \frac{1}{2}\ln\left(\frac{m+1}{m-1}\right). \tag{75}$$

So, from the equality of (73) and (75), the proof is completed. □

Corollary 3.27. *The unit vector $\overrightarrow{C_B}(t)$ in the direction of the Darboux vector $\overrightarrow{F_B}(t)$ of the timelike binormal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$\overrightarrow{C_B}(t) = (0, 0, 1). \tag{76}$$

Proof. We can easily see (76) by using (68) in $\overrightarrow{C_B}(t) = \dfrac{\overrightarrow{F_B}(t)}{\left\|\overrightarrow{F_B}(t)\right\|}$, but we want to show that (6) works. So, indeed, we pay regard to (57), (71) and (72) in the equation

$$\overrightarrow{C_B}(t) = \cosh\varphi_B\overrightarrow{T_B}(t) - \sinh\varphi_B\overrightarrow{B_B}(t),$$

we obtain (76). □

Corollary 3.28. *The binormal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$ is a helix on $\mathbb{S}^2_1$ and the axis of this helix is $\overrightarrow{C_B}(t)$, Figure 8.*

Proof. Let θ_B be the fixed angle between of $\overrightarrow{T_B}(t)$ and the axis $\overrightarrow{U_B}(t)$. From (62), the rate of the curvatures of the timelike curve $(\overrightarrow{B})$ is

$$\frac{\tau_B(t)}{\kappa_B(t)} = \coth\theta_B = -m = constant. \tag{77}$$

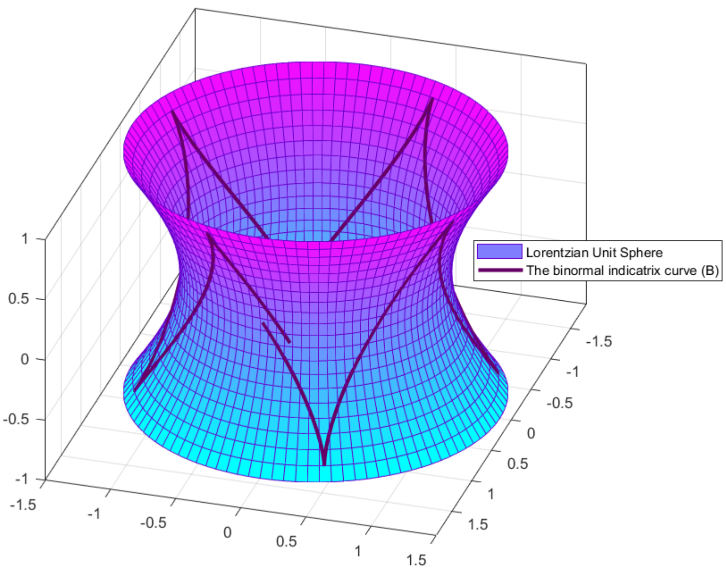

FIGURE 8. The Binormal Indicatrix Curve $(\vec{B})$ on $\mathbb{S}_1^2$

Then, the curve $(\overrightarrow{B})$ is a helix, [41]. The timelike axis $\overrightarrow{U_B}(t)$ of this helix is $\overrightarrow{C_B}(t)$. Indeed, from the definition of helix and (77), we write

$$\left\langle \overrightarrow{T_B}(t), \overrightarrow{U_B}(t) \right\rangle = -\cosh\theta_B = -n, \tag{78}$$
$$\left\langle \overrightarrow{B_T}(t), \overrightarrow{U_T}(t) \right\rangle = \sinh\theta_T = \frac{n}{m}.$$

And so, from (57) and (78), we obtain

$$\overrightarrow{U_B}(t) = \cosh\theta_B \overrightarrow{T_B}(t) - \sinh\theta_B \overrightarrow{B_B}(t) = (0,0,1) = \overrightarrow{C_B}(t),$$

where, we can say $\theta_B = \varphi_B$. □

Theorem 3.29. *The arc length s_B of the timelike principal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$ is as follows:*

$$s_B = \frac{\sin(nt) + n}{m}. \tag{79}$$

Proof. Considering (14), since $\overrightarrow{\gamma_m}(t)$ is in $\left[-\frac{\pi}{2n}, t\right]$, from (9), (59) and (60), the arc length of $(\overrightarrow{B})$ is

$$s_B = \int_{-\frac{\pi}{2n}}^{t} \left\| \frac{d\overrightarrow{B}(t)}{dt} \right\| dt = \int_{-\frac{\pi}{2n}}^{t} \left\| \overrightarrow{B'}(t) \right\| dt = \frac{n}{m} \int_{-\frac{\pi}{2n}}^{t} \cos(nt) dt = \frac{\sin(nt) + n}{m}.$$

From (79), $t = \dfrac{\arcsin(ms_B - n)}{n}$. Indeed, if this value is substituted into $(\overrightarrow{B})$ in (13), $\left\| \dfrac{d\overrightarrow{B}}{ds_B} \right\| = 1$ is achieved. □

Theorem 3.30. *The geodesic curvatures k_B and γ_T with regard to $\mathbb{R}^3_1$ and $\mathbb{S}^2_1$ of the timelike principal indicatrix curve $(\overrightarrow{B})$ of $\overrightarrow{\gamma_m}(t)$ are as follows:*

$$k_B(t) = \frac{1}{\sin(nt)} \qquad and \qquad \gamma_B(t) = \cot(nt).$$

Proof. From (10) and (63), the geodesic curvature k_B with regard to $\mathbb{R}^3_1$ of $(\overrightarrow{B})$ is given as follows:

$$k_B(t) = \sqrt{\left| \left\langle \overrightarrow{D_{T_B} T_B}(t), \overrightarrow{D_{T_B} T_B}(t) \right\rangle \right|} = \frac{1}{|\cos(nt)|}.$$

Also, from (11) and (63), we obtain

$$\begin{aligned} \overrightarrow{\overline{D}_{T_B} T_B}(t) &= \overrightarrow{D_{T_B} T_B}(t) - \overrightarrow{B}(t) \\ &= \Bigg(\frac{-\cos t \left(\sin^2(nt)\right) + n \sin t \cos(nt) \sin(nt)}{\cos(nt)}, \\ &\quad \frac{\sin t \left(\sin^2(nt)\right) + n \cos t \cos(nt) \sin(nt)}{\cos(nt)}, \frac{n \sin(nt)}{m} \Bigg). \end{aligned}$$

So, the geodesic curvature γ_B with regard to $\mathbb{S}^2_1$ of $(\overrightarrow{B})$ is achieved as follows:

$$\gamma_B(t) = \sqrt{\left| \left\langle \overrightarrow{\overline{D}_{T_B} T_B}(t), \overrightarrow{\overline{D}_{T_B} T_B}(t) \right\rangle \right|} = |\tan(nt)|.$$

□

4. Discussion and Conclusions

In this study, the various geometric properties of the spherical indicatrix curves obtained from the Frenet vectors of the spacelike Salkowski curve with the timelike principal normal in Lorentz 3-space are investigated. This work can be done for other Salkowski curve types or various other curve examples in the same space, similarly it can be studied in other well-known spaces. Eventually, we believe that the procedures and the path followed in this study will be useful for those who will do similar studies in Lorentz 3-space.

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