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A FUNCTION CONTAINING ALL LAGRANGE NUMBERS LESS THAN THREE

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Abstract. Given a real number α, the Lagrange number of α is the supremum of all real numbers $L > 0$ for which the inequality $|\alpha - p/q| < (Lq^2)^{-1}$ holds for infinitely many rational numbers p/q. All Lagrange numbers less than 3 can be arranged as a set $\{l_{p/q} : p/q \in \mathbb{Q} \cap [0, 1]\}$ using the Farey index. The present paper considers a function $C(\alpha)$ devised from Sturmian words. We demonstrate that the function $C(\alpha)$ contains all information on Lagrange numbers less than 3. More precisely, we prove that for any real number $\alpha \in (0, 1]$, the value $C(\alpha) - C(0)$ is equal to the sum of all numbers $3 - l_{p/q}$ where the Farey index p/q is less than α.

1. Introduction

For a real number $\alpha \in \mathbb{R}$, the *Lagrange number* $L(\alpha)$ of α is defined by the supremum of all real numbers $L > 0$ for which the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}$$

holds for infinitely many rational numbers $p/q \in \mathbb{Q}$.

Lagrange numbers less than 3 are particularly distinguished in view of Diophantine approximations. Let

$$\mathcal{L} = \left\{\sqrt{5}, \sqrt{8}, \frac{\sqrt{221}}{5}, \frac{\sqrt{1517}}{13}, \dots\right\}$$

be the set of Lagrange numbers less than 3. Markov showed in [14, 15] that $\mathcal{L}$ is a discrete set, and that each element in $\mathcal{L}$ characterizes some badly approximable real numbers. He also recognized that $\mathcal{L}$ has intimate connections to minimal values of indefinite quadratic forms on the integer lattice and a Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz$$

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$$(1, \underline{5}, 2)$$

$$(1, \underline{13}, 5) \qquad\qquad (5, \underline{29}, 2)$$

$$(1, \underline{34}, 13) \qquad (13, \underline{194}, 5) \qquad (5, \underline{433}, 29) \qquad (29, \underline{169}, 2)$$

FIGURE 1. Markov tree

$$\left(\tfrac{0}{1}, \underline{\tfrac{1}{2}}, \tfrac{1}{1}\right)$$

$$\left(\tfrac{0}{1}, \underline{\tfrac{1}{3}}, \tfrac{1}{2}\right) \qquad\qquad \left(\tfrac{1}{2}, \underline{\tfrac{2}{3}}, \tfrac{1}{1}\right)$$

$$\left(\tfrac{0}{1}, \underline{\tfrac{1}{4}}, \tfrac{1}{3}\right) \qquad \left(\tfrac{1}{3}, \underline{\tfrac{2}{5}}, \tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2}, \underline{\tfrac{3}{5}}, \tfrac{2}{3}\right) \qquad \left(\tfrac{2}{3}, \underline{\tfrac{3}{4}}, \tfrac{1}{1}\right)$$

FIGURE 2. Farey tree

which is now called the *Markov equation*.

Markov numbers are positive integers that appear in the solution triples $(x, y, z) \in \mathbb{Z}^3$ to the Markov equation. This triple (x, y, z) is termed the *Markov triple*, and we put

$$\mathcal{M} = \{1, 2, 5, 13, 29, \dots\}$$

to be the set of Markov numbers. Except for the *singular triples* $(1, 1, 1)$ and $(1, 1, 2)$, all the other *nonsingular* Markov triples have three different entries. The nonsingular triples are well arranged via a binary tree, called the *Markov tree*. Figure 1 depicts the first three rows of the tree. Here, the maximum entry is underlined and placed in the middle. The Markov tree is reminiscent of the *Farey tree*. See Figure 2. Children in both trees are born according to the recursive rules Figure 3 describes.

$$(x, \underline{z}, y) \qquad\qquad\qquad \left(\tfrac{a}{b}, \underline{\tfrac{a+c}{b+d}}, \tfrac{c}{d}\right)$$

$$(x, \underline{3xz - y}, z) \qquad (z, \underline{3zy - x}, y) \qquad \left(\tfrac{a}{b}, \underline{\tfrac{2a+c}{2b+d}}, \tfrac{a+c}{b+d}\right) \qquad \left(\tfrac{a+c}{b+d}, \underline{\tfrac{a+2c}{b+2d}}, \tfrac{c}{d}\right)$$

FIGURE 3. Generating rules of the Markov tree (left) and the Farey tree (right)

It is known that every Markov number appears in the Markov tree just as every rational number in $[0, 1]$ does in the Farey tree. ([2, Chapter 3]). Moreover, every rational number in $(0, 1)$ appears as an underlined entry in the Farey tree once and exactly once. Hence, $\mathbb{Q}_{[0,1]}$ is a well-defined index set for $\mathcal{M}$ — here and in what follows, if S is a set and I is an interval then S_I means $S \cap I$. To be more precise, let $t \in \mathbb{Q}_{[0,1]}$. Then we write m_t for the Markov number m so that the place of m in the Markov tree coincides with the corresponding place where t occupies in the Farey tree. We say that t is the *Farey index* of m_t. For instance, $m_0 = 1$, $m_1 = 2$, $m_{1/2} = 5$, $m_{1/3} = 13$, $m_{2/3} = 29$, etc. Frobenius' uniqueness conjecture on Markov numbers claims that the underlined entries in the Markov tree are all distinct [4]. Meanwhile, we have no general criterion on two distinct rational numbers $p/q, r/s \in \mathbb{Q}_{[0,1]}$ to determine which of $m_{p/q}$ and $m_{r/s}$ is greater than the other. Recently, however, remarkable partial results were obtained in [12, 11, 5].

Under the same index set with Markov numbers, Lagrange numbers less than 3 can be also arranged as $\mathcal{L} = \{l_{p/q} : p/q \in \mathbb{Q}_{[0,1]}\}$. Along this line, Markov [14, 15] proved that

$$l_{p/q} = \sqrt{9 - \frac{4}{m_{p/q}^2}}. \tag{1}$$

So, the uniqueness conjecture is equivalent to the statement that the values $l_{p/q}$ are all distinct.

Since the introduction, Markov and Lagrange numbers have been enjoying fertile and prolific connections to diverse area of mathematics, such as quadratic forms, Diophantine approximations, combinatorics, and hyperbolic geometry. For further details, the readers are nicely guided by [3, 2, 17].

We consider a function $C(\alpha)$ devised from Sturmian words, in fact, more generally from mechanical words. Closely speaking, continued fractions whose partial quotients constitute modified mechanical words are analyzed. Actually, a function similar to $C(\alpha)$ was also considered by the author. To be more concrete, let us denote the continued fraction expansion of a real number t by

$$t = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

where the *partial quotients* a_i for $i \geq 1$ are positive integers, while $a_0 \in \mathbb{Z}$. Let $a_1 a_2 \cdots$ be a mechanical word of slope α over an alphabet $\{1, 2\}$. In [8], a continued fraction of the form $[0, a_1, a_2, a_3, \ldots]$ was studied. By contrast, the work of Markov leads the present paper to examine a continued fraction of the form $[a_1, a_1, a_2, a_2, a_3, a_3, \ldots]$.

The primary objective of the paper is to reveal the overall connection between $C(\alpha)$ and Lagrange numbers. After combinatorial and analytical investigation, we specify that the function $C(\alpha)$ contains accumulated information on Lagrange numbers, and thus Markov numbers by (1). Finally, we are led to prove that for any real number $\alpha \in [0,1]$, the following identity holds:

$$C(\alpha+) - C(0) = \sum_{p/q \in \mathbb{Q}_{[0,\alpha]}} (3 - l_{p/q}),$$

where $C(\alpha+)$ means the right limit of C at α. Namely, each Lagrange number exists not in its own right, but in harmony with the whole Lagrange numbers. This generalizes the work in [6], where the authors proved the above identity of the case $\alpha = 1$. But the ones of the other cases $0 < \alpha < 1$ seem not simple to be proved in their context. They rearranged in increasing order the elements of the set $\mathcal{L} = \{l_{p/q} : p/q \in \mathbb{Q}_{[0,1]}\}$ as $L_1 \lneqq L_2 \lneqq L_3 \lneqq \cdots$, which consequently implies $C(1+) - C(0) \geq \sum_{n=1}^{\infty}(3 - L_n)$ with equality if and only if all $l_{p/q}$ are distinct, and declared that the uniqueness conjecture is equivalent to the identity $C(1+) - C(0) = \sum_{n=1}^{\infty}(3 - L_n)$. We will see below that the number $3 - l_{p/q}$ is realized as a discontinuous jump of $C(\alpha)$ at each rational $p/q \in \mathbb{Q}_{[0,1]}$. Thus we can say now that the uniqueness conjecture is equivalent to the statement that the discontinuous jumps of $C(\alpha)$ are all distinct.

2. Sturmian words and continued fractions

This section begins with some review of combinatorics on words, particularly focused on Sturmian and Christoffel words. Lothaire's book [13] will be a good substitute.

Let $\lfloor\cdot\rfloor$ (resp. $\lceil\cdot\rceil$) be the floor (resp. ceiling) function. Throughout the paper, $\mathcal{A}$ is a finite alphabet. Let $\mathcal{A}^*$ be the set of finite words over $\mathcal{A}$, while $\mathcal{A}^{\mathbb{N}}$ the set of (right) infinite words. For a nonempty word $w \in \mathcal{A}^*$, we mean by w^∞ the infinitely concatenated word $www\cdots$. If $w = a_1a_2\cdots a_n \in \mathcal{A}^*$, then $\tilde{w}$ denotes $a_na_{n-1}\cdots a_1$, i.e., the reversal of w. A word w satisfying $\tilde{w} = w$ is termed a *palindrome*.

Given real numbers $\alpha, \rho \in [0,1]$, we define two infinite words $s_{\alpha,\rho}, s'_{\alpha,\rho}$ over the alphabet $\mathcal{A} = \{a, b\}$ as follows. For $n \geq 0$, their n'th letters $s_{\alpha,\rho}(n)$ and $s'_{\alpha,\rho}(n)$ are given by

$$s_{\alpha,\rho}(n) := \begin{cases} a, & \text{if } \lfloor\alpha(n+1)+\rho\rfloor - \lfloor\alpha n+\rho\rfloor = 0, \\ b, & \text{if } \lfloor\alpha(n+1)+\rho\rfloor - \lfloor\alpha n+\rho\rfloor = 1, \end{cases}$$

$$s'_{\alpha,\rho}(n) := \begin{cases} a, & \text{if } \lceil\alpha(n+1)+\rho\rceil - \lceil\alpha n+\rho\rceil = 0, \\ b, & \text{if } \lceil\alpha(n+1)+\rho\rceil - \lceil\alpha n+\rho\rceil = 1. \end{cases}$$

Here, the infinite words $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are called *lower* and *upper mechanical words* with *slope* α and *intercept* ρ, respectively. One observes that $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are aperiodic if and only if α is irrational. On the other hand, a rational α forces both $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ to be purely periodic. Mechanical words with irrational slopes are called *Sturmian words* [16].

We restrict ourselves to mechanical words with zero intercept ($\rho = 0$), and so also write $s_\alpha := s_{\alpha,0}$ and $s'_\alpha := s'_{\alpha,0}$ for convenience of typography. If α is irrational, then only the first letters of s_α and s'_α are different, and a common infinite suffix c_α, called the *characteristic word* of slope α, follows:

$$s_\alpha = ac_\alpha, \quad s'_\alpha = bc_\alpha.$$

On the other hand, if $\alpha = p/q$ with $\gcd(p, q) = 1$, then s_α and s'_α have their shortest periodic words respectively as

$$s_\alpha = (az_{p,q}b)^\infty, \quad s'_\alpha = (bz_{p,q}a)^\infty,$$

where the common factor $z_{p,q}$ is a palindrome called the *central word*. If α is equal to 0 or 1, then both $az_{p,q}b$ and $bz_{p,q}a$ should read a when $\alpha = 0$, and b when $\alpha = 1$. We say that $az_{p,q}b$ (resp. $bz_{p,q}a$) is a *lower* (resp. an *upper*) *Christoffel word* of *slope* p/q. Note here that our slopes of Christoffel words are different from those of [3, 17, 11]. We adopt the current slopes to make the slopes of Christoffel words coherent with the ones of Sturmian words.

Lemma 2.1. *Let $p/q \in \mathbb{Q}_{[0,1)}$ be a rational number. Then the right limit*

$$\lim_{\alpha \to (p/q)+} s'_\alpha = b(z_{p,q}ba)^\infty$$

exists in the sense that s'_α and $b(z_{p,q}ba)^\infty$ have arbitrarily long common prefixes as α approaches p/q from the right.

Proof. See Lemma 2.3 of [9]. □

Remark 2.2. When $p/q = 0$, the infinite word $b(z_{p,q}ba)^\infty$ is understood as ba^∞ by convention.

If $a_0a_1a_2\cdots$ is a finite or infinite word over the positive integers, then we also write $[a_0a_1a_2\cdots]$ for the continued fraction $[a_0, a_1, a_2, \ldots]$. The i'th *convergent* of $[a_0a_1a_2\cdots]$ is the rational number $p_i/q_i := [a_0a_1\cdots a_i]$, whereas the i'th *complete quotient* is $\zeta_i := [a_ia_{i+1}a_{i+2}\cdots]$. The values of continued fractions are ordered according to the *alternating lexicographic order*.

Over $\mathcal{A} = \{a, b\}$, define $\chi(a) := 11$, $\chi(b) := 22$, and extend χ to $\mathcal{A}^* \cup \mathcal{A}^\mathbb{N}$ so that it is a morphism. For any finite or infinite word $u \in \mathcal{A}^* \cup \mathcal{A}^\mathbb{N}$, let us denote the continued fraction $[\chi(u)]$ by

$$[\![u]\!] := [\chi(u)].$$

Using this double bracket $[\![\cdot]\!]$, all elements in $\mathcal{L}$ and $\mathcal{M}$ can be neatly represented by the mechanical words as demonstrated in Theorem 8.2.1 and Theorem 10.3.5 of [17].

Proposition 2.3. *For $p/q \in \mathbb{Q}_{[0,1]}$,*

$$l_{p/q} = [\![s'_{p/q}]\!] + \frac{1}{[\![s_{p/q}]\!]},$$

and $m_{p/q}$ is equal to the denominator of the finite continued fraction $[\![bz_{p,q}a]\!]$.

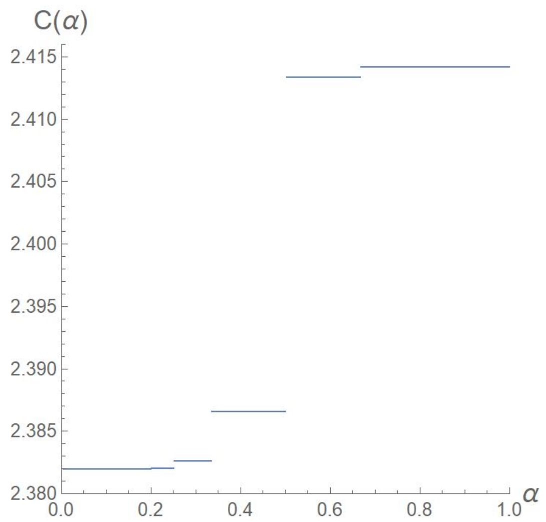

FIGURE 4. Graph of $C(\alpha)$

We are now able to define a function $C : [0, 1] \to \mathbb{R}$ by

$$C(\alpha) := [\![s'_\alpha]\!].$$

Accordingly, one has, e.g., $C(0) = [1^\infty] = \frac{1+\sqrt{5}}{2}$, $C(1) = [2^\infty] = 1 + \sqrt{2}$, and

$$C(1/2) = [\![s'_{1/2}]\!] = [(2211)^\infty] = \frac{9 + \sqrt{221}}{10}.$$

Given an order on $\{a, b\}$ by $a < b$, one notes that $s'_\alpha < s'_\beta$ lexicographically whenever $0 \leq \alpha < \beta \leq 1$. Hence, the alternating lexicographic order of continued fractions guarantees that C is strictly increasing. The basic properties of C are summarized in the next lemma. A similar argument to that of [8] also works well in the proof of this lemma.

Lemma 2.4. *The function C fulfills the following.*

(a) *C is strictly increasing.*
(b) *C is continuous at every irrational number.*
(c) *At every rational number, C is left-continuous but not right-continuous.*

(d) *For any rational $p/q \in \mathbb{Q}_{[0,1)}$,*

$$C(p/q+) := \lim_{\alpha \to (p/q)+} C(\alpha) = [\![b(z_{p,q}ba)^\infty]\!].$$

Figure 4 portrays the graph of $C(\alpha)$. It might look like a step function. The reason for this appearance is that the discontinuous jumps $C(p/q+) - C(p/q)$ at rational numbers $p/q \in \mathbb{Q}_{[0,1)}$ decay very quickly as q increases. More precisely, the value $C(p/q+)-C(p/q)$ is less than $10/\sigma^{4q}$, where $\sigma = \frac{1+\sqrt{5}}{2}$. See Theorem 3.6 below.

3. Lagrange numbers and $C(\alpha)$

This section reveals that the function $C(\alpha)$ intrinsically contains all information on Lagrange numbers. We begin with a simple but influential observation.

Lemma 3.1. *For any real number $c > 0$, we have*

$$[2, 2, c] + [0, 1, 1, c] = 3.$$

Proof. This lemma reflects the abuse of notation for the following meaning:

$$[2, 2, c] + [0, 1, 1, c] = 2 + \cfrac{1}{2 + \cfrac{1}{c}} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{c}}} = 3.$$

□

For any Lagrange number $l_{p/q}$, the next theorem declares that the number $3 - l_{p/q}$ is nothing but the discontinuous jump of the function $C(\alpha)$ at $\alpha = p/q$.

Theorem 3.2. *Let $p/q \in \mathbb{Q}_{[0,1)}$ be a rational number. Then*

$$3 - l_{p/q} = [\![b(z_{p,q}ba)^\infty]\!] - [\![(bz_{p,q}a)^\infty]\!] = C(p/q+) - C(p/q).$$

Proof. Plugging $c = [\![(z_{p,q}ba)^\infty]\!]$, Lemma 3.1 implies

$$3 = [\![b(z_{p,q}ba)^\infty]\!] + \frac{1}{[\![(az_{p,q}b)^\infty]\!]},$$

while

$$l_{p/q} = [\![(bz_{p,q}a)^\infty]\!] + \frac{1}{[\![(az_{p,q}b)^\infty]\!]},$$

by Proposition 2.3. □

The Fibonacci sequence $\{F_n\}_{n\geq -1}$ defined by

$$F_{-1} = 0,\ F_0 = 1,\ \text{and } F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 0$$

obeys, for any integer $n \geq 0$,

$$F_n = \frac{\sigma^{n+1} - \overline{\sigma}^{n+1}}{\sqrt{5}},\ \text{where } \sigma = \frac{1+\sqrt{5}}{2} \text{ and } \overline{\sigma} = \frac{1-\sqrt{5}}{2}.$$

We are now in a position to state our main theorem. Figure 4 makes us suspect that all increases of the function $C(\alpha)$ are caused only by discontinuous jumps at rational numbers. This is indeed the case as the next theorem describes. From now on, we coherently use lower case for the convergents p_k/q_k of α, while upper case for the convergents P_k/Q_k of $C(\alpha)$.

Theorem 3.3. *For any real number* $\alpha \in (0,1]$, *we have*

$$C(\alpha) - C(0) = \sum_{r/s \in \mathbb{Q}_{[0,\alpha)}} (3 - l_{r/s}),$$

where the summation runs over all reduced rational numbers r/s *in* $\mathbb{Q}_{[0,\alpha)}$.

Proof. Since $C(\alpha)$ is strictly increasing, we prove only the case of $\alpha = 1$. At first, Theorem 3.2 guarantees that

$$C(1) - C(0) \geq \sum_{r/s \in \mathbb{Q}_{[0,1)}} (3 - l_{r/s}).$$

Let $\varepsilon > 0$ be any positive real number. Given a rational number $r/s \in \mathbb{Q}_{[0,1)}$, we define an open interval $I_{r/s}$ by

$$\begin{aligned} I_{r/s} &:= \left(C(r/s) - \frac{\varepsilon}{2s^3}, C(r/s+) + \frac{\varepsilon}{2s^3} \right) \\ &= \left\{ t \in \mathbb{R} : [\![(bz_{r,s}a)^\infty]\!] - \frac{\varepsilon}{2s^3} < t < [\![b(z_{r,s}ba)^\infty]\!] + \frac{\varepsilon}{2s^3} \right\}. \end{aligned}$$

We claim that the collection $\{I_{r/s} : r/s \in \mathbb{Q}_{[0,1)}\}$ is an open cover of the interval $(C(0), C(1)) = \left(\frac{1+\sqrt{5}}{2}, 1+\sqrt{2}\right)$. Suppose $\beta \in (C(0), C(1))$. If β belongs to a closed interval $[C(r/s), C(r/s+)]$ for some $r/s \in \mathbb{Q}_{[0,1)}$, then we are done. Thus, we assume $\beta = C(\alpha_0)$ for some irrational number $\alpha_0 \in (0,1)$. Let p_k/q_k be the k'th convergent of α_0, and P_k/Q_k the k'th convergent of $C(\alpha_0)$. Because of convergents' best approximation property [7], both s'_{α_0} and s'_{p_k/q_k} have a common prefix of length at least $q_k - 1$, which makes the corresponding convergents of $C(\alpha_0)$ and $C(p_k/q_k)$ coincide. Consequently, one derives

$$\begin{aligned} |C(\alpha_0) - C(p_k/q_k)| &= \left| \left(C(\alpha_0) - \frac{P_{2q_k-3}}{Q_{2q_k-3}} \right) - \left(C(p_k/q_k) - \frac{P_{2q_k-3}}{Q_{2q_k-3}} \right) \right| \\ &< \frac{1}{Q_{2q_k-3}Q_{2q_k-2}} < \frac{1}{(Q_{2q_k-3})^2} \\ &< \frac{1}{(F_{2q_k-3})^2} = \frac{5}{(\sigma^{2q_k-2} - \overline{\sigma}^{2q_k-2})^2}. \end{aligned}$$

The value $|C(\alpha_0) - C(p_k/q_k)|$ is eventually less than $\frac{\varepsilon}{2q_k^3}$ as k increases, and therefore $\beta = C(\alpha_0)$ lies in the interval I_{p_k/q_k}.

Owing to Theorem 3.2, the length $|I_{r/s}|$ of the interval $I_{r/s}$ is given by

$$|I_{r/s}| = 3 - l_{r/s} + \frac{\varepsilon}{s^3}.$$

Since $\{I_{r/s} : r/s \in \mathbb{Q}_{[0,1)}\}$ is an open cover of $(C(0), C(1))$, one has

$$\begin{aligned} C(1) - C(0) &< \sum_{r/s\in\mathbb{Q}_{[0,1)}} |I_{r/s}| = \sum_{r/s\in\mathbb{Q}_{[0,1)}} \Big(3 - l_{r/s} + \frac{\varepsilon}{s^3}\Big) \\ &= \sum_{r/s\in\mathbb{Q}_{[0,1)}} (3 - l_{r/s}) + \sum_{r/s\in\mathbb{Q}_{[0,1)}} \frac{\varepsilon}{s^3} \\ &= \sum_{r/s\in\mathbb{Q}_{[0,1)}} (3 - l_{r/s}) + \sum_{s=1}^{\infty} \frac{\varepsilon\varphi(s)}{s^3} \\ &< \sum_{r/s\in\mathbb{Q}_{[0,1)}} (3 - l_{r/s}) + \sum_{s=1}^{\infty} \frac{\varepsilon}{s^2} = \sum_{r/s\in\mathbb{Q}_{[0,1)}} (3 - l_{r/s}) + \frac{\pi^2}{6}\varepsilon, \end{aligned}$$

where φ is the Euler totient function. Since ε is arbitrary, we conclude

$$C(1) - C(0) = \sum_{r/s\in\mathbb{Q}_{[0,1)}} (3 - l_{r/s}).$$

□

A special case $\alpha = 1$ of the theorem implies the result in [6], where its proof appealed to hyperbolic geometry.

Corollary 3.4. *For any real number $\alpha \in [0, 1]$, we have*

$$C(\alpha+) - C(0) = \sum_{r/s\in\mathbb{Q}_{[0,\alpha]}} (3 - l_{r/s}).$$

In particular, the sum of $3 - l_{p/q}$ for all $p/q \in \mathbb{Q}_{[0,1]}$ is given by

$$\sum_{p/q\in\mathbb{Q}_{[0,1]}} (3 - l_{p/q}) = \frac{7 - 2\sqrt{2} - \sqrt{5}}{2}.$$

Proof. Since C is continuous at irrational numbers by Lemma 2.4, the case of irrational α boils down to Theorem 3.3.

Let $\alpha = p/q \in \mathbb{Q}_{[0,1]}$ be a rational number. One notes that

$$\begin{aligned} C(p/q+) - C(0) &= C(p/q) - C(0) + C(p/q+) - C(p/q) \\ &= \sum_{r/s\in\mathbb{Q}_{[0,p/q)}} (3 - l_{r/s}) + 3 - l_{p/q} = \sum_{r/s\in\mathbb{Q}_{[0,p/q]}} (3 - l_{r/s}), \end{aligned}$$

which is followed by

$$\begin{aligned} \sum_{p/q\in\mathbb{Q}_{[0,1]}} (3 - l_{p/q}) &= 3 - l_1 + \sum_{p/q\in\mathbb{Q}_{[0,1)}} (3 - l_{p/q}) \\ &= 3 - l_1 + C(1) - C(0) = \frac{7 - 2\sqrt{2} - \sqrt{5}}{2}. \end{aligned}$$

□

The following are also immediate consequences of the theorem. For any real number $\alpha \in (0,1]$ and a positive integer n, let us denote by $\varphi_\alpha(n)$ the number of positive integers m for which $\gcd(m,n)=1$ and $0<m/n<\alpha$. This nonnegative integer $\varphi_\alpha(n)$ was called the *fractional totient function* in [10].

Corollary 3.5. (a) *The sum* $\sum_{p/q\in\mathbb{Q}_{[0,\alpha)}}(3-l_{p/q})$ *is a transcendental number if and only if* $\alpha\in(0,1)$ *is an irrational number.*

(b) *For any real number* $\alpha\in(0,1]$,

$$\sum_{\substack{p/q\in\mathbb{Q}_{[0,\alpha)}\\ q\le x}} l_{p/q} = \frac{9\alpha x^2}{\pi^2} + O(x\log x).$$

Proof. (a) In the formula

$$\sum_{p/q\in\mathbb{Q}_{[0,\alpha)}} (3-l_{p/q}) = [\![s'_\alpha]\!] - \frac{1+\sqrt{5}}{2},$$

the sequence of partial quotients of $[\![s'_\alpha]\!]$ is so called *quasi-Sturmian*. A real number whose partial quotients form a quasi-Sturmian sequence is known to be transcendental [1].

(b) From

$$\sum_{\substack{p/q\in\mathbb{Q}_{[0,\alpha)}\\ q\le x}} (3-l_{p/q}) = O(1),$$

it follows that

$$\sum_{\substack{p/q\in\mathbb{Q}_{[0,\alpha)}\\ q\le x}} l_{p/q} = 3\sum_{q\le x}\varphi_\alpha(q) + O(1).$$

Now [10, Theorem 2.2] applies. □

By computations, (b) of Corollary 3.5 is verified.

Example 1. Let

$$S(\alpha,x) := \frac{\pi^2}{9\alpha x^2}\times \sum_{\substack{p/q\in\mathbb{Q}_{[0,\alpha)}\\ q\le x}} l_{p/q}, \quad E(\alpha,x) := \frac{\pi^2\log x}{9\alpha x},$$

and $N(\alpha,x)$ be the number of Lagrange numbers involved in the above sum. The computation produces the following table.

α	$S(\alpha, 1000)$	$E(\alpha, 1000)$	$N(\alpha, 1000)$
$1/3$	1.000775	0.022726	101400
$1/2$	1.000750	0.015150	152096
1	1.000751	0.007575	304192

Next, we turn to the converging rate of the series in Theorem 3.3. Our series is independent of rearrangement, so we can safely consider its truncation in the form of

$$C(\alpha) - C(0) - \sum_{\substack{p/q \in \mathbb{Q}_{[0,\alpha)} \\ q \le n}} (3 - l_{p/q}) = \sum_{\substack{p/q \in \mathbb{Q}_{[0,\alpha)} \\ q > n}} (3 - l_{p/q}).$$

Theorem 3.6. *Let $\alpha \in (0, 1]$ be a real number. Then, for any integer $n \geq 7$,*

$$\sum_{\substack{p/q \in \mathbb{Q}_{[0,\alpha)} \\ q > n}} (3 - l_{p/q}) < \frac{2\alpha n}{\sigma^{4n}}.$$

Proof. Recalling that $3 - l_{p/q} = [\![b(z_{p,q}ba)^\infty]\!] - [\![(bz_{p,q}a)^\infty]\!]$, we let η_k and ζ_k be the complete quotients of $[\![b(z_{p,q}ba)^\infty]\!]$ and $[\![(bz_{p,q}a)^\infty]\!]$, respectively. Since $[\![b(z_{p,q}ba)^\infty]\!]$ and $[\![(bz_{p,q}a)^\infty]\!]$ have common convergents P_i/Q_i for all $i = 1, 2, \ldots, 2q - 3$. This fact leads us to derive

$$\begin{aligned} [\![b(z_{p,q}ba)^\infty]\!] - [\![(bz_{p,q}a)^\infty]\!] &= \frac{\eta_{2q-2}P_{2q-3} + P_{2q-4}}{\eta_{2q-2}Q_{2q-3} + Q_{2q-4}} - \frac{\zeta_{2q-2}P_{2q-3} + P_{2q-4}}{\zeta_{2q-2}Q_{2q-3} + Q_{2q-4}} \\ &= \frac{\eta_{2q-2} - \zeta_{2q-2}}{(\eta_{2q-2}Q_{2q-3} + Q_{2q-4})(\zeta_{2q-2}Q_{2q-3} + Q_{2q-4})} \end{aligned}$$

With the identities

$$\eta_{2q-2} = [\![(baz_{p,q})^\infty]\!] = 2 + \cfrac{1}{2 + \cfrac{1}{[\![(az_{p,q}b)^\infty]\!]}},$$

and

$$\zeta_{2q-2} = [\![(abz_{p,q})^\infty]\!] = 1 + \cfrac{1}{1 + \cfrac{1}{[\![(bz_{p,q}a)^\infty]\!]}}$$

in mind, one has $\eta_{2q-2} - \zeta_{2q-2} < 1$, and also finds

$$\eta_{2q-2}Q_{2q-3} + Q_{2q-4} > 2Q_{2q-3} + Q_{2q-4} > 2F_{2q-3} + F_{2q-4} = F_{2q-1}$$

and

$$\zeta_{2q-2}Q_{2q-3} + Q_{2q-4} > Q_{2q-3} + Q_{2q-4} > F_{2q-3} + F_{2q-4} = F_{2q-2}.$$

Gathering these, we deduce

$$
\begin{aligned}
[\![b(z_{p,q}ba)^\infty]\!] - [\![(bz_{p,q}a)^\infty]\!] &< \frac{1}{F_{2q-1}F_{2q-2}} = \frac{5}{(\sigma^{2q}-\overline{\sigma}^{2q})(\sigma^{2q-1}-\overline{\sigma}^{2q-1})} \\
&< \frac{5}{(\sigma^{2q}-\overline{\sigma}^{2q})\sigma^{2q-1}} = \frac{5\sigma}{\sigma^{4q}-1} < \frac{10}{\sigma^{4q}},
\end{aligned}
$$

where $-1 < \overline{\sigma} < 0$ and $\sigma\overline{\sigma} = -1$ are used. Therefore, we find for $n \geq 7$ that

$$
\begin{aligned}
\sum_{\substack{p/q \in \mathbb{Q}_{[0,\alpha)} \\ q>n}} (3 - l_{p/q}) &< \sum_{q=n+1}^{\infty} \frac{10\varphi_\alpha(q)}{\sigma^{4q}} < \sum_{q=n+1}^{\infty} \frac{10\alpha q}{\sigma^{4q}} \\
&= \frac{10\alpha(n(\sigma^4-1)+\sigma^4)}{(\sigma^4-1)^2\sigma^{4n}} < \frac{10\alpha\sigma^4}{(\sigma^4-1)^2} \cdot \frac{n}{\sigma^{4n}} = \frac{2\alpha n}{\sigma^{4n}}.
\end{aligned}
$$

□

In the proof of Theorem 3.6, most brutal estimations are from $F_n < Q_n$ for every $n \geq 1$, and also from $\varphi_\alpha(q) < \alpha q$. If q is a prime number, then we have $\varphi_\alpha(q) = \lceil \alpha q \rceil - 1$. However, if q has many prime factors, then $\varphi_\alpha(q)$ is much smaller than αq. The next example indicates that the bound in the theorem admits of a further improvement.

Example 2. For a real number $\alpha \in (0, 1]$ and a positive integer n, let

$$
T(\alpha, n) := \sum_{\substack{p/q \in \mathbb{Q}_{[0,\alpha)} \\ q>n}} (3 - l_{p/q}), \text{ and } R(\alpha, n) := \frac{2\alpha n}{\sigma^{4n}}.
$$

Computations for $n = 10$ yield the following table.

α	$T(\alpha, 10)$	$R(\alpha, 10)$
1/3	1.164044×10^{-9}	2.913420×10^{-8}
1/2	1.173600×10^{-9}	4.370130×10^{-8}
1	1.174009×10^{-9}	8.740261×10^{-8}

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