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# SHARP INEQUALITIES INVOLVING THE CHEN-RICCI INEQUALITY FOR SLANT RIEMANNIAN SUBMERSIONS

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Abstract. Main objective of the present paper is to establish Chen inequalities for slant Riemannian submersions in contact geometry. In this manner, we give some examples for slant Riemannian submersions and also investigate some curvature relations between the total space, the base space and fibers. Moreover, we establish Chen-Ricci inequalities on the vertical and the horizontal distributions for slant Riemannian submersions from Sasakian space forms.

## 1. Introduction

In differential geometry, a sharp inequality for a submanifold in a real space form involving intrinsic invariants of submanifolds and squared mean curvature, the main extrinsic invariant were established by B. Y. Chen in [12] and [13]. Many related results have been introduced by various geometers for different submanifolds in different ambient spaces, see [5, 6, 17, 21–23, 26, 29, 30, 34, 37, 41, 44]. In 2011, B. Y. Chen [16] published a book which covered an extensive and comprehensive survey on Riemannian (pseudo) submanifolds and $\delta$-invariants as well as their applications.

One of the main flaw in Riemannian geometry is to define suitable maps between Riemannian manifolds that will enable to compare their geometric properties. In 1960s, Riemannian submersions were independently introduced by B. O'Neill [32] and A. Gray [20] as follows:

A differentiable map $\Psi : (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ is called a Riemannian submersion if $\Psi_*$ is onto and it satisfies

$$g_2(\Psi_* X_1, \Psi_* X_2) = g_1(X_1, X_2) \tag{1}$$

for $X_1, X_2 \in \Gamma(TM_1)$, where $\Psi_*$ denotes the derivative map.

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We know that Riemannian submersions are related with physics and have applications in Yang-Mills theory [9, 43], Kaluza-Klein theory [8, 24], Supergravity and superstring theories [25, 31]. In 2005, B. Y. Chen proved optimal relationship between Riemannian submersions and minimal immersions in [14] and [15]. In 2012, a sharp relationship between the $\delta$-invariants and Riemannian submersions with totally geodesic fibers was established by in [4]. In 2017, sharp inequalities involving the Ricci curvature for Riemannian submersions were obtained in [21].

In 2010, B. Şahin [35] introduced the notions of anti-invariant Riemannian submersions as a natural generalization of almost Hermitian submersions which was defined by B. Watson in [42]. J. W. Lee [28] introduced the notions of anti-invariant Riemannian submersions in contact geometry. After that, B. Şahin [36] defined slant submersions from almost Hermitian manifolds into Riemannian manifolds as a generalization both almost Hermitian submersions and anti-invariant Riemannian submersions. Then, the notion of slant submersions from Sasakian manifolds defined by İ. K. Erken and C. Murathan in [18]. Many results related to Riemannian submersions have been studied by various geometers for different total spaces, see [1–3, 33, 39]. Most of the studies related to Riemannian or almost Hermitian submersions can be found in [19, 38].

The paper is organized as follows. In Section 2, we give brief introduction about Riemannian submersions, curvature relations and Sasakian space forms. In Section 3, we obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for slant Riemannian submersions from Sasakian space forms. The equality cases are also discussed. Moreover, we prove Chen-Ricci inequalities on the vertical and horizontal distributions for slant Riemannian submersions from Sasakian space forms. Finally, we find relationships between the intrinsic and extrinsic invariants using fundamental tensors. The equality cases are also considered.

## 2. Preliminaries

### 2.1. Riemannian submersions and O'Neill tensors

Now, we give the definition of Riemannian submersions and the O'Neill tensors.

**Definition.** Let $(\widetilde{M}, g_1)$ and $(\overline{M}, g_2)$ be Riemannian manifolds, where $\dim(\widetilde{M}) = m_1$, $\dim(\overline{M}) = m_2$ and $m_1 > m_2$. A Riemannian submersion $\Psi : \widetilde{M} \to \overline{M}$ is a map of $\widetilde{M}$ onto $\overline{M}$ satisfying the following conditions:

- $\Psi$ has maximal rank.
- The metric of horizontal vectors is preserved by the differential $\Psi_*$.

For each $q \in \overline{M}$, $\Psi^{-1}(q)$ is an $(m_1 - m_2)$-dimensional submanifold of $\widetilde{M}$. The submanifolds $\Psi^{-1}(q)$, $q \in \overline{M}$, are called fibers. A vector field on $\widetilde{M}$ is called vertical and horizontal if it is always tangent to fibers and orthogonal to fibers, respectively. A vector field $\xi$ on $\widetilde{M}$ is called basic if $\xi$ is horizontal and $\Psi$-related to a vector field $\xi'$ on $\overline{M}$, that is, $\Psi_*\xi_p = \xi'_{\Psi_*}(p)$ for all $p \in \widetilde{M}$. The projection morphisms on the distributions ker $\Psi_*$ and $(\ker\Psi_*)^\perp$ are denoted by $\mathcal{V}$ and $\mathcal{H}$, respectively. The sections of $\mathcal{V}$ and $\mathcal{H}$ are called the vertical vector fields and horizontal vector fields, respectively. Then, one has:

$$T_p\widetilde{M} = \mathcal{V}_p \oplus \mathcal{H}_p.$$

The O'Neill tensors are defined by means of the vertical and horizontal projections

$$v : \Gamma(T\widetilde{M}) \to \Gamma^v(T\widetilde{M}),\ h : \Gamma(T\widetilde{M}) \to \Gamma^h(T\widetilde{M})$$

according to the formulas:

$$\mathcal{T}(E,F) = \mathcal{T}_E F = h\widetilde{\nabla}_{vE}vF + v\widetilde{\nabla}_{vE}hF, \tag{2}$$

$$\mathcal{A}(E,F) = \mathcal{A}_E F = v\widetilde{\nabla}_{hE}hF + h\widetilde{\nabla}_{hE}vF \tag{3}$$

for any $E, F \in \Gamma(T\widetilde{M})$. Here $\widetilde{\nabla}$ denotes the Levi-Civita connection of $(\widetilde{M}, g_1)$. It is easy to prove that $\mathcal{T}$ and $\mathcal{A}$ are, respectively, vertical and horizontal tensor fields, that is:

$$\mathcal{T}_E F = \mathcal{T}_{vE} F, \quad \mathcal{A}_E F = \mathcal{A}_{hE} F, \quad E, F \in \mathcal{X}(M).$$

We also note that $\mathcal{T}_U V = \mathcal{T}_V U$ for any $U, V \in \Gamma^v(T\widetilde{M})$ and $\mathcal{A}_X Y = -\mathcal{A}_Y X$ for any $X, Y \in \Gamma^h(T\widetilde{M})$.

*Remark* 2.1. The following formulas are also an immediate consequence of (2) and (3)

$$\widetilde{\nabla}_U V = \mathcal{T}_U V + v\left(\hat{\nabla}_U V\right), \tag{4}$$

$$\widetilde{\nabla}_U \xi = \mathcal{T}_U \xi + h\left(\widetilde{\nabla}_U \xi\right), \tag{5}$$

$$\widetilde{\nabla}_\xi U = v\left(\widetilde{\nabla}_\xi U\right) + \mathcal{A}_\xi U, \tag{6}$$

$$\widetilde{\nabla}_\xi \eta = \mathcal{A}_\xi \eta + h\left(\widetilde{\nabla}_\xi \eta\right) \tag{7}$$

for any $\xi, \eta \in \Gamma^h(T\widetilde{M})$, $U, V \in \Gamma^v(T\widetilde{M})$.

Moreover, if $\xi$ is basic, then $h(\nabla_U \xi) = h(\nabla_\xi U) = \mathcal{A}_\xi U$, $[\xi, U]$ being vertical.

The following lemma shows that the O'Neill tensors are anti-symmetric with respect to $g_1$.

**Lemma 2.2.** *Let $(\widetilde{M}, g_1)$ and $(\overline{M}, g_2)$ be Riemannian manifolds admitting a Riemannian submersion $\Psi : \widetilde{M} \to \overline{M}$. For $E, F, G \in \Gamma(T\widehat{M})$, we have*

$$g_1(\mathcal{T}_E F, G) = -g_1(F, \mathcal{T}_E G), \tag{8}$$

$$g_1(\mathcal{A}_E F, G) = -g_1(F, \mathcal{A}_E G). \tag{9}$$

*Remark* 2.3. In this paper, we will assume all horizontal vector fields as basic vector fields.

### 2.2. Curvature relations on Riemannian submersions

Denote by $\widetilde{R}$, $\overline{R}$, $\hat{R}$ and $R^*$ the Riemannian curvature tensor of Riemannian manifolds $\widetilde{M}$, $\overline{M}$, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, respectively. Then the Gauss-Codazzi type equations are given by

$$\widetilde{R}(U,V,F,W) = \hat{R}(U,V,F,W) + g_1(\mathcal{T}_U W, \mathcal{T}_V F) - g_1(\mathcal{T}_V W, \mathcal{T}_U F), \tag{10}$$

$$\begin{aligned} \widetilde{R}(X,Y,Z,H) = {} & R^*(X,Y,Z,H) - 2g_1(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ & + g_1(\mathcal{A}_Y Z, \mathcal{A}_X H) - g_1(\mathcal{A}_X Z, \mathcal{A}_Y H), \end{aligned} \tag{11}$$

$$\begin{aligned} \widetilde{R}(X,V,Y,W) = {} & g_1((\widetilde{\nabla}_X \mathcal{T})(V,W),Y) + g_1((\widetilde{\nabla}_V \mathcal{A})(X,Y),W) \\ & - g_1(\mathcal{T}_V X, \mathcal{T}_W Y) + g_1(\mathcal{A}_Y W, \mathcal{A}_X V), \end{aligned} \tag{12}$$

where

$$\Psi_*(R^*(X,Y)Z) = \overline{R}(\Psi_* X, \Psi_* Y)\Psi_* Z \tag{13}$$

for all $U,V,F,W \in \Gamma^v(T\widetilde{M})$ and $X,Y,Z,H \in \Gamma^h(T\widetilde{M})$.

Moreover, the mean curvature vector field $H$ of any fiber of Riemannian submersion $\Psi$ is given by

$$N = rH, \; N = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j, \tag{14}$$

where $\{U_1, \ldots, U_r\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$. Furthermore, $\Psi$ has totally geodesic fibers if $\mathcal{T}$ vanishes on $\Gamma^h(T\widetilde{M})$ and $\Gamma^v(T\widetilde{M})$.

### 2.3. Sasakian space forms

A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}, g_1)$ is said to be a Sasakian manifold if it admits an endomorphism $\phi$ of its tangent bundle $T\widetilde{M}$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$\phi^2 = -Id + \eta \otimes \xi, \; \eta(\xi) = 1, \; \phi\xi = 0, \; \eta \circ \phi = 0, \tag{15}$$

$$g_1(\phi X, \phi Y) = g_1(X,Y) - \eta(X)\eta(Y), \; \eta(X) = g_1(X,\xi), \tag{16}$$

$$(\nabla_X \phi)Y = g_1(X,Y)\xi - \eta(Y)X, \; \nabla_X \xi = -\phi X \tag{17}$$

for any vector fields $X$, $Y$ on $T\widetilde{M}$, where $\nabla$ denotes the Riemannian connection with respect to $g_1$.

A plane section $\pi$ in $T_p\widetilde{M}$ is called a $\phi$-section if it is spanned by $X$ and $\phi X$, where $X$ is a unit tangent vector orthogonal to $\xi$. The sectional curvature of a $\phi$-section is called a $\phi$-sectional curvature. A Sasakian manifold with constant $\phi$-sectional curvature $c$ is said to be a Sasakian space form and is denoted by $\widetilde{M}(c)$.

The curvature tensor of $\widetilde{R}$ of a Sasakian space form $\widetilde{M}(c)$ is given by [7]

$$\widetilde{R}(X,Y)Z = \frac{(c+3)}{4}\{g_1(Y,Z)X - g_1(X,Z)Y\} + \frac{(c-1)}{4}\{\eta(X)\eta(Z)Y$$

$$- \eta(Y)\eta(Z)X + g_1(X,Z)\eta(Y)\xi - g_1(Y,Z)\eta(X)\xi$$
$$+ g_1(\phi Y,Z)\phi X - g_1(X,\phi Z)\phi Y - 2g_1(\phi X,Y)\phi Z\} \tag{18}$$

for any tangent vector fields $X$, $Y$, $Z$ on $\widetilde{M}(c)$.

## 3. The geometry of slant Riemannian submersions

In this section, we are going to introduce the notion of slant Riemannian submersions from Sasakian manifolds onto Riemannian submersions. We mention examples and giving the characterization equations that we will use in the next future.

**Definition.** Let $\widetilde{M}(\phi,\xi,\eta,g_1)$ be a Sasakian manifold and $(\overline{M},g_2)$ be a Riemannian manifold. A Riemannian submersion $\Psi : M(\phi,\xi,\eta,g_1) \to (\overline{M},g_2)$ is said to be slant if for any nonzero vector $X \in \Gamma((\ker\Psi_*) - \{\xi\})$, the angle $\theta(X)$ between $\phi X$ and the space $\ker\Psi_*$ is a constant (which is independent of the choice of $p \in \widetilde{M}$ and of $X \in \Gamma((\ker\Psi_*) - \{\xi\}))$. The angle $\theta$ is called the slant angle of the slant submersion.

Let $\Psi : (\widetilde{M},g_1,\phi,\xi,\eta) \to (\overline{M},g_2)$ be a slant Riemannian submersion from a Sasakian manifold $(\widetilde{M},g_1,\phi,\xi,\eta)$ to a Riemannian manifold $(\overline{M},g_2)$. Then for any $U \in \Gamma(\ker\Psi_*)$, we put

$$\phi U = \psi U + \omega U, \tag{19}$$

where $\psi U$ and $\omega U$ are vertical and horizontal components of $\phi U$, respectively. Similarly, for any $X \in \Gamma(\ker\Psi_*)^{\perp}$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X, \tag{20}$$

where $\mathcal{B}X$ (resp. $\mathcal{C}X$) is vertical part (resp. horizontal part) of $\phi X$.

First of all, if $\xi$ is orthogonal to $\ker\Psi_*$, then we give the following theorem.

**Theorem 3.1.** *Let $\Psi$ be a slant Riemannian submersion from a Sasakian manifold $\widetilde{M}(\phi,\xi,\eta,g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$. If $\xi$ is orthogonal to* $\ker F_*$*, then $\Psi$ is anti-invariant.*

The following theorem is a characterization for slant submersions of a Sasakian manifold. The proof of it exactly same with slant immersions, see [10]. Therefore we omit its proof.

**Theorem 3.2.** *Let $\Psi$ be a Riemannian submersion from a Sasakian manifold $\widetilde{M}(\phi,\xi,\eta,g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$. Then, $\pi$ is a slant Riemannian submersion if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$\psi^2 = -\lambda(I - \eta\otimes\xi). \tag{21}$$

Furthermore, in such a case, if $\theta$ is the slant angle of $\Psi$, it satisfies that $\lambda = \cos^2\theta$. By using (16), (19) and (21), we have the following lemma.

**Lemma 3.3.** *Let $\Psi$ be a slant Riemannian submersion from a Sasakian manifold $\widetilde{M}(\phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ with slant angle $\theta$. Then the following relations are valid*

$$g_1(\psi U, \psi V) = \cos^2\theta\,(g_1(U, V) - \eta(U)\eta(V)), \tag{22}$$

$$g_1(\omega U, \omega V) = \sin^2\theta\,(g_1(U, V) - \eta(U)\eta(V)) \tag{23}$$

*for any $U, V \in \Gamma(\ker \Psi_*)$.*

Let $(\widetilde{M}(c), g_1)$, $(\overline{M}, g_2)$ be a Sasakian space form and a Riemannian manifold, respectively and $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion. Furthermore, let $\{U_1, \ldots, U_r, X_1, \ldots, X_n\}$ be an orthonormal basis of $T_p\widetilde{M}(c)$ such that $\mathcal{V} = span\{U_1, \ldots, U_r = \xi\}$, $\mathcal{H} = span\{X_1, \ldots, X_n\}$. Then we may consider a slant orthonormal frame as follows:

$$U_1,\ U_2 = \frac{1}{\cos\theta}\psi U_1,\ \ldots,\ U_{2k} = \frac{1}{\cos\theta}\psi U_{2k},\ U_r = \xi.$$

We have

$$\begin{aligned} g_1(\phi U_1, U_2) &= g_1(\phi U_1, \frac{1}{\cos\theta}\psi U_1) = \frac{1}{\cos\theta} g_1(\phi U_1, \psi U_1) \\ &= \frac{1}{\cos\theta} g_1(\psi U_1, \psi U_1) = \cos\theta \end{aligned}$$

and, in same way,

$$g^2(\phi U_i, U_{i+1}) = \cos^2\theta$$

then

$$\sum_{i=1}^{r} g^2(\phi U_i, U_{i+1}) = (r-1)\cos^2\theta.$$

Similarly, if $\xi$ is horizontal, then we obtain

$$g^2(\phi U_i, U_{i+1}) = \cos^2\theta$$

and

$$\sum_{i=1}^{r} g^2(\phi U_i, U_{i+1}) = r\cos^2\theta.$$

## 4. Chen-Ricci inequality and Chen inequalities

Let $(\widetilde{M}(c), g_1)$, $(\overline{M}, g_2)$ be a Sasakian space form and a Riemannian manifold, respectively and $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion. Furthermore, let $\{U_1, \ldots, U_r, X_1, \ldots, X_n\}$ be an orthonormal basis of $T_p\widetilde{M}(c)$ such that $\mathcal{V} = span\{U_1, \ldots, U_r\}$, $\mathcal{H} = span\{X_1, \ldots, X_n\}$. Then using (10), (11) and (18), we have

$$\begin{aligned} \hat{R}(U, V, F, W) = &\frac{(c+3)}{4}\{g_1(V, F)g_1(U, W) - g_1(U, F)g_1(V, W)\} \\ &+ \frac{(c-1)}{4}\{\eta(U)\eta(F)g_1(V, W) - \eta(V)\eta(F)g_1(U, W) \end{aligned}$$

$$\begin{aligned}&+\eta(V)\eta(W)g_1(U,F)-\eta(U)\eta(W)g_1(V,F)\\&+g_1(\phi V,F)g_1(\phi U,W)-g_1(\phi U,F)g_1(\phi V,W)\\&-2g_1(\phi U,V)g_1(W,\phi F)\}-g_1(\mathcal{T}_UW,\mathcal{T}_VF)+g_1(\mathcal{T}_VW,\mathcal{T}_UF),\end{aligned}\tag{24}$$

$$\begin{aligned}R^*(X,Y,Z,H)=&\frac{(c+3)}{4}\{g_1(Y,Z)g_1(X,H)-g_1(X,Z)g_1(Y,H)\}\\&+\frac{(c-1)}{4}\{\eta(X)\eta(Z)g_1(Y,H)-\eta(Y)\eta(Z)g_1(X,H)\\&+\eta(Y)\eta(H)g_1(X,Z)-\eta(X)\eta(H)g_1(Y,Z)\\&+g_1(\phi Y,Z)g_1(\phi X,H)-g_1(\phi Y,H)g_1(\phi X,Z)\\&-2g_1(\phi X,Y)g_1(H,\phi Z)\}+2g_1(\mathcal{A}_XY,\mathcal{A}_ZH)\\&-g_1(\mathcal{A}_YZ,\mathcal{A}_XH)+g_1(\mathcal{A}_XZ,\mathcal{A}_YH).\end{aligned}\tag{25}$$

**Theorem 4.1.** *Let $\Psi:\widetilde{M}(c)\to\overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c),g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$ such that $\xi$ is vertical. Then*

$$\begin{aligned}\widehat{Ric}(U)\geq&\frac{(c+3)}{4}(r-1)+\frac{(c-1)}{4}\{(2-r-3\cos^2\theta)(\eta(U))^2\\&+2-3\sin^2\theta\}-rg_1(\mathcal{T}_UU,H).\end{aligned}\tag{26}$$

*The equality case of* (26) *holds for a unit vertical vector $U\in\mathcal{V}_p(\widetilde{M}(c))$ if and only if each fiber is totally geodesic.*

*Proof.* From (24) we obtain

$$\begin{aligned}\widehat{Ric}(U)=&\frac{(c+3)}{4}(r-1)g_1(U,U)-\frac{(c-1)}{4}\{(2-r)(\eta(U))^2\\&-g_1(U,U)+3\sum_{i=1}^{r}g^2(\phi U,U_i)\}-rg_1(\mathcal{T}_UU,H)+\left\|\mathcal{T}_UU_i\right\|^2,\end{aligned}\tag{27}$$

where

$$\widehat{Ric}(U)=\sum_{i=1}^{r}\hat{R}(U,U_i,U_i,U).$$

Since

$$\sum_{i=1}^{r}g^2(\phi U,U_i)=\cos^2\theta(g_1(U,U)-(\eta(U))^2),\tag{28}$$

using last equation in (27), we get (26). □

**Theorem 4.2.** *Let $\Psi:\widetilde{M}(c)\to\overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c),g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$ such that $\xi$ is vertical. Then*

$$2\hat{\tau}\geq\frac{(c+3)}{4}r(r-1)+\frac{(c-1)}{4}(r-1)(1-3\sin^2\theta)-r^2\left\|H\right\|^2.\tag{29}$$

*The equality case of* (29) *holds if and only if each fiber is totally geodesic.*

*Proof.* Using the symmetry of $\mathcal{T}$ in (24), we have

$$2\hat{\tau} = \frac{(c+3)}{4}r(r-1) + \frac{(c-1)}{4}\{(2-2r) + 3(r-1)\cos^2\theta\}$$
$$- r^2\|H\|^2 + \sum_{i,j=1}^{r} g_1(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_i}U_j), \tag{30}$$

which implies (29), where

$$\hat{\tau} = \sum_{1\le i<j\le r} \hat{R}(U_i, U_j, U_j, U_i).$$

□

For the horizontal distribution, in view of (25), since $\Psi$ is a slant Riemannian submersion and $\xi$ is vertical, using the anti-symmetry of $\mathcal{A}$, we find

$$2\tau^* = \frac{(c+3)}{4}n(n-1) + 3\sum_{i,j=1}^{n}\{\frac{(c-1)}{4}g_1(\mathcal{C}X_i, X_j)g_1(\mathcal{C}X_i, X_j)$$
$$- g_1(\mathcal{A}_{X_i}X_j, \mathcal{A}_{X_i}X_j)\}, \tag{31}$$

where

$$\tau^* = \sum_{1\le i<j\le r} \hat{R}(X_i, X_j, X_j, X_i). \tag{32}$$

Now we define

$$\|\mathcal{C}\|^2 = \sum_{i=1}^{n} g^2(\mathcal{C}X_i, X_j), \tag{33}$$

then from (31) and (33) we obtain

$$2\tau^* = \frac{(c+3)}{4}n(n-1) + \frac{3(c-1)}{4}\|\mathcal{C}\|^2 - 3\sum_{i,j=1}^{n} g_1(\mathcal{A}_{X_i}X_j, \mathcal{A}_{X_i}X_j). \tag{34}$$

From (34) we obtain the following theorem.

**Theorem 4.3.** *Let* $\Psi : \widetilde{M}(c) \to \overline{M}$ *be a slant Riemannian submersion from a Sasakian space form* $(\widetilde{M}(c), g_1)$ *onto a Riemannian manifold* $(\overline{M}, g_2)$ *such that* $\xi$ *is vertical. Then*

$$2\tau^* \le \frac{(c+3)}{4}n(n-1) + \frac{3(c-1)}{4}\|\mathcal{C}\|^2. \tag{35}$$

*The equality case of* (35) *holds if and only if* $\mathcal{H}(\widetilde{M})$ *is integrable.*

Now, we suppose that $\xi$ is horizontal.

**Theorem 4.4.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that $\xi$ is horizontal. Then we have*

$$2\hat{\tau} \geq \frac{(c+3)}{4} r(r-1) + \frac{3(c-1)}{4} r \cos^2\theta - r^2 \|H\|^2 . \tag{36}$$

*The equality case of* (36) *holds if and only if each fiber is totally geodesic.*

*Proof.* Using the symmetry of $\mathcal{T}$ in (24), we have

$$2\hat{\tau} = \frac{(c+3)}{4} r(r-1) + \frac{3(c-1)}{4} r \cos^2\theta - r^2 \|H\|^2 + \sum_{i,j=1}^{r} g_1(\mathcal{T}_{U_i} U_j, \mathcal{T}_{U_i} U_j), \tag{37}$$

which implies (36). □

For the horizontal distribution, from (25), since $\xi$ is horizontal and $\mathcal{A}$ is anti-symmetric, after some computations, we have

$$\begin{aligned} 2\tau^* = {} & \frac{(c+3)}{4} n(n-1) + \frac{(c-1)}{4} [(2-2n) + 3\|\mathcal{C}\|^2] \\ & - 3 \sum_{i,j=1}^{n} g_1(\mathcal{A}_{X_i} X_j, \mathcal{A}_{X_i} X_j). \end{aligned} \tag{38}$$

From (38) we obtain the following theorem.

**Theorem 4.5.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that $\xi$ is horizontal. Then we have*

$$2\tau^* \leq \frac{(c+3)}{4} n(n-1) + \frac{(c-1)}{4} [(2-2n) + 3\|\mathcal{C}\|^2]. \tag{39}$$

*The equality case of* (39) *holds if and only if $\mathcal{H}(\widetilde{M})$ is integrable.*

Let $(\widetilde{M}(c), g_1)$ be a Sasakian space form and $(\overline{M}, g_2)$ a Riemannian manifold. Assume that $\Psi : \widetilde{M}(c) \to \overline{M}$ is a slant Riemannian submersion and $\{U_1, \dots, U_r, X_1, \dots, X_n\}$ is an orthonormal basis of $Tp\widetilde{M}(c)$ such that $\mathcal{V}p(\widetilde{M}) = span\{U_1, \dots, U_r\}$, $\mathcal{H}p(\widetilde{M}) = span\{X_1, \dots, X_n\}$. Now we denote $\mathcal{T}_{ij}^s$ by

$$\mathcal{T}_{ij}^s = g_1(\mathcal{T}_{U_i} U_j, X_s), \tag{40}$$

where $1 \leq i, j \leq r$ and $1 \leq s \leq n$ (see [21]).

Similarly, we denote $\mathcal{A}_{ij}^{\alpha}$ by

$$\mathcal{A}_{ij}^{\alpha} = g_1(\mathcal{A}_{X_i} X_j, U_\alpha), \tag{41}$$

where $1 \leq i, j \leq n$ and $1 \leq \alpha \leq r$. From [21], we use

$$\delta(N) = \sum_{i=1}^{n} \sum_{k=1}^{r} g_1((\nabla_{X_i} \mathcal{T})_{U_k} U_k, X_i). \tag{42}$$

From the binomial theorem there is such as the following equation between the tensor fields $\mathcal{T}$:

$$\sum_{s=1}^{n}\sum_{i,j=1}^{r}(\mathcal{T}_{ij}^{s})^{2}=\frac{1}{2}r^{2}\|H\|^{2}+\frac{1}{2}(\mathcal{T}_{11}^{s}-\mathcal{T}_{22}^{s}-\cdots-\mathcal{T}_{rr}^{s})^{2}$$
$$+2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^{s})^{2}-2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(\mathcal{T}_{ii}^{s}\mathcal{T}_{jj}^{s}-(\mathcal{T}_{ij}^{s})^{2}). \tag{43}$$

**Theorem 4.6.** *Let $\Psi:\widetilde{M}(c)\to\overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c),g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$ such that $\xi$ is vertical. Then we have*

$$\widehat{Ric}(U_1)\geq\frac{(c+3)}{4}(r-1)+\frac{(c-1)}{4}[(2-r-3\cos^{2}\theta)(\eta(U_1))^{2}$$
$$+2-3\sin^{2}\theta]-\frac{1}{4}r^{2}\|H\|^{2}. \tag{44}$$

*The equality case of* (44) *holds if and only if*

$$\mathcal{T}_{11}^{s}=\mathcal{T}_{22}^{s}+\cdots+\mathcal{T}_{rr}^{s},$$
$$\mathcal{T}_{1j}^{s}=0,\ j=2,\dots,r.$$

*Proof.* Using (40) in (30) and the symmetry of $\mathcal{T}$, we can write

$$2\hat{\tau}=\frac{(c+3)}{4}r(r-1)+\frac{(c-1)}{4}(r-1)(1-3\sin^{2}\theta)$$
$$-r^{2}\|H\|^{2}+\sum_{s=1}^{n}\sum_{i,j=1}^{r}(\mathcal{T}_{ij}^{s})^{2}. \tag{45}$$

Thus using (43) in (45) we obtain

$$2\hat{\tau}=\frac{(c+3)}{4}r(r-1)+\frac{(c-1)}{4}(r-1)(1-3\sin^{2}\theta)-\frac{1}{2}r^{2}\|H\|^{2}$$
$$+\frac{1}{2}(\mathcal{T}_{11}^{s}-\mathcal{T}_{22}^{s}-\cdots-\mathcal{T}_{rr}^{s})^{2}+2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^{s})^{2}$$
$$-2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(\mathcal{T}_{ii}^{s}\mathcal{T}_{jj}^{s}-(\mathcal{T}_{ij}^{s})^{2}). \tag{46}$$

Then from (46) we have

$$2\hat{\tau}\geq\frac{(c+3)}{4}r(r-1)+\frac{(c-1)}{4}(r-1)(1-3\sin^{2}\theta)-\frac{1}{2}r^{2}\|H\|^{2}$$
$$-2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(\mathcal{T}_{ii}^{s}\mathcal{T}_{jj}^{s}-(\mathcal{T}_{ij}^{s})^{2}). \tag{47}$$

Moreover, taking $U = W = U_i$, $V = F = U_j$ in (10) and using (40), we obtain

$$2\sum_{2\leq i<j\leq r}\widetilde{R}(U_i,U_j,U_j,U_i) = 2\sum_{2\leq i<j\leq r}\hat{R}(U_i,U_j,U_j,U_i) + 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(\mathcal{T}_{ii}^s\mathcal{T}_{jj}^s - (\mathcal{T}_{ij}^s)^2). \tag{48}$$

Using (48) in (47), we get

$$2\hat{\tau} \geq \frac{(c+3)}{4}r(r-1) + \frac{(c-1)}{4}(r-1)(1-3\sin^2\theta) - \frac{1}{2}r^2\|H\|^2 + 2\sum_{2\leq i<j\leq r}\hat{R}(U_i,U_j,U_j,U_i) - 2\sum_{2\leq i<j\leq r}\widetilde{R}(U_i,U_j,U_j,U_i). \tag{49}$$

Furthermore, we have

$$2\hat{\tau} = 2\sum_{2\leq i<j\leq r}\hat{R}(U_i,U_j,U_j,U_i) + 2\sum_{j=1}^{r}\hat{R}(U_1,U_j,U_j,U_1). \tag{50}$$

Considering (50) in (49), we get

$$2\widehat{Ric}(U_1) \geq \frac{(c+3)}{4}r(r-1) + \frac{3(c-1)}{4}(r-1)(1-3\sin^2\theta) - \frac{1}{2}r^2\|H\|^2 - 2\sum_{2\leq i<j\leq r}\widetilde{R}(U_i,U_j,U_j,U_i). \tag{51}$$

Since $\widetilde{M}(c)$ is a Sasakian space form, its curvature tensor $\widetilde{R}$ satisfies the equality (18), we have

$$\sum_{2\leq i<j\leq r}\widetilde{R}(U_i,U_j,U_j,U_i) = \frac{(c+3)}{8}(r-2)(r-1) + \frac{(c-1)}{4}[(r-2+3\cos^2\theta)(\eta(U_1))^2 + 2 - r + \frac{3(r-3)}{2}\cos^2\theta]. \tag{52}$$

From (51) and (52) we obtain (44). □

**Theorem 4.7.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that $\xi$ is vertical. Then we have*

$$Ric^*(X_1) \leq \frac{(c+3)}{4}(n-1) + \frac{3(c-1)}{4}\|\mathcal{C}X_1\|^2. \tag{53}$$

*The equality case of* (53) *holds if and only if*

$$\mathcal{A}_{1j}^{\alpha} = 0,\ j = 2,\ldots,n.$$

*Proof.* From (34), we have

$$2\tau^* = \frac{(c+3)}{4}n(n-1) + \frac{3(c-1)}{4}\|C\|^2 - 3\sum_{\alpha=1}^{r}\sum_{i,j=1}^{n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{54}$$

Since $\mathcal{A}$ is anti-symmetric on $\mathcal{H}(\widetilde{M}(c))$, (54) can be written as

$$2\tau^* = \frac{(c+3)}{4}n(n-1) + \frac{3(c-1)}{4}\|C\|^2 - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 - 6\sum_{\alpha=1}^{r}\sum_{2\leq i<j\leq n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{55}$$

Furthermore, taking $X = H = X_i, \ Y = Z = X_j$ in (11) and using (41), we obtain

$$2\sum_{2\leq i<j\leq n}\widetilde{R}(X_i, X_j, X_j, X_i) = 2\sum_{2\leq i<j\leq n}R^*(X_i, X_j, X_j, X_i) + 6\sum_{\alpha=1}^{r}\sum_{2\leq i<j\leq n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{56}$$

Using (56) in (55), we get

$$2\tau^* = \frac{(c+3)}{4}n(n-1) + \frac{3(c-1)}{4}\|\mathcal{C}\|^2 - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 + 2\sum_{2\leq i<j\leq n}R^*(X_i, X_j, X_j, X_i) - 2\sum_{2\leq i<j\leq n}\widetilde{R}(X_i, X_j, X_j, X_i). \tag{57}$$

Besides, from (18) we obtain

$$\sum_{2\leq i<j\leq n}\widetilde{R}(X_i, X_j, X_j, X_i) = \frac{(c+3)}{8}(n-2)(n-1) + \frac{3(c-1)}{4}\sum_{2\leq i<j\leq n}g^2(\mathcal{C}X_i, X_j). \tag{58}$$

Then from (57) and (58)

$$2Ric^*(X_1) = \frac{(c+3)}{2}(n-1) + \frac{3(c-1)}{2}\|\mathcal{C}X_1\|^2 - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2, \tag{59}$$

which completes the proof. □

Now, we compute the Chen-Ricci inequality between the vertical and horizontal distributions for the case of $\xi$ is vertical. For the scalar curvature $\widetilde{\tau}$ of

$\widetilde{M}(c)$, we obtain

$$2\widetilde{\tau} = \sum_{s=1}^{n} \widetilde{Ric}(X_s, X_s) + \sum_{k=1}^{r} \widetilde{Ric}(U_k, U_k), \tag{60}$$

$$\begin{aligned} 2\widetilde{\tau} = & \sum_{j,k=1}^{r} \widetilde{R}(U_j, U_k, U_k, U_j) + \sum_{i=1}^{n}\sum_{k=1}^{r} \widetilde{R}(X_i, U_k, U_k, X_i) \\ & + \sum_{i,s=1}^{n} \widetilde{R}(X_i, X_s, X_s, X_i) + \sum_{s=1}^{n}\sum_{j=1}^{r} \widetilde{R}(U_j, X_s, X_s, U_j). \end{aligned} \tag{61}$$

Let denote

$$\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g_1(\mathcal{T}_{U_k} X_i, \mathcal{T}_{U_k} X_i), \tag{62}$$

$$\left\|\mathcal{T}^{\mathcal{H}}\right\|^2 = \sum_{j,k=1}^{r} g_1(\mathcal{T}_{U_j} U_k, \mathcal{T}_{U_j} U_k), \tag{63}$$

$$\left\|\mathcal{A}^{\mathcal{V}}\right\|^2 = \sum_{i,j=1}^{n} g_1(\mathcal{A}_{X_i} X_j, \mathcal{A}_{X_i} X_j), \tag{64}$$

$$\left\|\mathcal{A}^{\mathcal{H}}\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g_1(\mathcal{A}_{X_i} U_k, \mathcal{A}_{X_i} U_k). \tag{65}$$

**Theorem 4.8.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that $\xi$ is vertical. Then we have*

$$\begin{aligned} & \frac{(c+3)}{4}(nr + n + r - 2) + \frac{(c-1)}{4}[-1 - n + (2 - r - 3\cos^2\theta)(\eta(U_1))^2 \\ & + 3(\cos^2\theta + \|\mathcal{B}\|^2 + \|\mathcal{C}X_1\|^2)] \\ \leq & \widehat{Ric}(U_1) + Ric^*(X_1) + \frac{1}{4}r^2\|H\|^2 + 3\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2 - \delta(N) \\ & + \left\|\mathcal{T}^{\mathcal{V}}\right\|^2 - \left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \end{aligned} \tag{66}$$

*The equality case of* (66) *holds if and only if*

$$\mathcal{T}_{11}^s = \mathcal{T}_{22}^s + \cdots + \mathcal{T}_{rr}^s,$$
$$\mathcal{T}_{1j}^s = 0, \; j = 2, \ldots, r.$$

*Proof.* Since $\widetilde{M}(c)$ is a Sasakian space form, from (61) we obtain

$$2\widetilde{\tau} = \frac{(c+3)}{4}(n+r)(n+r-1) + \frac{(c-1)}{4}[2(1-r-n)$$

$$+3\{(r-1)\cos^2\theta+\|\mathcal{C}\|^2+2\sum_{i=1}^{n}\sum_{k=1}^{r}g^2(\mathcal{B}X_i,U_k)\}]. \tag{67}$$

Now, we define

$$\|\mathcal{B}\|^2=\sum_{i=1}^{n}\sum_{k=1}^{r}g^2(\mathcal{B}X_i,U_k). \tag{68}$$

On the other hand, using the Gauss-Codazzi type equations (10), (11) and (12), we get

$$\begin{aligned}2\widetilde{\tau}=\ &2\hat{\tau}+2\tau^*+r^2\|H\|^2-\sum_{k,j=1}^{r}g_1(\mathcal{T}_{U_k}U_j,\mathcal{T}_{U_k}U_j)\\&+3\sum_{i,s=1}^{n}g_1(\mathcal{A}_{X_i}X_s,\mathcal{A}_{X_i}X_s)-\sum_{i=1}^{n}\sum_{k=1}^{r}g_1((\nabla_{X_i}\mathcal{T})_{U_k}U_k,X_i)\\&+\sum_{i=1}^{n}\sum_{k=1}^{r}(g_1(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g_1(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k))\\&-\sum_{s=1}^{n}\sum_{j=1}^{r}g_1((\nabla_{X_s}\mathcal{T})_{U_j}U_j,X_s)+\sum_{s=1}^{n}\sum_{j=1}^{r}(g_1(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)\\&-g_1(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j)).\end{aligned} \tag{69}$$

Thus from (43) and (69), we derive

$$\begin{aligned}2\widetilde{\tau}=\ &2\hat{\tau}+2\tau^*+\frac{1}{2}r^2\|H\|^2-\frac{1}{2}(\mathcal{T}_{11}^s-\mathcal{T}_{22}^s-\cdots-\mathcal{T}_{rr}^s)^2\\&-2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^s)^2+2\sum_{s=1}^{n}\sum_{2\le i<j\le r}(\mathcal{T}_{ii}^s\mathcal{T}_{jj}^s-(\mathcal{T}_{ij}^s)^2)\\&+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^\alpha)^2+6\sum_{\alpha=1}^{r}\sum_{2\le i<s\le n}(\mathcal{A}_{is}^\alpha)^2\\&+\sum_{i=1}^{n}\sum_{k=1}^{r}(g_1(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g_1(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k))\\&-2\delta(N)+\sum_{s=1}^{n}\sum_{j=1}^{r}(g_1(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)-g_1(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j)).\end{aligned} \tag{70}$$

Using (48), (56), (67) and (68) in (70), we obtain

$$\begin{aligned}&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)+3\{(r-1)\cos^2\theta\\&+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}]\\=\ &2\widehat{Ric}(U_1)+2Ric^*(X_1)+\frac{1}{2}r^2\|H\|^2-\frac{1}{2}(\mathcal{T}_{11}^s-\mathcal{T}_{22}^s-\cdots-\mathcal{T}_{rr}^s)^2\end{aligned}$$

$$
\begin{aligned}
&-2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^{s})^2+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2\\
&+\sum_{i=1}^{n}\sum_{k=1}^{r}\{g_1(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g_1(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k)\}\\
&-2\delta(N)+\sum_{s=1}^{n}\sum_{j=1}^{r}(g_1(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)-g_1(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j))\\
&+2\sum_{2\leq i<j\leq r}R(U_i,U_j,U_j,U_i)+2\sum_{2\leq i<j\leq n}R(X_i,X_j,X_j,X_i). \qquad (71)
\end{aligned}
$$

Moreover, (52) and (58) in (71) we obtain (66). □

**Theorem 4.9.** *Let $\Psi:\widetilde{M}(c)\to\overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c),g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$ such that $\xi$ is horizontal. Then we have*

$$\widehat{Ric}(U_1)\geq\frac{(c+3)}{4}(r-1)-\frac{3(c-1)}{4}\cos^2\theta-\frac{1}{4}r^2\|H\|^2. \qquad (72)$$

*The equality case of* (72) *holds if and only if*

$$
\begin{aligned}
&\mathcal{T}_{11}^{s}=\mathcal{T}_{22}^{s}+\cdots+\mathcal{T}_{rr}^{s},\\
&\mathcal{T}_{1j}^{s}=0,\ j=2,\ldots,r.
\end{aligned}
$$

*Proof.* Similar to proof of Theorem 4.6, we obtain our theorem. □

**Theorem 4.10.** *Let $\Psi:\widetilde{M}(c)\to\overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c),g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$ such that $\xi$ is horizontal. Then we have*

$$Ric^*(X_1)\leq\frac{(c+3)}{4}(n-1)+\frac{(c-1)}{4}[-1-(n-2)(\eta(X_1))^2+3\|\mathcal{C}X_1\|^2]. \qquad (73)$$

*The equality case of* (73) *holds if and only if*

$$\mathcal{A}_{1j}^{\alpha}=0,\ j=2,\ldots,n.$$

*Proof.* Similar to proof of Theorem 4.7, we get our theorem. □

**Theorem 4.11.** *Let $\Psi:\widetilde{M}(c)\to\overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c),g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$ such that $\xi$ is horizontal. Then*

$$
\begin{aligned}
&\frac{(c+3)}{4}(nr+n+r-2)+\frac{(c-1)}{4}[(2-n)(\eta(X_1))^2-r-1\\
&+3(\cos^2\theta+\|\mathcal{B}\|^2+\|\mathcal{C}X_1\|^2)]\\
\leq\ &\widehat{Ric}(U_1)+Ric^*(X_1)+\frac{1}{4}r^2\|H\|^2+3\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2-\delta(N)
\end{aligned}
$$

$$+\left\|\mathcal{T}^{\mathcal{V}}\right\|^{2}-\left\|\mathcal{A}^{\mathcal{H}}\right\|^{2}. \tag{74}$$

*The equality case of* (74) *holds if and only if*

$$\mathcal{T}_{11}^{s}=\mathcal{T}_{22}^{s}+\cdots+\mathcal{T}_{rr}^{s},$$
$$\mathcal{T}_{1j}^{s}=0,\ j=2,\ldots,r.$$

*Proof.* Since $\widetilde{M}(c)$ is a Sasakian space form, from (61) we obtain

$$\begin{aligned}2\widetilde{\tau}=&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}\{-2(r-1+n)\\&+3(r\cos^{2}\theta+2\left\|\mathcal{B}\right\|^{2}+\left\|\mathcal{C}\right\|^{2})\}.\end{aligned} \tag{75}$$

Using (48), (56) and (75) in (70) we get,

$$\begin{aligned}&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}\{-2(r-1+n)\\&+3(r\cos^{2}\theta+2\left\|\mathcal{B}\right\|^{2}+\left\|\mathcal{C}\right\|^{2})\}\\=&\,2\widehat{Ric}(U_1)+2Ric^{*}(X_1)+\frac{1}{2}r^{2}\left\|H\right\|^{2}-\frac{1}{2}(\mathcal{T}_{11}^{s}-\mathcal{T}_{22}^{s}-\cdots-\mathcal{T}_{rr}^{s})^{2}\\&-2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^{s})^{2}+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^{2}\\&+\sum_{i=1}^{n}\sum_{k=1}^{r}(g_1(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g_1(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k))\\&-2\delta(N)+\sum_{s=1}^{n}\sum_{j=1}^{r}(g_1(\mathcal{T}_{U_j}Xs,\mathcal{T}_{U_j}Xs)-g_1(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j))\\&+2\sum_{2\le i<j\le r}\widetilde{R}(U_i,U_j,U_j,U_i)+2\sum_{2\le i<j\le n}\widetilde{R}(X_i,X_j,X_j,X_i).\end{aligned} \tag{76}$$

Since $\widetilde{M}(c)$ is a Sasakian space form, we obtain

$$\begin{aligned}\sum_{2\le i<j\le r}\widetilde{R}(U_i,U_j,U_j,U_i)=&\frac{(c+3)}{8}(r-2)(r-1)\\&+\frac{3(c-1)}{8}(r-2)\cos^{2}\theta\end{aligned} \tag{77}$$

and

$$\begin{aligned}&\sum_{2\le i<j\le n}\widetilde{R}(X_i,X_j,X_j,X_i)\\&=\frac{(c+3)}{8}(n-2)(n-1)+\frac{(c-1)}{4}(-(n-2)(1-\eta(X_1)^{2})\\&\quad+3\sum_{2\le i<j\le n}g^{2}(\mathcal{C}X_i,X_j)).\end{aligned} \tag{78}$$

From (77), (78) in (76), we derive

$$\begin{aligned}
&\frac{(c+3)}{2}(nr+n+r-2)+\frac{(c-1)}{4}[2(2-n)(\eta(X_1))^2-2r-2+6(\cos^2\theta \\
&+\|\mathcal{B}\|^2+\|\mathcal{C}X_1\|^2)] \\
&=2\widehat{Ric}(U_1)+2Ric^*(X_1)+\frac{1}{2}r^2\|H\|^2-\frac{1}{2}(\mathcal{T}_{11}^s-\mathcal{T}_{22}^s-\cdots-\mathcal{T}_{rr}^s)^2 \\
&-2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^s)^2+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^\alpha)^2 \\
&+\sum_{i=1}^{n}\sum_{k=1}^{r}(g_1(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g_1(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k)) \\
&-2\delta(N)+\sum_{s=1}^{n}\sum_{j=1}^{r}(g_1(\mathcal{T}_{U_j}Xs,\mathcal{T}_{U_j}Xs)-g_1(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j)) \qquad (79)
\end{aligned}$$

which gives (74). □

From (62), (65), (67), (68) and (69) we obtain

$$\begin{aligned}
&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)+3\{(r-1)\cos^2\theta \\
&+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}] \\
&=2\hat{\tau}+2\tau^*+r^2\|H\|^2-\left\|\mathcal{T}^{\mathcal{H}}\right\|^2+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(N)+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \qquad (80)
\end{aligned}$$

From (80) we obtain following theorem.

**Theorem 4.12.** *Let $\Psi:\widetilde{M}(c)\to\overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c),g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$ such that $\xi$ is vertical. Then we have*

$$\begin{aligned}
2\hat{\tau}+2\tau^* \le\ &\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n) \\
&+3\{(r-1)\cos^2\theta+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}]-r^2\|H\|^2+\left\|\mathcal{T}^{\mathcal{H}}\right\|^2 \\
&+2\delta(N)-2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2+2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2, \qquad (81)
\end{aligned}$$

$$\begin{aligned}
2\hat{\tau}+2\tau^* \ge\ &\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n) \\
&+3\{(r-1)\cos^2\theta+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}]-r^2\|H\|^2+\left\|\mathcal{T}^{\mathcal{H}}\right\|^2 \\
&-3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2+2\delta(N)-2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2. \qquad (82)
\end{aligned}$$

*Equality cases of (81) and (82) hold for all $p\in\widetilde{M}$ if and only if horizontal distribution $\mathcal{H}$ is integrable.*

From Theorem 4.12, we have the following corollary.

**Corollary 4.13.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that $\xi$ is vertical and each fiber is totally geodesic. Then we have*

$$2\hat{\tau} + 2\tau^* \leq \frac{(c+3)}{4}(n+r)(n+r-1) + \frac{(c-1)}{4}[2(1-r-n) + 3\{(r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|\mathcal{C}\|^2\}] + 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2, \tag{83}$$

$$2\hat{\tau} + 2\tau^* \geq \frac{(c+3)}{4}(n+r)(n+r-1) + \frac{(c-1)}{4}[2(1-r-n) + 3\{(r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|\mathcal{C}\|^2\}] - 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2. \tag{84}$$

*Equality cases of* (83) *and* (84) *hold for all $p \in \widetilde{M}$ if and only if horizontal distribution $\mathcal{H}$ is integrable.*

From (80) we obtain following theorem.

**Theorem 4.14.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that $\xi$ is vertical. Then we have*

$$2\hat{\tau} + 2\tau^* \geq \frac{(c+3)}{4}(n+r)(n+r-1) + \frac{(c-1)}{4}[2(1-r-n) + 3\{(r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|\mathcal{C}\|^2\}] - r^2\|H\|^2 + 2\delta(N) - 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 + 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2 - 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2, \tag{85}$$

$$2\hat{\tau} + 2\tau^* \leq \frac{(c+3)}{4}(n+r)(n+r-1) + \frac{(c-1)}{4}[2(1-r-n) + 3\{(r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|\mathcal{C}\|^2\}] - r^2\|H\|^2 + \left\|\mathcal{T}^{\mathcal{H}}\right\|^2 + 2\delta(N) + 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2 - 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2. \tag{86}$$

*Equality cases of* (85) *and* (86) *hold for all $p \in \widetilde{M}$ if and only if the fiber through $p$ of $\Psi$ is a totally geodesic submanifold of $\widetilde{M}$.*

From Theorem 4.14 we get following corollary.

**Corollary 4.15.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that $\xi$ is vertical and $\mathcal{H}$ is integrable. Then we have*

$$2\hat{\tau} + 2\tau^* \geq \frac{(c+3)}{4}(n+r)(n+r-1) + \frac{(c-1)}{4}[2(1-r-n) + 3\{(r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|\mathcal{C}\|^2\}] - r^2\|H\|^2 + 2\delta(N) - 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2, \tag{87}$$

$$2\hat{\tau}+2\tau^{*}\leq\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)+3\{(r-1)\cos^{2}\theta+2\left\|\mathcal{B}\right\|^{2}+\left\|\mathcal{C}\right\|^{2}\}]-r^{2}\left\|H\right\|^{2}+2\delta(N)+\left\|\mathcal{T}^{\mathcal{H}}\right\|^{2}. \tag{88}$$

*Equality cases of* (87) *and* (88) *hold for all* $p\in\widetilde{M}$ *if and only if the fiber through* $p$ *of* $\Psi$ *is a totally geodesic submanifold of* $\widetilde{M}$.

**Lemma 4.16.** *Let* $a$ *and* $b$ *be non-negative real numbers. Then*

$$\frac{a+b}{2}\geq\sqrt{ab}$$

*with equality if and only if* $a=b$.

Using Lemma 4.16 in (80), we obtain following theorems.

**Theorem 4.17.** *Let* $\Psi:\widetilde{M}(c)\to\overline{M}$ *be a slant Riemannian submersion from a Sasakian space form* $(\widetilde{M}(c),g_1)$ *onto a Riemannian manifold* $(\overline{M},g_2)$ *such that* $\xi$ *is vertical. Then we have*

$$\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)+3\{(r-1)\cos^{2}\theta+2\left\|\mathcal{B}\right\|^{2}+\left\|\mathcal{C}\right\|^{2}\}]\leq 2\hat{\tau}+2\tau^{*}+r^{2}\left\|H\right\|^{2}+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^{2}+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^{2}-2\delta(N)-2\sqrt{2}\left\|\mathcal{A}^{\mathcal{H}}\right\|\left\|\mathcal{T}^{\mathcal{H}}\right\|. \tag{89}$$

*Equality case of* (89) *holds for all* $p\in\widetilde{M}$ *if and only if* $\left\|\mathcal{A}^{\mathcal{H}}\right\|=\left\|\mathcal{T}^{\mathcal{H}}\right\|$.

**Theorem 4.18.** *Let* $\Psi:\widetilde{M}(c)\to\overline{M}$ *be a slant Riemannian submersion from a Sasakian space form* $(\widetilde{M}(c),g_1)$ *onto a Riemannian manifold* $(\overline{M},g_2)$ *such that* $\xi$ *is vertical. Then we have*

$$\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)+3\{(r-1)\cos^{2}\theta+2\left\|\mathcal{B}\right\|^{2}+\left\|\mathcal{C}\right\|^{2}\}]\geq 2\hat{\tau}+2\tau^{*}+r^{2}\left\|H\right\|^{2}-\left\|\mathcal{T}^{\mathcal{H}}\right\|^{2}-2\delta(N)-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^{2}+2\sqrt{6}\left\|\mathcal{A}^{\mathcal{V}}\right\|\left\|\mathcal{T}^{\mathcal{V}}\right\|. \tag{90}$$

*Equality case of* (90) *holds for all* $p\in\widetilde{M}$ *if and only if* $\left\|\mathcal{A}^{\mathcal{V}}\right\|=\left\|\mathcal{T}^{\mathcal{V}}\right\|$.

**Lemma 4.19** ([40])**.** *Let* $a_1,a_2,\dots,a_n$ *be* $n$*-real numbers* $(n>1)$. *Then*

$$\frac{1}{n}(\sum_{i=1}^{n}a_i)^2\leq\sum_{i=1}^{n}a_i^2$$

*with equality if and only if* $a_1=a_2=\cdots=a_n$.

**Theorem 4.20.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that $\xi$ is vertical. Then we have*

$$\begin{aligned}&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)+3\{(r-1)\cos^2\theta\\&+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}]\\&\leq 2\hat{\tau}+2\tau^*+r(r-1)\|H\|^2+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(N)+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2.\end{aligned}\tag{91}$$

*Equality case of* (91) *holds for all $p \in \widetilde{M}$ if and only if we have the following statements*:

i) $\Psi$ *is a Riemannian submersion that has a totally umbilical fiber.*
ii) $\mathcal{T}_{ij} = 0$ *for $i \neq j \in \{1, 2, \ldots, r\}$.*

*Proof.* From (80) we have

$$\begin{aligned}&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)+3\{(r-1)\cos^2\theta\\&+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}]\\&=2\hat{\tau}+2\tau^*+r^2\|H\|^2-\sum_{i=1}^{n}\sum_{j=1}^{r}(\mathcal{T}_{jj}^s)^2-\sum_{i=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^s)^2\\&\quad+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(N)+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2.\end{aligned}\tag{92}$$

Using Lemma 4.19 in (92), we obtain

$$\begin{aligned}&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)+3\{(r-1)\cos^2\theta\\&+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}]\\&\leq 2\hat{\tau}+2\tau^*+r^2\|H\|^2-\frac{1}{r}\sum_{s=1}^{n}(\sum_{j=1}^{r}\mathcal{T}_{jj}^s)^2-\sum_{s=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^s)^2\\&\quad+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(N)+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2\end{aligned}\tag{93}$$

which is equivalent to (91). Equality case of (91) holds for all $p \in \widetilde{M}$ if and only if

$$\mathcal{T}_{11} = \mathcal{T}_{22} = \cdots = \mathcal{T}_{rr} \quad \text{and} \quad \sum_{s=1}^{n}\sum_{j\neq k}^{r}(T_{jk}^s)^2 = 0$$

which completes proof of the theorem. □

Using by similar proof way of Theorem 4.20, we have the following theorem.

**Theorem 4.21.** *Let $\Psi : \widetilde{M}(c) \to \overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c), g_1)$ onto a Riemannian manifold $(\overline{M}, g_2)$ such that*

*$\xi$ is vertical. Then we have*

$$\begin{aligned}&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)\\&+3\{(r-1)\cos^2\theta+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}]\\&\geq 2\hat{\tau}+2\tau^*+r^2\|H\|^2-\left\|\mathcal{T}^{\mathcal{H}}\right\|^2+\frac{3}{n}tr(\mathcal{A}^{\mathcal{V}})^2-2\delta(N)\\&+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2.\end{aligned}\tag{94}$$

*Equality case of (94) holds for all $p\in\widetilde{M}$ if and only if $\mathcal{A}_{11}=\mathcal{A}_{22}=\cdots=\mathcal{A}_{nn}$ and $\mathcal{A}_{ij}=0$ for $i\neq j\in\{1,2,\ldots,n\}$ .*

From Theorem 4.21, we get following corollary.

**Corollary 4.22.** *Let $\Psi:\widetilde{M}(c)\to\overline{M}$ be a slant Riemannian submersion from a Sasakian space form $(\widetilde{M}(c),g_1)$ onto a Riemannian manifold $(\overline{M},g_2)$ such that $\xi$ is vertical and each fiber is totally geodesic. Then we have*

$$\begin{aligned}&\frac{(c+3)}{4}(n+r)(n+r-1)+\frac{(c-1)}{4}[2(1-r-n)\\&+3\{(r-1)\cos^2\theta+2\|\mathcal{B}\|^2+\|\mathcal{C}\|^2\}]\\&\geq 2\hat{\tau}+2\tau^*+r^2\|H\|^2+\frac{3}{n}tr(\mathcal{A}^{\mathcal{V}})^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2.\end{aligned}\tag{95}$$

*Equality case of (95) holds for all $p\in\widetilde{M}$ if and only if $\mathcal{A}_{11}=\mathcal{A}_{22}=\cdots=\mathcal{A}_{nn}$ and $\mathcal{A}_{ij}=0$ for $i\neq j\in\{1,2,\ldots,n\}$ .*

## 5. Examples

Now, we are going to mention some examples for slant Riemannian submersions in the following.

**Example 5.1** ([7])**.** We consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_i,y_i,z)$ $(i=1,2,\ldots,n)$ and its usual contact form

$$\eta=\frac{1}{2}(dz-\sum_{i=1}^{n}y_i dx_i).$$

The characteristic vector field is given $\xi$ by $2\frac{\partial}{\partial z}$ and its Riemannian metric and tensor field are given by

$$g=\frac{1}{4}\eta\otimes\eta+\sum_{i=1}^{n}((dx_i)^2+(dy_i)^2),\ \phi=\begin{pmatrix}0&\delta_{ij}&0\\-\delta_{ij}&0&0\\0&y_j&0\end{pmatrix},\ i=1,2,\ldots,n.$$

This gives a contact metric structure on $\mathbb{R}^{2n+1}$. The vector fields $E_i=2\frac{\partial}{\partial y_i}$, $E_{n+i}=2(\frac{\partial}{\partial x_i}+y_i\frac{\partial}{\partial z})$, $\xi$ form a $\phi$-basis for the contact metric structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\phi,\xi,\eta,g_1)$ is a Sasakian manifold.

**Example 5.2.** Every invariant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds are slant Riemannian submersions with $\theta = \{0\}$.

**Example 5.3** ([18, 27])**.** Every anti-invariant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds are slant Riemannian submersions with $\theta = \{\frac{\pi}{2}\}$.

A slant Riemannian submersion, which is neither invariant nor anti-invariant, is called a proper slant Riemannian submersion. In the following example, the characteristic vector field $\xi$ is a vertical vector field.

**Example 5.4.** $\mathbb{R}^5$ has got a Sasakian structure as in Example 5.1. Let $\Psi : \mathbb{R}^5 \to \mathbb{R}^2$ be a map defined by $\Psi(x_1, y_1, x_2, y_2, z) = (-\frac{1}{4}x_1 + \frac{1}{\sqrt{5}}y_1 + \frac{1}{4}x_2, -\frac{1}{4}x_1 + \frac{1}{2\sqrt{5}}y_1 + \frac{1}{4}x_2)$. Then, by direct calculations we derives

$$\ker \Psi_* = span\{V_1 = \frac{E_3 + E_4}{\sqrt{2}}, V_2 = E_2, V_3 = \xi = E_5\}$$

and

$$(\ker \Psi_*)^{\perp} = span\{H_1 = \frac{-E_3 + E_4}{\sqrt{2}}, H_2 = E_1\}.$$

Then it is easy to see that $\Psi$ is a Riemannian submersion. Moreover, $\phi V_1 = \frac{-E_1 - E_2}{\sqrt{2}}$ and $\phi V_2 = E_4$ imply that $|g_1(\phi V_1, V_2)| = \frac{1}{\sqrt{2}}$. So $\Psi$ is a slant submersion with slant angle $\theta = \frac{\pi}{4}$. We derive immediately that fibers of the submersions are totally geodesic and the horizontal distribution is integrable. Hence, we have again that $\Psi$ satisfies the equality case of the inequalities stated in Theorems 4.1, 4.2, 4.3, 4.6 and 4.7.

Now, we construct a non-trivial example of slant submersion from a Sasakian manifold with the characteristic vector field $\xi$ is a horizontal vector field. Let $(\mathbb{R}^5, g, \phi, \xi, \eta)$ denote the manifold $\mathbb{R}^5$ with the structure given by

$$\phi(X_1, X_2, X_3, X_4, Z) = (-X_2, X_1, -X_4, X_3, \sqrt{2}x^1 X_2 + x^2 X_1),$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{4}(dx^i \otimes dx^i),$$

$$\eta = \frac{1}{2}(dz - \sqrt{2}x^1 dx^1 - x^2 dx^2), \;\; \xi = 2\frac{\partial}{\partial z},$$

where $(x^1, x^2, x^3, x^4, z)$ are the Cartesian coordinates. We will use this notation in the following example.

**Example 5.5.** Let $F$ be a submersion defined by

$$\begin{array}{llll} F: & \mathbb{R}^5 & \longrightarrow & \mathbb{R}^3 \\ & (x^1, x^2, x^3, x^4, z) & & (\frac{-x^1 + x^3}{\sqrt{2}}, x^4, \frac{(-x^1)^2}{\sqrt{2}} - \frac{(x^2)^2}{2} + z). \end{array}$$

Then it follows that

$$kerF_* = span\{Z_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3} + x^1\frac{\partial}{\partial z},\ Z_2 = \frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial z}\}$$

and

$$(kerF_*)^{\perp} = span\{H_1 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x^1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3},\ H_2 = \frac{\partial}{\partial x^4},\ H_3 = \xi = 2\frac{\partial}{\partial z}\}.$$

A straight computations, $F$ is a slant submersion with slant angle $\theta = \frac{\pi}{4}$. Also by direct computations, we obtain

$$\begin{aligned} g_2(F_*H_1, F_*H_1) &= g_1(H_1, H_1), \\ g_2(F_*H_2, F_*H_2) &= g_1(H_2, H_2), \\ g_2(F_*\xi, F_*\xi) &= g_1(\xi, \xi), \end{aligned}$$

where $g_1 = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{4}(dx^i \otimes dx^i)$ and $g_2 = \frac{1}{4}\sum_{i=1}^{4}(dx^i \otimes dx^i)$ denote the inner products of $\mathbb{R}^5$ and $\mathbb{R}^3$. As a result, $F$ is a slant Riemannian submersion. We derive immediately that the horizontal distribution is integrable. Hence, we have again that $F$ satisfies the equality case of the inequality stated in Theorem 4.5.

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