

## CERTAIN PROPERTIES OF THE CLASS OF UNIVALENT FUNCTIONS WITH REAL COEFFICIENTS

MILUTIN OBRADOVIĆ AND NIKOLA TUNESKI

ABSTRACT. Let  $\mathcal{U}^+$  be the class of analytic functions  $f$  such that  $\frac{z}{f(z)}$  has real and positive coefficients and  $f^{-1}$  be its inverse. In this paper we give sharp estimates of the initial coefficients and initial logarithmic coefficients for  $f$ , as well as, sharp estimates of the second and the third Hankel determinant for  $f$  and  $f^{-1}$ . We also show that the Zalcman conjecture holds for functions  $f$  from  $\mathcal{U}^+$ .

### 1. Introduction

Let  $\mathcal{A}$  be the class of functions  $f$  that are analytic in the open unit disc  $\mathbb{D} = \{z : |z| < 1\}$  of the form

$$(1) \quad f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$

let  $\mathcal{S}$  be its subclass consisting of univalent functions from  $\mathcal{A}$ , and  $\mathcal{S}^+$  consists of functions  $f$  from  $\mathcal{S}$  with representation

$$(2) \quad \frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots, \quad b_n \geq 0, \quad n = 1, 2, 3, \dots$$

Note that the Silverman class of univalent functions with negative coefficients, i.e., with expansion

$$f(z) = z - a_2 z^2 - a_3 z^3 + \cdots, \quad a_n \geq 0, \quad n = 1, 2, 3, \dots,$$

is subclass of  $\mathcal{S}^+$  since  $z/f(z)$  satisfies (2). Also, the Koebe function  $k(z) = \frac{z}{(1+z)^2}$  is in  $\mathcal{S}^+$ .

Further, with  $\mathcal{U}(\lambda)$ ,  $0 \leq \lambda < 1$ , we will denote the class of functions  $f$  from  $\mathcal{A}$  satisfying the condition

$$(3) \quad \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \quad z \in \mathbb{D},$$

---

Received September 19, 2022; Accepted April 21, 2023.

2020 *Mathematics Subject Classification.* 30C45, 30C50, 30C55.

*Key words and phrases.* Univalent, real coefficients, logarithmic coefficients, coefficient estimates, Hankel determinant, Zalcman conjecture.

while  $\mathcal{U} \equiv \mathcal{U}(1)$ . This class of functions attracts significant interest in the past decades. The class is intriguing because it doesn't follow the usual pattern to embed or be embedded in the class of starlike functions (functions that map the unit disk onto a starlike region). A collection of the more significant once can be found in [19, Chapter 12].

If we denote with  $\mathcal{U}^+(\lambda)$  the class of functions that satisfy (2) and (3), and additionally  $\mathcal{U}^+ \equiv \mathcal{U}^+(1)$ , then [12] we have the following equivalence

$$f \in \mathcal{S}^+ \iff f \in \mathcal{U}^+ \iff \sum_{n=2}^{\infty} (n-1)b_n \leq 1.$$

Next lemma makes an extension of the second equivalence.

**Lemma 1.1.**  $f \in \mathcal{U}^+(\lambda) \iff \sum_{n=2}^{\infty} (n-1)b_n \leq \lambda.$

*Proof.* If  $f \in \mathcal{U}^+(\lambda)$ , then by definition

$$\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| = \left| \frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' - 1 \right| = \left| - \sum_{n=2}^{\infty} (n-1)b_n z^n \right| < \lambda,$$

$z \in \mathbb{D}$ . For real  $z$  and  $z \rightarrow 1$  from left, the last inequality gives  $\sum_{n=2}^{\infty} (n-1)b_n \leq \lambda$ .

If  $\sum_{n=2}^{\infty} (n-1)b_n \leq \lambda$ , then

$$\begin{aligned} \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| &= \left| - \sum_{n=2}^{\infty} (n-1)b_n z^n \right| \leq \sum_{n=2}^{\infty} (n-1)b_n |z|^n \\ &< \sum_{n=2}^{\infty} (n-1)b_n \leq \lambda, \end{aligned}$$

$z \in \mathbb{D}$ , which means that  $f \in \mathcal{U}^+(\lambda)$ . □

In [16] the authors gave sharp bounds of the first five coefficients in the expansion of  $f$ . The result for the case  $\lambda = 1$  reduces to the following.

**Theorem A.** *Let  $f(z) = z + a_2z^2 + a_3z^3 + \dots \in \mathcal{S}^+ = \mathcal{U}^+$ . Then the following estimates are sharp*

$$-2 \leq a_2 \leq 0, \quad -1 \leq a_3 \leq 3, \quad -4 \leq a_4 \leq \frac{4}{3}\sqrt{\frac{2}{3}}, \quad -\frac{9}{4} \leq a_5 \leq 5.$$

Let note that for the general class  $\mathcal{U}$  we haven  $|a_n| \leq n$ ,  $n = 2, 3, \dots$ , independently from the de Branges theorem. Namely, for the functions  $f$  from  $\mathcal{U}$ , we have

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^2} = 1 + \sum_{n=1}^{\infty} n z^{n-1},$$

and the rest follows from the Rogosinski theorem ([19, Theorem 3.2.9, p. 35]).

Further, in [14], the authors proved that for  $f$  in the general class  $\mathcal{U}(\lambda)$ ,  $0 < \lambda \leq 1$ ,  $|a_2| \leq 1 + \lambda$ , and if  $|a_2| = 1 + \lambda$ , then  $f$  must be of the form

$$f_\theta(z) = \frac{z}{1 - (1 + \lambda)e^{i\theta}z + \lambda e^{2i\theta}z^2}.$$

For the case  $f \in \mathcal{U}^+(\lambda)$  we have the following corresponding result.

**Theorem B.** *If  $f$  is given by (1) with (2), and  $f \in \mathcal{U}^+(\lambda)$ ,  $0 < \lambda \leq 1$ , then  $-(1 + \lambda) \leq a_2 \leq 0$  (or  $0 \leq b_1 \leq 1 + \lambda$ ). Moreover, if  $a_2 = -(1 + \lambda)$ , then  $f$  must be of the form*

$$(4) \quad f_\lambda(z) = \frac{z}{1 + (1 + \lambda)z + \lambda z^2},$$

*i.e.,  $b_2 = \lambda$ , and  $b_3 = b_4 = \dots = 0$ ,*

In this paper we study coefficient problems for inverse functions of functions in  $\mathcal{U}^+(\lambda)$ , more precisely we will give sharp upper bounds of the leading coefficients and leading logarithmic coefficients of the inverse functions, as well as estimates of the modulus of the second and the third Hankel determinant.

### 2. Coefficient estimates

The famous Koebe 1/4 theorem guaranties that each function  $f$  from  $\mathcal{S}$  has an inverse at least on a disk with radius 1/4. Let the inverse function has an expansion

$$(5) \quad f^{-1}(w) = w + A_2w^2 + A_3w^3 + \dots$$

on the disk  $|w| < \frac{1}{4}$ . By using the identity  $f(f^{-1}(w)) = w$ , and representations (1) and (5), we obtain the following relations:

$$(6) \quad \begin{aligned} A_2 &= -a_2, \\ A_3 &= -a_3 + 2a_2^2, \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3. \end{aligned}$$

We now give sharp bounds of  $A_2$ ,  $A_3$  and  $A_4$ .

**Theorem 2.1.** *Let  $f \in \mathcal{U}^+(\lambda)$ ,  $0 < \lambda \leq 1$ , and  $f^{-1}$  is given by (5). Then, the following estimates are sharp:*

$$(7) \quad \begin{aligned} 0 &\leq A_2 \leq 1 + \lambda, \\ 0 &\leq A_3 \leq 1 + 3\lambda + \lambda^2, \\ 0 &\leq A_4 \leq (1 + \lambda)(1 + 5\lambda + \lambda^2). \end{aligned}$$

*Proof.* Since  $f \in \mathcal{U}^+(\lambda)$ ,  $0 < \lambda \leq 1$ , then by definition

$$f(z) = z + a_2z^2 + a_3z^3 + \dots = \frac{z}{1 + b_1z + b_2z^2 + \dots},$$

where  $b_n \geq 0$ ,  $n = 1, 2, 3, \dots$ . After comparing the coefficients we receive,

$$(8) \quad \begin{aligned} a_2 &= -b_1, \\ a_3 &= -b_2 + b_1^2, \\ a_4 &= -b_3 + 2b_1b_2 - b_1^3, \\ a_5 &= -b_4 + b_2^2 + 2b_1b_3 - 3b_1^2b_2 + b_1^4, \end{aligned}$$

and by combining (8) with (6),

$$(9) \quad \begin{aligned} A_2 &= b_1, \\ A_3 &= b_2 + b_1^2, \\ A_4 &= b_3 + 3b_1b_2 + b_1^3. \end{aligned}$$

Since  $a_2$  is real (so are  $a_3, a_4, \dots$ ), and  $|a_2| \leq 1 + \lambda$  for all functions from  $\mathcal{U}(\lambda)$  (see [20] or [19, Theorem 12.3.1, p. 188]), we receive  $-(1 + \lambda) \leq a_2 \leq 1 + \lambda$ .

From  $A_2 = -a_2 = b_1$ ,  $b_1 \geq 0$  and  $a_2 \leq 1 + \lambda$ , easily follows the estimate

$$(10) \quad 0 \leq A_2 = b_1 \leq 1 + \lambda.$$

Further, from Lemma 1.1 we have

$$b_2 + 2b_3 + 3b_4 + \dots \leq \lambda,$$

which implies

$$(11) \quad \begin{aligned} 0 &\leq b_2 \leq \lambda, \\ b_2 + 2b_3 &\leq \lambda \quad \left( \Leftrightarrow b_3 \leq \frac{1}{2}(\lambda - b_2) \right), \\ b_2 + 2b_3 + 3b_4 &\leq \lambda \quad \left( \Leftrightarrow b_4 \leq \frac{1}{3}(\lambda - 2b_3 - b_2) \right). \end{aligned}$$

Combining (10), (11) and (9), we receive

$$0 \leq A_3 \leq \lambda + (1 + \lambda)^2 = 1 + 3\lambda + \lambda^2,$$

and for  $A_4$ ,

$$\begin{aligned} 0 \leq A_4 &\leq \frac{1}{2}(\lambda - b_2) + 3(1 + \lambda)b_2 + (1 + \lambda)^3 \\ &= \frac{\lambda}{2} + \left( \frac{5}{2} + 3\lambda \right) b_2 + (1 + \lambda)^3 \\ &\leq \frac{\lambda}{2} + \left( \frac{5}{2} + 3\lambda \right) \lambda + (1 + \lambda)^3 \\ &= (1 + \lambda)(1 + 5\lambda + \lambda^2). \end{aligned}$$

For the sharpness of the upper bounds, for the function  $f_\lambda(z)$  defined by (4), we have

$$f_\lambda(z) = \frac{z}{1 + (1 + \lambda)z + \lambda z^2} = z - (1 + \lambda)z^2 + (1 + \lambda + \lambda^2)z^3 - (1 + \lambda + \lambda^2 + \lambda^3)z^4 + \dots,$$

i.e.,  $f_\lambda \in \mathcal{U}^+(\lambda)$  and

$$z = f_\lambda^{-1}(w) = w + (1 + \lambda)w^2 + (1 + 3\lambda + \lambda^2)w^3 + (1 + \lambda)(1 + 5\lambda + \lambda^2)w^4 + \dots.$$

For the sharpness of the lower bounds it is enough to consider the inverse function of the function

$$f(z) = \frac{z}{1 + \lambda z^4/3},$$

where  $b_1 = b_2 = b_3 = 0$ . □

*Remark 2.2.* In [18], the authors obtained the same result, but with an additional condition

$$(12) \quad \frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)}.$$

In our proof we didn't use that condition.

*Remark 2.3.* From (8) we have  $a_3 = b_1^2 - b_2 \equiv \psi_1(b_1)$ ,  $0 \leq b_1 \leq 1 + \lambda$ . It is evident that  $\psi_1(b_1) \geq -b_2 \geq -\lambda$ , while  $\max \psi_1 = \psi_1(1 + \lambda)$ . But,  $b_1 = 1 + \lambda$  implies  $b_2 = \lambda$ , so

$$\psi_1(1 + \lambda) = (1 + \lambda)^2 - \lambda = 1 + \lambda + \lambda^2.$$

Similarly, by (8) we have

$$-a_4 = b_1^3 - 2b_1b_2 + b_3 \equiv \psi_2(b_1),$$

$0 \leq b_1 \leq 1 + \lambda$ , and by using the previous comments,

$$\max \psi_2 = \psi_2(1 + \lambda) = (1 + \lambda)^3 - 2(1 + \lambda)\lambda = 1 + \lambda + \lambda^2 + \lambda^3.$$

It means that  $a_4 \geq -(1 + \lambda + \lambda^2 + \lambda^3)$ . If we use the upper bound for  $a_4$  given in [16], then

$$|a_4| \leq 1 + \lambda + \lambda^2 + \lambda^3$$

for  $f \in \mathcal{U}^+(\lambda)$ . It means that the results given in Theorem 3 from [16] are valid independently of the condition (12).

This is the reason we pose the next conjecture.

**Conjecture 2.4.** For the functions in the class  $\mathcal{U}^+(\lambda)$ ,

$$|a_n| \leq \frac{1 - \lambda^n}{1 - \lambda} = 1 + \lambda + \lambda^2 + \lambda^3 + \dots + \lambda^n, \quad n = 2, 3, 4, \dots$$

This was proven to hold for the general class  $\mathcal{U}(\lambda)$  under the condition (12) in the cases  $n = 2, 3, 4$  ([13] or [19, Theorem 12.3.2, p. 191]).

In [16] the authors studied the logarithmic coefficients for the class  $\mathcal{U}^+$  ( $\mathcal{S}^+$ ), and here we will continue and give estimates for the logarithmic coefficients of the inverse functions of functions in the classes  $\mathcal{U}^+(\lambda)$ . For the logarithmic coefficients  $(\Gamma_1, \Gamma_2, \Gamma_3, \dots)$  we have

$$\log \frac{f^{-1}(w)}{w} = \log(1 + A_2w + A_3w^2 + \dots),$$

or

$$\sum_{n=1}^{\infty} 2\Gamma_n w^n = A_2w + \left(A_3 - \frac{1}{2}A_2^2\right)w^2 + \left(A_4 - A_2A_3 + \frac{1}{3}A_2^3\right)w^3 + \dots.$$

From the last relation we get

$$\begin{aligned} \Gamma_1 &= \frac{1}{2}A_2 = \frac{1}{2}b_1, \\ (13) \quad \Gamma_2 &= \frac{1}{2}\left(A_3 - \frac{1}{2}A_2^2\right) = \frac{1}{2}\left(b_2 + \frac{1}{2}b_1^2\right), \\ \Gamma_3 &= \frac{1}{2}\left(A_4 - A_2A_3 + \frac{1}{3}A_2^3\right) = \frac{1}{2}\left(b_3 + 2b_1b_2 + \frac{1}{3}b_1^3\right), \end{aligned}$$

where we used relations (9).

**Theorem 2.5.** *Let  $f \in \mathcal{U}^+(\lambda)$ ,  $0 < \lambda \leq 1$ . Then*

- (a)  $0 \leq \Gamma_1 \leq \frac{1+\lambda}{2}$ ;
- (b)  $0 \leq \Gamma_2 \leq \frac{1}{4}(1 + 4\lambda + \lambda^2)$ ;
- (c)  $0 \leq \Gamma_3 \leq \frac{1}{6}(1 + \lambda)(1 + 8\lambda + \lambda^2)$ ,

and all results are sharp.

*Proof.* The proof is similar to the one of Theorem 2.1. □

**Corollary 2.6.** *For  $\lambda = 1$  we receive sharp estimates for the inverse function of functions from  $\mathcal{U}^+(1) = \mathcal{U}^+ = \mathcal{S}^+$ ;*

$$0 \leq \Gamma_1 \leq 1, \quad 0 \leq \Gamma_2 \leq \frac{3}{2}, \quad 0 \leq \Gamma_3 \leq \frac{10}{3}.$$

### 3. The second and the third Hankel determinant

Another concept, rediscovered few years ago, attracting the attention of mathematicians working in the field of univalent functions is so called Hankel determinant  $H_q(n)(f)$  of a given function  $f(z) = z + a_2z^2 + a_3z^3 + \dots$ , for  $q \geq 1$  and  $n \geq 1$ , defined by

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

Main interest is to find upper bound (preferably sharp) of the modulus of  $H_q(n)(f)$ . The general Hankel determinant is hard to deal with, so the second

and the third ones,

$$H_2(2)(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

and

$$H_3(1)(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

respectively, are studied instead. The research is focused on the subclasses of univalent functions (starlike, convex,  $\alpha$ -convex, close-to-convex, spirallike, . . .) since the general class of normalised univalent functions is also hard to deal with. Some of the more significant results can be found in [1–6, 8–10, 15, 21].

In this section we will give sharp bounds of the second and the third Hankel determinant for the functions in  $\mathcal{U}^+(\lambda)$  and for their inverse.

Using (8), (10) and (11), after some calculations, we receive

$$(14) \quad \begin{aligned} H_2(2)(f) &= b_1b_3 - b_2^2, \\ H_3(1)(f) &= b_2b_4 - b_3^2. \end{aligned}$$

**Theorem 3.1.** *Let  $f \in \mathcal{U}^+(\lambda)$ ,  $0 < \lambda \leq 1$ . Then the following estimates are sharp:*

- (i)  $-\lambda^2 \leq H_2(2)(f) \leq (1 - \frac{\lambda}{2}) \frac{\lambda}{2}$ ;
- (ii)  $-\frac{\lambda^2}{4} \leq H_3(1)(f) \leq \frac{\lambda^2}{12}$ .

*Proof.* (i) First, let note that from (10) and (11), we have  $b_1, b_3 \geq 0$  and  $b_2 \leq \lambda$ , which easily implies

$$H_2(2)(f) = b_1b_3 - b_2^2 \geq -b_2^2 \geq -\lambda^2.$$

The estimate is sharp due to the function  $f(z) = \frac{z}{1+\lambda z^2}$ .

For the upper bound, from (14) we have  $H_2(2)(f) = b_1b_3 - b_2^2 \leq b_1b_3$ , and further from  $0 \leq b_1 \leq 1 + \lambda$  and  $0 \leq b_3 \leq \frac{\lambda}{2}$  (see (10) and (11)) we need to find  $\max\{b_1b_3\}$ . For  $b_3 = \frac{\lambda}{2}$  ( $\Rightarrow b_2 = b_4 = \dots = 0$ ) we have

$$H_2(2)(f) \leq \frac{\lambda}{2}b_1.$$

If we consider the function  $f$  defined by

$$\frac{z}{f(z)} = 1 + b_1z + \frac{\lambda}{2}z^3,$$

then  $f$  is in  $\mathcal{U}^+(\lambda)$  if  $1 + b_1z + \frac{\lambda}{2}z^3 \neq 0$ . Since

$$\frac{z}{f(z)} \Big|_{z=1} = 1 + b_1 + \frac{\lambda}{2} > 0 \quad \text{and} \quad \frac{z}{f(z)} \Big|_{z=-1} = 1 - b_1 - \frac{\lambda}{2},$$

then it is necessary  $1 - b_1 - \frac{\lambda}{2} \geq 0$ . In contrary, if  $1 - b_1 - \frac{\lambda}{2} < 0$ , then for  $z = r$  real in  $-1 < r < 1$ , we have

$$\left. \frac{z}{f(z)} \right|_{z=r} = 1 + b_1 r + \frac{\lambda}{2} r^3,$$

has zero in  $(-1, 1)$ , i.e.,  $f$  is not analytic on the unit disk.

The previous condition is also a sufficient one. Namely, for  $0 \leq b_1 \leq 1 - \frac{\lambda}{2}$  and  $0 \leq b_3 \leq \frac{\lambda}{2}$ :

$$\begin{aligned} \left| \frac{z}{f(z)} \right| &= \left| 1 + b_1 z + \frac{\lambda}{2} z^3 \right| \geq 1 - b_1 |z| - \frac{\lambda}{2} |z|^3 \\ &> 1 - \left( 1 - \frac{\lambda}{2} \right) - \frac{\lambda}{2} = 0. \end{aligned}$$

So,  $0 \leq b_1 \leq 1 - \frac{\lambda}{2}$  and  $H_2(2)(f) \leq (1 - \frac{\lambda}{2}) \frac{\lambda}{2}$ . The upper bound is sharp due to the function  $f(z) = \frac{z}{1 + (1 - \lambda/2)z + \lambda/2 z^3}$ .

(ii) Similarly as in the proof of part (a), from (14) and (11),

$$\begin{aligned} H_3(1)(f) &\leq b_2 \cdot \frac{1}{3} (\lambda - b_2 - 2b_3) - b_3^2 \\ &= \frac{1}{3} \lambda b_2 - \frac{1}{3} b_2^2 - \frac{2}{3} b_2 b_3 - b_3^2 \\ &\leq \frac{1}{3} \lambda b_2 - \frac{1}{3} b_2^2 \leq \frac{\lambda^2}{12}. \end{aligned}$$

Also,  $H_3(1)(f) \geq -b_3^2 \geq -(\frac{\lambda}{2})^2 = -\frac{\lambda^2}{4}$ , since by (11),  $b_3 \leq \frac{\lambda - b_2}{2} \leq \frac{\lambda}{2}$ . The functions  $f(z) = \frac{z}{1 + \lambda/2 z^3}$  and  $f(z) = \frac{z}{1 + \lambda/2 z^2 + \lambda/6 z^4}$  show that the estimates are sharp.  $\square$

For the sharp estimates of the Hankel determinant for the inverse of the functions from  $\mathcal{U}^+(\lambda)$  we have the following theorem.

**Theorem 3.2.** *Let  $f \in \mathcal{U}^+(\lambda)$ ,  $0 < \lambda \leq 1$ . Then the following estimates are sharp:*

- (i)  $-\lambda^2 \leq H_2(2)(f^{-1}) \leq \lambda(1 + \lambda + \lambda^2)$ ;
- (ii)  $-\frac{\lambda^2}{4} \leq H_3(1)(f^{-1}) \leq \lambda^3$ .

*Proof.* (i) From (9) we have

$$H_2(2)(f^{-1}) = A_2 A_4 - A_3^2 = b_1 b_3 + b_1^2 b_2 - b_2^2 \equiv \psi_1(b_1),$$

where  $\psi_1$  is an increasing function of  $b_1$ ,  $0 \leq b_1 \leq 1 + \lambda$ , and  $\psi_1(b_1) \leq \psi(1 + \lambda)$ . But, Lemma 1.1, implies  $b_2 = \lambda$  and  $b_3 = 0$  when for  $b_1 = 1 + \lambda$ . So,  $\psi_1(b_1) \leq \psi(1 + \lambda) = (1 + \lambda) \cdot 0 + (1 + \lambda)^2 \lambda - \lambda^2 = \lambda(1 + \lambda + \lambda^2)$ , and the result follows. This upper bound is sharp due to the function  $f_\lambda$  defined in (4).

Also,  $H_2(2)(f^{-1}) \geq -b_2^2 \geq -\lambda^2$ . This bound is also sharp as the function  $f(z) = \frac{z}{1 + \lambda z^2}$  shows.



(ii) For the third Hankel determinant of  $f^{-1}$  we have

$$H_3(1)(f^{-1}) = H_3(1)(f) - (a_3 - a_2^2)^3 = b_2b_4 - b_3^2 + b_2^3,$$

which is an increasing function of  $b_2$  on the interval  $(0, \lambda)$ , reaching the maximum value on the interval for  $b_2 = \lambda$ . At the same moment, due to Lemma 1.1 we receive that  $b_3 = b_4 = 0$ , leading to

$$H_3(1)(f^{-1}) \leq \lambda \cdot 0 - 0^2 + \lambda^3 = \lambda^3.$$

The function  $\frac{z}{1+\lambda z^3}$  shows that this estimate is the best possible.

Also,  $H_3(1)(f^{-1}) \geq -b_3^2 \geq -\frac{\lambda^2}{4}$  since from Lemma 1.1, under the condition  $b_n = 0$  for all positive integers  $n \neq 3$ , we receive  $b_3 \leq \frac{\lambda}{2}$ . This estimate is also the best possible with an extremal function  $\frac{z}{1+\lambda/2z^3}$ .  $\square$

#### 4. Zalcman conjecture

In the early 1970's Zalcman posed the following conjecture for the class of univalent functions:

$$|a_n^2 - a_{2n-1}| \leq (n - 1)^2 \quad (n \in \mathbb{N}, n \geq 2).$$

There is a manuscript by Krushkal ([7]) that uses complex geometry of the universal Teichmüller space claiming to have proven the conjecture, but this work is not widely reckognized to be correct. Later, in 1999, Ma ([11]) proposed a generalized Zalcman conjecture,

$$|a_m a_n - a_{m+n-1}| \leq (m - 1)(n - 1) \quad (m, n \in \mathbb{N}, m \geq 2, n \geq 2),$$

and closed it for the class of starlike functions and for the class of univalent functions with real coefficients. The general case is still an open problem.

The Zalcman conjecture for the class  $\mathcal{U}$  when  $n = 2$  and  $n = 3$  was proven in [17].

**Theorem C.** *Let  $f \in \mathcal{U}$  be of the form (1). Then*

- (i)  $|a_2 a_3 - a_4| \leq 2$ ;
- (ii)  $|a_2 a_4 - a_5| \leq 3$ .

*These inequalities are sharp with equality for the Koebe function  $k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n$  and its rotations.*

We now give direct proof of the Zalcman conjecture for the class  $\mathcal{U}^+(\lambda)$  for the cases when  $n = 2$  and  $n = 3$ .

**Theorem 4.1.** *Let  $f \in \mathcal{U}^+(\lambda)$ ,  $0 < \lambda \leq 1$ , be of the form (1). Then*

- (i)  $-(1 + \lambda)\lambda \leq a_2 a_3 - a_4 \leq \frac{1}{2}\lambda$ ;
- (ii)  $|a_2 a_4 - a_5| \leq \lambda + \lambda^2 + \lambda^3$ .

*These inequalities are sharp.*

*Proof.* (i) Using (8) we have

$$a_2a_3 - a_4 = -b_1b_2 + b_3 \leq b_3 \leq \frac{1}{2}\lambda,$$

(where we used Lemma 1.1 or (11)). Also,

$$-(a_2a_3 - a_4) = b_1b_2 - b_3 \leq b_3 \leq (1 + \lambda)\lambda,$$

(see Remark 2.3), i.e.,  $a_2a_3 - a_4 \leq (1 + \lambda)\lambda$ . The functions  $f_\lambda$  and  $f(z) = \frac{z}{1 + \frac{\lambda}{2}z^3}$  show that the result is sharp.

(ii) Using (8) we obtain that

$$a_2a_4 - a_5 = b_1^2b_2 - b_1b_3 - b_2^2 + b_4 \equiv \psi_2(b_1),$$

$0 \leq b_1 \leq 1 + \lambda$ . Since  $\max \psi_2 = \psi_2(1 + \lambda) = (1 + \lambda)^2\lambda - \lambda^2 = \lambda + \lambda^2 + \lambda^3$ , then

$$a_2a_4 - a_5 \leq \lambda + \lambda^2 + \lambda^3.$$

On the other hand,

$$a_2a_4 - a_5 \geq -b_1b_3 - b_2^2 \geq -(1 + \lambda)\frac{\lambda}{2} - \lambda^2 > -(\lambda + \lambda^2 + \lambda^3).$$

The result is sharp due to the function  $f_\lambda$ .  $\square$

### References

- [1] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, and Y. J. Sim, *Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha*, J. Math. Inequal. **11** (2017), no. 2, 429–439. <https://doi.org/10.7153/jmi-11-36>
- [2] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, and Y. J. Sim, *The bounds of some determinants for starlike functions of order alpha*, Bull. Malays. Math. Sci. Soc. **41** (2018), no. 1, 523–535. <https://doi.org/10.1007/s40840-017-0476-x>
- [3] W. K. Hayman, *On the second Hankel determinant of mean univalent functions*, Proc. London Math. Soc. (3) **18** (1968), 77–94. <https://doi.org/10.1112/plms/s3-18.1.77>
- [4] A. Janteng, S. Abdul Halim, and M. Darus, *Coefficient inequality for a function whose derivative has a positive real part*, JIPAM. J. Inequal. Pure Appl. Math. **7** (2006), no. 2, Article 50, 5 pp.
- [5] A. Janteng, S. Abdul Halim, and M. Darus, *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal. (Ruse) **1** (2007), no. 13-16, 619–625.
- [6] B. Kowalczyk, A. Lecko, and Y. J. Sim, *The sharp bound for the Hankel determinant of the third kind for convex functions*, Bull. Aust. Math. Soc. **97** (2018), no. 3, 435–445. <https://doi.org/10.1017/S0004972717001125>
- [7] S. Krushkal, *A short geometric proof of the zalcman and bieberbach conjectures*, arXiv:1109.4646
- [8] O. S. Kwon, A. Lecko, and Y. J. Sim, *The bound of the Hankel determinant of the third kind for starlike functions*, Bull. Malays. Math. Sci. Soc. **42** (2019), no. 2, 767–780. <https://doi.org/10.1007/s40840-018-0683-0>
- [9] A. Lecko, Y. J. Sim, and B. Śmiarowska, *The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2*, Complex Anal. Oper. Theory **13** (2019), no. 5, 2231–2238. <https://doi.org/10.1007/s11785-018-0819-0>
- [10] S. K. Lee, V. Ravichandran, and S. Supramaniam, *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl. **2013** (2013), 281, 17 pp. <https://doi.org/10.1186/1029-242X-2013-281>

- [11] W. C. Ma, *Generalized Zalcman conjecture for starlike and typically real functions*, J. Math. Anal. Appl. **234** (1999), no. 1, 328–339. <https://doi.org/10.1006/jmaa.1999.6378>
- [12] M. Obradović and S. Ponnusamy, *Coefficient characterization for certain classes of univalent functions*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), no. 2, 251–263. <http://projecteuclid.org/euclid.bbms/1244038137>
- [13] M. Obradović, S. Ponnusamy, and P. Vasundhara, *Univalence, strong starlikeness and integral transforms*, Ann. Polon. Math. **86** (2005), no. 1, 1–13. <https://doi.org/10.4064/ap86-1-1>
- [14] M. Obradović, S. Ponnusamy, and K.-J. Wirths, *Geometric studies on the class  $\mathcal{U}(\lambda)$* , Bull. Malays. Math. Sci. Soc. **39** (2016), no. 3, 1259–1284. <https://doi.org/10.1007/s40840-015-0263-5>
- [15] M. Obradović and N. Tuneski, *Some properties of the class  $\mathcal{U}$* , Ann. Univ. Mariae Curie-Skłodowska Sect. A **73** (2019), no. 1, 49–56.
- [16] M. Obradović and N. Tuneski, *A class of univalent functions with real coefficients*, Bull. Malays. Math. Sci. Soc. **43** (2020), no. 3, 2875–2886. <https://doi.org/10.1007/s40840-019-00842-5>
- [17] M. Obradović and N. Tuneski, *Zalcman and generalized Zalcman conjecture for the class  $\mathcal{U}$* , Novi Sad J. Math. **52** (2022), no. 1, 173–184.
- [18] M. Obradović and N. Tuneski, *Coefficients of the inverse of functions for the subclass of the class  $\mathcal{U}(\lambda)$* , J. Anal. **30** (2022), no. 1, 399–404. <https://doi.org/10.1007/s41478-021-00326-5>
- [19] D. K. Thomas, N. Tuneski, and A. Vasudevarao, *Univalent functions*, De Gruyter Studies in Mathematics, 69, De Gruyter, Berlin, 2018. <https://doi.org/10.1515/9783110560961>
- [20] A. Vasudevarao and H. Yanagihara, *On the growth of analytic functions in the class  $\mathcal{U}(\lambda)$* , Comput. Methods Funct. Theory **13** (2013), no. 4, 613–634. <https://doi.org/10.1007/s40315-013-0045-8>
- [21] P. Zaprawa, *Third Hankel determinants for subclasses of univalent functions*, Mediterr. J. Math. **14** (2017), no. 1, Paper No. 19, 10 pp. <https://doi.org/10.1007/s00009-016-0829-y>

MILUTIN OBRADOVIĆ  
DEPARTMENT OF MATHEMATICS  
FACULTY OF CIVIL ENGINEERING  
UNIVERSITY OF BELGRADE  
BULEVAR KRALJA ALEKSANDRA 73  
11000, BELGRADE, SERBIA  
Email address: obrad@grf.bg.ac.rs

NIKOLA TUNESKI  
DEPARTMENT OF MATHEMATICS AND INFORMATICS  
FACULTY OF MECHANICAL ENGINEERING  
SS. CYRIL AND METHODIUS UNIVERSITY IN SKOPJE  
KARPOŠ II B.B., 1000 SKOPJE  
REPUBLIC OF NORTH MACEDONIA  
Email address: nikola.tuneski@mf.edu.mk