

## ON THE CONSTRUCTION OF OPTIMAL LINEAR CODES OF DIMENSION FOUR

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ABSTRACT. A fundamental problem in coding theory is to find  $n_q(k, d)$ , the minimum length  $n$  for which an  $[n, k, d]_q$  code exists. We show that some  $q$ -divisible optimal linear codes of dimension 4 over  $\mathbb{F}_q$ , which are not of Belov type, can be constructed geometrically using hyperbolic quadrics in  $\text{PG}(3, q)$ . We also construct some new linear codes over  $\mathbb{F}_q$  with  $q = 7, 8$ , which determine  $n_7(4, d)$  for 31 values of  $d$  and  $n_8(4, d)$  for 40 values of  $d$ .

### 1. Introduction

We denote by  $\mathbb{F}_q$  the field of  $q$  elements. Let  $\mathbb{F}_q^n$  be the vector space of  $n$ -tuples over  $\mathbb{F}_q$ . An  $[n, k, d]_q$  code  $\mathcal{C}$  is a  $k$ -dimensional subspace of  $\mathbb{F}_q^n$  with minimum Hamming weight  $d = \min\{wt(c) \mid c \in \mathcal{C}, c \neq (0, \dots, 0)\}$ , where  $wt(c)$  is the number of non-zero entries in the vector  $c$ . The weight distribution of  $\mathcal{C}$  is the list of non-zero integers  $A_i$ , where  $A_i$  is the number of codewords of weight  $i$ ,  $0 \leq i \leq n$ . A fundamental problem in coding theory is to find  $n_q(k, d)$ , the minimum length  $n$  for which an  $[n, k, d]_q$  code exists [12, 13]. An  $[n, k, d]_q$  code satisfies the inequality called the *Griesmer bound* [11, 12]:

$$n \geq g_q(k, d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil,$$

where  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ . The values of  $n_q(k, d)$  are determined for all  $d$  only for some small values of  $q$  and  $k$ . For  $k = 3$ ,  $n_q(3, d)$  is known for all  $d$  for  $q \leq 9$  [1]. See [25] for the updated table of  $n_q(k, d)$  for some small  $q$  and  $k$ . See also [10] for the updated linear codes bound. We mainly deal with linear codes of dimension  $k = 4$ . See [7] for the construction of optimal linear codes of dimension  $k = 5$ . The following theorems give some known values of  $n_q(4, d)$ .

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**Theorem 1.1** ([4, 22, 24]). For all prime power  $q$ ,  $n_q(4, d) = g_q(4, d)$  for  $1 \leq d \leq q - 2$ ,  $q^2 - 2q + 1 \leq d \leq q^2 - q$ ,  $q^3 - 2q^2 + 1 \leq d \leq q^3 - 2q^2 + q$ ,  $q^3 - q^2 - q + 1 \leq d \leq q^3 + q^2 - q$ ,  $2q^3 - 5q^2 + 1 \leq d \leq 2q^3 - 5q^2 + 3q$ ,  $2q^3 - 4q^2 + 1 \leq d \leq 2q^3 - 4q^2 + 2q$  and any  $d \geq 2q^3 - 3q^2 + 1$ .

**Theorem 1.2** ([4, 18, 19, 22, 24]). For  $q \geq 7$ ,  $n_q(4, d) = g_q(4, d) + 1$  for  $q^2 - q + 1 \leq d \leq q^2 - 1$ ,  $q^3 - 2q^2 - q + 1 \leq d \leq q^3 - 2q^2 - \lfloor (q+1)/2 \rfloor$ ,  $2q^3 - 5q^2 - q + 1 \leq d \leq 2q^3 - 5q^2$ ,  $2q^3 - 4q^2 - 3q + 1 \leq d \leq 2q^3 - 4q^2$ ,  $2q^3 - 3q^2 - 3q + 1 \leq d \leq 2q^3 - 3q^2$ .

See [25] for  $q \leq 5$ . An  $[n, k, d]_q$  code is called  $m$ -divisible if all codewords have weights divisible by an integer  $m > 1$ . We show that some optimal linear codes of dimension 4 can be constructed as  $q$ -divisible codes using hyperbolic quadrics in  $\text{PG}(3, q)$  as follows.

**Theorem 1.3.** Assume  $q \geq 7$ .

- (a) There exist  $q$ -divisible  $[g_q(4, d), 4, d]_q$  codes for  $d = q^3 - 2q^2 + q$ ,  $2q^3 - 5q^2 + q$ ,  $2q^3 - 5q^2 + 2q$ ,  $2q^3 - 5q^2 + 3q$ ,  $2q^3 - 4q^2 + q$ ,  $2q^3 - 4q^2 + 2q$ .
- (b) There exist  $q$ -divisible  $[g_q(4, d) + 1, 4, d]_q$  codes for  $d = q^3 - 3q^2 + 3q$ ,  $2q^3 - 6q^2 + 6q$ ,  $2q^3 - 5q^2 - \alpha q$  with  $0 \leq \alpha \leq q - 4$ ,  $2q^3 - 5q^2 + 4q$ ,  $2q^3 - 4q^2 - \beta q$  with  $0 \leq \beta \leq q - 3$ .

It can be proved that  $[g_q(4, d), 4, d]_q$  codes do not exist for  $d = q^3 - 3q^2 + 3q$  for  $7 \leq q \leq 13$ , giving the following.

**Theorem 1.4.**  $n_q(4, d) = g_q(4, d) + 1$  for  $d = q^3 - 3q^2 + 3q$  for  $7 \leq q \leq 13$ .

As for  $q = 8$ , it is known that  $n_8(4, 337) = g_8(4, 337) + 1$  or  $g_8(4, 337) + 2$ . Since one can obtain an  $[n - 1, k, d - 1]_q$  code from a given  $[n, k, d]_q$  code by puncturing, Theorem 1.4 implies that  $n_8(4, d) = g_8(4, d) + 1$  for  $337 \leq d \leq 344$ , which is a new result, see [25].

We also construct some new codes over  $\mathbb{F}_q$  with  $q = 7, 8$ , which determine  $n_7(4, d)$  for 31 values of  $d$  and  $n_8(4, d)$  for 40 values of  $d$ .

**Theorem 1.5.**

- (a)  $n_7(4, d) = g_7(4, d)$  for  $113 \leq d \leq 119$  and  $400 \leq d \leq 413$ .
- (b)  $n_7(4, d) = g_7(4, d) + 1$  for  $124 \leq d \leq 126$  and  $127 \leq d \leq 133$ .
- (c)  $n_7(4, d) \leq g_7(4, d) + 1$  for  $d = 105$ ,  $120 \leq d \leq 123$  and  $148 \leq d \leq 154$ .

**Theorem 1.6.**

- (a)  $n_8(4, d) = g_8(4, d)$  for  $641 \leq d \leq 672$ .
- (b)  $n_8(4, d) = g_8(4, d) + 1$  for  $345 \leq d \leq 352$ .
- (c)  $n_8(4, d) \leq g_8(4, d) + 1$  for  $329 \leq d \leq 336$ .

Our main results are Theorems 1.3-1.6. This paper is organized as follows. In Section 2, we give the geometric methods to construct  $q$ -divisible codes or to investigate a given code through projective geometry. In Section 3, we present some construction of optimal  $q$ -divisible codes using hyperbolic quadrics, which yields Theorem 1.3. In Section 4, we prove the non-existence of some Griesmer

codes, giving Theorem 1.4 by Theorem 1.3(b). In Section 5, we present some new linear codes over  $\mathbb{F}_q$  for  $q = 7, 8$  to prove Theorems 1.5 and 1.6.

### 2. Geometric methods

In this section, we give geometric methods to construct new codes from old ones or to prove the non-existence of linear codes with certain parameters through projective geometry. We denote by  $\text{PG}(r, q)$  the projective geometry of dimension  $r$  over  $\mathbb{F}_q$ . A  $j$ -flat is a projective subspace of dimension  $j$  in  $\text{PG}(r, q)$ . The 0-flats, 1-flats, 2-flats,  $(r - 2)$ -flats and  $(r - 1)$ -flats are called *points*, *lines*, *planes*, *secundums* and *hyperplanes*, respectively. We denote by  $\theta_j$  the number of points in a  $j$ -flat, i.e.,  $\theta_j = (q^{j+1} - 1)/(q - 1)$ .

Let  $\mathcal{C}$  be an  $[n, k, d]_q$  code having no coordinate which is identically zero. The columns of a generator matrix of  $\mathcal{C}$  can be considered as a multiset of  $n$  points in  $\Sigma = \text{PG}(k-1, q)$  denoted by  $\mathcal{M}_{\mathcal{C}}$ . A point  $P$  in  $\Sigma$  is called an  $i$ -point if it has multiplicity  $m_{\mathcal{C}}(P) = i$  in  $\mathcal{M}_{\mathcal{C}}$ . Denote by  $\gamma_0$  the maximum multiplicity of a point from  $\Sigma$  in  $\mathcal{M}_{\mathcal{C}}$  and let  $C_i$  be the set of  $i$ -points in  $\Sigma$ ,  $0 \leq i \leq \gamma_0$ . We denote by  $\Delta_1 + \dots + \Delta_s$  the multiset consisting of the  $s$  sets  $\Delta_1, \dots, \Delta_s$  in  $\Sigma$ . We write  $s\Delta$  for  $\Delta_1 + \dots + \Delta_s$  when  $\Delta_1 = \dots = \Delta_s$ . If a multiset  $\mathcal{M}$  is written as  $\mathcal{M}_1 + \mathcal{M}_2$  with two multisets  $\mathcal{M}_1, \mathcal{M}_2$ , we also write  $\mathcal{M}_2 = \mathcal{M} - \mathcal{M}_1$ . Note that  $\mathcal{M}_{\mathcal{C}} = \sum_{i=1}^{\gamma_0} iC_i$ . For any subset  $S$  of  $\Sigma$ , the *multiplicity of  $S$* , denoted by  $m_{\mathcal{C}}(S)$ , is defined as

$$m_{\mathcal{C}}(S) = \sum_{P \in S} m_{\mathcal{C}}(P) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|,$$

where  $|T|$  denotes the number of elements in a set  $T$ . Then we obtain the partition  $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$  such that  $n = m_{\mathcal{C}}(\Sigma)$  and

$$(1) \quad n - d = \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\},$$

where  $\mathcal{F}_j$  denotes the set of  $j$ -flats in  $\Sigma$ . Such a partition of  $\Sigma$  is called an  $(n, n - d)$ -arc of  $\Sigma$ . Conversely an  $(n, n - d)$ -arc of  $\Sigma$  gives an  $[n, k, d]_q$  code in the natural manner. A line  $l$  with  $t = m_{\mathcal{C}}(l)$  is called a  $t$ -line. A  $t$ -plane, a  $t$ -hyperplane and so on are defined similarly. Let

$$\gamma_j = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \in \mathcal{F}_j\}, \quad 0 \leq j \leq k - 1,$$

and let  $\lambda_s$  be the number of  $s$ -points in  $\Sigma$ . It holds that  $\gamma_{k-2} = n - d$ ,  $\gamma_{k-1} = n$ . When  $\mathcal{C}$  is Griesmer, the values  $\gamma_0, \gamma_1, \dots, \gamma_{k-3}$  are also uniquely determined ([23]) as follows:

$$(2) \quad \gamma_j = \sum_{u=0}^j \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \text{ for } 0 \leq j \leq k - 1.$$

Denote by  $a_i$  the number of  $i$ -hyperplanes in  $\Sigma$ . Note that

$$(3) \quad a_i = A_{n-i}/(q - 1) \text{ for } 0 \leq i \leq n - d.$$

The list of  $a_i$ 's is called the *spectrum* of  $\mathcal{C}$ . Simple counting arguments yield the following [21]:

$$(4) \quad \sum_{i=0}^{\gamma_{k-2}} a_i = \theta_{k-1},$$

$$(5) \quad \sum_{i=1}^{\gamma_{k-2}} i a_i = n \theta_{k-2},$$

$$(6) \quad \sum_{i=2}^{\gamma_{k-2}} i(i-1) a_i = n(n-1) \theta_{k-3} + q^{k-2} \sum_{s=2}^{\gamma_0} s(s-1) \lambda_s.$$

**Lemma 2.1.** *Let  $\mathcal{C}$  be an  $[n, k, d]_q$  code with multiset  $\mathcal{M}_{\mathcal{C}}$  over  $\Sigma = \text{PG}(k - 1, q)$ . Let  $\Delta$  be a  $t$ -flat in  $\Sigma$  with  $1 \leq t \leq k - 1$  and let  $P$  be a point of  $\Sigma$ . Then,*

- (a) *A code  $\mathcal{C}'$  with multiset  $\mathcal{M}_{\mathcal{C}'} = \mathcal{M}_{\mathcal{C}} + \Delta$  is  $q$ -divisible if and only if  $\mathcal{C}$  is  $q$ -divisible.*
- (b) *A code  $\mathcal{C}'$  with multiset  $\mathcal{M}_{\mathcal{C}'} = \mathcal{M}_{\mathcal{C}} + qP$  is  $q$ -divisible if and only if  $\mathcal{C}$  is  $q$ -divisible.*

*Proof.* It follows from (3) that  $\mathcal{C}$  is  $q$ -divisible if and only if  $m_{\mathcal{C}}(H) \equiv n \pmod{q}$  for any hyperplane  $H$ . Since every hyperplane meets  $\Delta$  in exactly  $\theta_{t-1}$  or  $\theta_t$  points,  $\mathcal{C}$  is  $q$ -divisible if and only if  $m_{\mathcal{C}'}(H) \equiv n + 1 \equiv n + \theta_t \pmod{q}$  for any hyperplane  $H$ . Hence, (a) follows. The part (b) can be proved similarly.  $\square$

**Lemma 2.2.** *Let  $\mathcal{C}$  be an  $[n, k, d]_q$  code with multiset  $\mathcal{M}_{\mathcal{C}}$  over  $\Sigma = \text{PG}(k - 1, q)$ . Assume  $d > q^t$  and that  $\mathcal{M}_{\mathcal{C}}$  contains a  $t$ -flat  $\Delta$  with  $1 \leq t \leq k - 1$ . Let  $\mathcal{C}'$  be an  $[n - \theta_t, k, d']_q$  code with multiset  $\mathcal{M}_{\mathcal{C}'} = \mathcal{M}_{\mathcal{C}} - \Delta$ . Then,*

- (a)  $d' \geq d - q^t$ .
- (b)  $\mathcal{C}'$  is  $q$ -divisible if  $\mathcal{C}$  is  $q$ -divisible.

*Proof.* See [26] for (a). The part (b) follows from Lemma 2.1(a).  $\square$

The method to construct new codes from a given  $[n, k, d]_q$  code by deleting the coordinates corresponding to some geometric object in  $\text{PG}(k - 1, q)$  is called *geometric puncturing*, see [24].

**Lemma 2.3** (Cf. [5]). *Let  $\mathcal{C}_1$  be an  $[n_1, k, d_1]_q$  code with a codeword of weight  $d_1 + m$ ,  $m > 0$  and let  $\mathcal{C}_2$  be an  $[n_2, k - 1, d_2]_q$  code. Then, there exists an  $[n_1 + n_2, k, d]_q$  code  $\mathcal{C}$  with  $\mathcal{M}_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}_1} + \mathcal{M}_{\mathcal{C}_2}$  satisfying  $d = d_1 + m$  if  $m < d_2$  and  $d = d_1 + d_2$  if  $m \geq d_2$ . Moreover,  $\mathcal{C}$  is  $q$ -divisible if  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are  $q$ -divisible.*

*Proof.* Let  $\mathcal{C}_1$  be an  $[n_1, k, d_1]_q$  code with generator matrix  $G$  whose  $i$ -th row is  $g_i$  for  $1 \leq i \leq k$ . Assume that a codeword  $a_1 g_1 + \dots + a_k g_k$  with  $a_1, \dots, a_k \in \mathbb{F}_q$  has weight  $d_1 + m$ ,  $m > 0$ . Then, the hyperplane  $H$  defined by the equation  $a_1 X_1 + \dots + a_k X_k = 0$  has multiplicity  $n_1 - d_1 - m$ . Since  $\mathcal{C}_2$  has dimension

$k - 1$ , one can take  $\mathcal{M}_{\mathcal{C}_2}$  as a multiset on  $H$ . Then, the  $[n_1 + n_2, k, d]_q$  code  $\mathcal{C}$  with  $\mathcal{M}_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}_1} + \mathcal{M}_{\mathcal{C}_2}$  satisfies

$$n_1 + n_2 - d = \max\{n_1 - d_1 - m + n_2, n_1 - d_1 + n_2 - d_2\}$$

from (1), and our assertion follows. The divisibility is easily checked by (3).  $\square$

Let  $d$  be a positive integer to construct an  $[n, k, d]_q$  code with dimension  $k \geq 3$ . Since  $[g_q(k, sq^{k-1}), k, sq^{k-1}]_q$  codes (called *s-fold simplex codes*) exist for any positive integer  $s$ , we assume that  $d$  is not divisible by  $q^{k-1}$ . Then,  $d$  can be uniquely expressed with  $s = \lceil d/q^{k-1} \rceil$  as

$$(7) \quad d = sq^{k-1} - \sum_{j=1}^r q^{u_j-1},$$

where  $r$  and  $u_j$ 's are integers satisfying

$$(8) \quad k - 1 \geq u_1 \geq u_2 \geq \dots \geq u_r \geq 1$$

and

$$(9) \quad u_j > u_{j+q-1} \text{ for } 1 \leq j \leq r - q + 1.$$

The condition (9) means that at most  $q - 1$  of  $u_1, \dots, u_r$  can take any given value. If  $r \leq s$  and if  $u_r \geq 2$ , it is easy to construct Griesmer codes from an  $s$ -fold simplex code by geometric puncturing and the resulting codes are  $q$ -divisible since an  $s$ -fold simplex code is  $q^{k-1}$ -divisible.

**Theorem 2.4.** *There exists a  $q$ -divisible  $[g_q(k, d), k, d]_q$  code if  $r \leq s$  and if  $q$  divides  $d$ .*

Assume  $r \geq s + 1$  and let  $u = \sum_{i=1}^{s+1} u_i$ . The following theorem was proved by Belov et al. [2] for binary linear codes and by Hill [12] and Dodunekov [9] for codes over  $\mathbb{F}_q$ .

**Theorem 2.5** ([12]). *There exists a  $[g_q(k, d), k, d]_q$  code if  $u \leq sk$ .*

The positive integer  $d$  can also be uniquely expressed as

$$(10) \quad d = sq^{k-1} - \sum_{i=0}^{k-2} d_i q^i,$$

where  $d_i$ 's are integers satisfying  $r = \sum_i d_i$  and

$$(11) \quad 0 \leq d_i \leq q - 1 \text{ for all } i.$$

To construct a code of length  $s\theta_{k-1} - \sum_{i=0}^{k-2} d_i\theta_i$ , we shall make a multiset

$$(12) \quad s \text{PG}(k - 1, q) - \sum_{i=0}^{k-2} \left( \sum_{j=1}^{d_i} \Delta_{ij} \right)$$

with  $d_i$   $i$ -flats  $\Delta_{ij}$  for  $0 \leq i \leq k - 2$  and  $1 \leq j \leq d_i$  if possible. Such a construction of Griesmer codes is possible if  $u \leq sk$  with  $u = \sum_{i=1}^{s+1} u_i$ , where

$u_1, \dots, u_r$  satisfy (7)-(9), and the resulting codes are called the *Griesmer codes of Belov type* [13]. In [17], a *GPS code*  $\mathcal{C}$  (obtained by geometric puncturing from an  $s$ -fold simplex code) is defined as a linear code whose multiset  $\mathcal{M}_{\mathcal{C}}$  is of the form (12) even for the case that  $u \geq sk + 1$ , and many optimal codes are constructed as GPS. GPS codes are  $q$ -divisible if  $d$  is divided by  $q$ . Note that Theorem 2.5 is proved by constructing GPS codes. Hence we get the following.

**Corollary 2.6.** *There exists a  $q$ -divisible  $[g_q(k, d), k, d]_q$  code if  $u \leq sk$  and  $d_0 = 0$ .*

In the next section, we consider  $q$ -divisible linear codes  $\mathcal{C}$  with multiset of the form

$$\mathcal{M}_{\mathcal{C}} = (A_1 + \dots + A_a) - (B_1 + \dots + B_b)$$

with geometric objects  $A_i$  and  $B_j$ , each of which could be the whole space  $\Sigma = \text{PG}(k - 1, q)$ ,  $t$ -flats with  $t \geq 1$  or  $qP$  with some point  $P$  so that the code  $\mathcal{C}$  is  $q$ -divisible by Lemmas 2.1 and 2.2. We show that some  $q$ -divisible optimal codes of dimension 4 can be constructed using hyperbolic quadrics in  $\text{PG}(3, q)$ .

**Lemma 2.7** (Cf. [16, 28]). *Let  $\mathcal{C}$  be an  $m$ -divisible  $[n, k, d]_q$  code with  $q = p^h$ ,  $p$  prime, whose spectrum is*

$$(a_{n-d-(w-1)m}, a_{n-d-(w-2)m}, \dots, a_{n-d-m}, a_{n-d}) = (\alpha_{w-1}, \alpha_{w-2}, \dots, \alpha_1, \alpha_0),$$

where  $m = p^r$  for some  $1 \leq r < h(k - 2)$  satisfying  $\lambda_0 > 0$  and

$$(13) \quad n > \frac{dq}{q - 1}.$$

Then there exists a  $t$ -divisible  $[n^*, k, d^*]_q$  code  $\mathcal{C}^*$  with  $t = q^{k-2}/m$ ,  $n^* = \sum_{j=0}^{w-1} j\alpha_j = ntq - \frac{d}{m}\theta_{k-1}$ ,  $d^* = ((n - d)q - n)t$  whose spectrum is

$$(a_{n^*-d^*-\gamma_0 t}, a_{n^*-d^*-(\gamma_0-1)t}, \dots, a_{n^*-d^*-t}, a_{n^*-d^*}) = (\lambda_{\gamma_0}, \lambda_{\gamma_0-1}, \dots, \lambda_1, \lambda_0).$$

$\mathcal{C}^*$  is called a *projective dual* of  $\mathcal{C}$ , see also [6] and [13].

*Remark 1.* The multiset  $\mathcal{M}_{\mathcal{C}^*}$  for the projective dual of  $\mathcal{C}$  is given by considering  $(n - d - jm)$ -hyperplanes as  $j$ -points in the dual space  $\Sigma^*$  of  $\Sigma = \text{PG}(k - 1, q)$  for  $0 \leq j \leq w - 1$  [28]. To guarantee that  $\mathcal{C}^*$  has dimension  $k$ , the condition

$$(14) \quad \bigcap_{H \in \mathcal{F}_{k-2}, m_{\mathcal{C}}(H) < n-d} H = \emptyset$$

was added in [16] to the original one in [28]. It follows from the proof of Lemma 5.1 in [28] that the maximum value of  $m_{\mathcal{C}^*}(H^*)$  with a hyperplane  $H^*$  of  $\Sigma^*$  is  $nt - d\theta_{k-2}/m$  when  $H^*$  is a 0-point of  $\Sigma$  for  $\mathcal{C}$ . Hence, we have  $\dim \mathcal{C}^* = k$  if  $nt - d\theta_{k-2}/m < n^*$ , equivalently, if (13) holds.

The codes NOT satisfying (13) are very special. Indeed, the Plotkin bound for codes with very large  $d$  requires the condition ' $n < dq/(q - 1)$ ' [3, 15]. Some optimal codes satisfying ' $n = dq/(q - 1)$ ' can be constructed as Griesmer codes of Belov type. For example, let  $\mathcal{M}_{\mathcal{C}} = \text{PG}(3, q) - \ell$  with a line  $\ell$ . Then,  $\mathcal{C}$  is

TABLE 1. Multiset  $\mathcal{M}_C$  for the code  $\mathcal{C}$ .

Lemma	$\mathcal{M}_C$
3.2	$2\text{PG}(3, q) + 2 \sum_{i=1}^2 (l_i + m_i) - (\mathcal{H}_1 + \mathcal{H}_2 + \Delta_{11} + \Delta_{12} + \Delta_{21} + \Delta_{22})$
3.3	$2\text{PG}(3, q) + \sum_{i=1}^2 (l_i + m_i) - (\mathcal{H}_1 + \Delta_{11} + \Delta_{12} + \Delta_{21} + \Delta_{22})$
3.4	$2\text{PG}(3, q) + (l_1 + m_1 + m_2) - (\mathcal{H}_1 + \Delta_{11} + \Delta_{12} + \Delta_{22})$

a  $q$ -divisible  $[q^3 + q^2, 4, q^3 - q]_q$  code with spectrum  $(a_{q^2}, a_{q^2+q}) = (q + 1, q^2)$ . Obviously, the LHS of (14) is equal to  $\ell$  and  $n(q - 1) = dq$  holds.

**3. Construction of  $q$ -divisible codes using hyperbolic quadrics**

Let  $\Sigma = \text{PG}(3, q)$  throughout this section. Assume  $q \geq 7$ . In this section, we construct some  $q$ -divisible codes, many of which are optimal. Let us take four points  $Q_0(0, 0, 0, 1), Q_1(0, 0, 1, 0), Q_2(0, 1, 0, 0), Q_3(1, 0, 0, 0)$ , six lines  $l_1 = \langle Q_0, Q_3 \rangle, l_2 = \langle Q_1, Q_2 \rangle, m_1 = \langle Q_0, Q_2 \rangle, m_2 = \langle Q_1, Q_3 \rangle, \ell = \langle Q_2, Q_3 \rangle, \ell' = \langle Q_0, Q_1 \rangle$  and four planes  $\Delta_{ij} = \langle l_i, m_j \rangle$  with  $1 \leq i \leq j \leq 2$ . We also take two hyperbolic quadrics  $\mathcal{H}_1, \mathcal{H}_2$  in  $\Sigma$  through the four lines  $l_1, l_2, m_1, m_2$  as

$$\mathcal{H}_1 = V(x_0x_1 + x_2x_3), \mathcal{H}_2 = V(x_0x_1 + \epsilon x_2x_3),$$

where  $\epsilon \in \mathbb{F}_q$  with  $\epsilon \neq 0, 1$ . We recall that a hyperbolic quadric in  $\text{PG}(3, q)$  consists of a set of  $q + 1$  skew lines  $L$ , called *regulus*, and has another regulus each line of which meets all lines of  $L$ , called the *complementary regulus* [14].

For  $d = q^3 - 2q^2 + q = q^3 - q^2 - (q - 1)q$ , we have  $u = 3 + 2 > sk = 4$ . Hence, there exists no Griesmer code of Belov type although a Griesmer code exists [22]. Such a Griesmer code can be constructed as a  $q$ -divisible code.

**Lemma 3.1.** *There exists a  $q$ -divisible  $[n = q^3 - q^2 + 1, 4, d = q^3 - 2q^2 + q]_q$  code with spectrum  $(a_1, a_{q^2-2q+1}, a_{q^2-q+1}) = (1, q^2, q^3 + q)$ .*

*Proof.* Let  $\mathcal{C}$  be an  $[n, 4, d]_q$  code with

$$\mathcal{M}_C = \text{PG}(3, q) + (l_1 + m_1) - (\mathcal{H}_1 + \Delta_{11}).$$

Then,  $n = \theta_3 + 2\theta_1 - (q + 1)^2 - \theta_2 = q^3 - q^2 + 1$ . It is easy to see that  $m(\Delta_{11}) = 1$ . Let  $\delta$  be a plane in  $\Sigma$ . If  $\delta$  is a tangent plane not containing  $l_1$  nor  $m_1$ , then  $m(\delta) = \theta_2 + 2 - (2q + 1 + \theta_1) = q^2 - 2q + 1$ . If  $\delta$  is a tangent plane containing  $l_1$  or  $m_1$ , then  $m(\delta) = \theta_2 + \theta_1 + 1 - (2q + 1 + \theta_1) = q^2 - q + 1$ . If  $\delta$  is not a tangent plane, then  $m(\delta) = \theta_2 + 2 - (q + 1 + \theta_1) = q^2 - q + 1$ . Hence, we get the spectrum as stated and we have  $d = n - (q^2 - q + 1) = q^3 - 2q^2 + q$ .  $\square$

The following Lemmas 3.2-3.4 can be proved similarly, see Table 1.

**Lemma 3.2.** *There exists a  $q$ -divisible  $[2q^3 - 4q^2 + 2q + 4, 4, 2q^3 - 6q^2 + 6q]_q$  code  $\mathcal{C}$  with spectrum  $(a_{q^2-2q+4}, a_{2q^2-5q+4}, a_{2q^2-4q+4}) = (4, 2q^2 - 4q + 2, q^3 - q^2 + 5q - 5)$ .*

**Lemma 3.3.** *There exists a  $q$ -divisible  $[2q^3 - 3q^2 + 1, 4, 2q^3 - 5q^2 + 3q]_q$  code  $\mathcal{C}$  with spectrum  $(a_{q^2-2q+1}, a_{2q^2-4q+1}, a_{2q^2-3q+1}) = (4, q^2 - 2q + 1, q^3 + 3q - 4)$ .*

**Lemma 3.4.** *There exists a  $q$ -divisible  $[2q^3 - 2q^2 + 1, 4, 2q^3 - 4q^2 + 2q]_q$  code  $\mathcal{C}$  with spectrum  $(a_{q^2-2q+1}, a_{q^2-q+1}, a_{2q^2-3q+1}, a_{2q^2-2q+1}) = (1, 2, q^2 - q, q^3 + 2q - 2)$ .*

Note that the code  $\mathcal{C}$  is Griesmer in Lemmas 3.3, 3.4 and that the code in Lemma 3.2 satisfies  $n = g_q(4, d) + 1$ .

Let  $\mathcal{C}$  be the Griesmer  $[2q^3 - 3q^2 + 1, 4, 2q^3 - 5q^2 + 3q]_q$  code in Lemma 3.3. Note that the set of 0-points is  $\ell \cup \ell' \cup l_1 \cup l_2 \cup m_1 \cup m_2$ . Assume that  $L = \{l_1, \dots, l_{q+1}\}$  and  $M = \{m_1, \dots, m_{q+1}\}$  are reguli of  $\mathcal{H}_1$ . Then, for  $3 \leq j \leq q + 1$ , the tangent plane  $\Delta_{3j} = \langle l_3, m_j \rangle$  contains six 0-points:  $P_{31} = l_3 \cap m_1$  and  $P_{32} = l_3 \cap m_2$  on  $l_3$ ,  $P_{1j} = l_1 \cap m_j$  and  $P_{2j} = l_2 \cap m_j$  on  $m_j$  and  $\langle P_{31}, P_{1j} \rangle \cap \langle P_{32}, P_{2j} \rangle$ ,  $\langle P_{31}, P_{2j} \rangle \cap \langle P_{32}, P_{1j} \rangle$ . Since  $q \geq 7$ , one can find a line  $l_{3j}$  on the plane  $\Delta_{3j}$  through the point  $l_3 \cap m_j$  which has no 0-point. Hence, the multiset  $\mathcal{M}_{\mathcal{C}}$  contains  $q - 1$  skew lines  $l_{3j}$  with  $3 \leq j \leq q + 1$ . It follows from Lemma 2.2 that there exists a  $q$ -divisible  $[n = 2q^3 - 3q^2 + 1 - t\theta_1, 4, d = 2q^3 - 5q^2 + 3q - tq]_q$  code for  $1 \leq t \leq q - 1$ , which are Griesmer for  $t = 1, 2$  and satisfy  $n = g_q(4, d) + 1$  for  $3 \leq t \leq q - 1$ . Similarly, we can prove that  $\mathcal{M}_{\mathcal{C}}$  for the code in Lemma 3.4 contains  $q - 1$  skew lines, giving the following.

**Lemma 3.5.** *There exists a  $q$ -divisible  $[g_q(4, d), 4, d]_q$  code for  $d = 2q^3 - 4q^2 + q, 2q^3 - 5q^2 + 2q, 2q^3 - 5q^2 + q$  and a  $q$ -divisible  $[g_q(4, d) + 1, 4, d]_q$  code for  $d = 2q^3 - 5q^2 - \alpha q$  with  $0 \leq \alpha \leq q - 4$  and for  $d = 2q^3 - 4q^2 - \beta q$  with  $0 \leq \beta \leq q - 3$ .*

*Remark 2.* It is known that a  $q$ -divisible  $[q^2, 3, q^2 - q]_q$  code  $\mathcal{C}'$  with spectrum  $(a_0, a_q) = (1, q^2 + q)$  exists (which can be obtained as a projection of an elliptic quadric  $\mathcal{E}_3$  from a point  $P \in \mathcal{E}_3$  onto a plane  $\delta$  not through  $P$ ). Adding  $\mathcal{M}_{\mathcal{C}'}$  to a  $(q^2 - 2q + 1)$ -plane for the code  $\mathcal{C}$  in Lemma 3.3, one can construct a  $q$ -divisible Griesmer  $[2q^3 - 2q^2 + 1, 4, 2q^3 - 4q^2 + 2q]_q$  code by Lemma 2.3.

**Lemma 3.6.** *There exists a  $[g_q(4, d^*) + 1 = q^3 - 2q^2 + q + 3, 4, d^* = q^3 - 3q^2 + 3q]_q$  code  $\mathcal{C}^*$  with spectrum  $(a_3, a_{q^2-3q+3}, a_{q^2-2q+3}) = (1, 2q^2 - q - 1, q^3 - q^2 + 2q)$ .*

*Proof.* Let  $\mathcal{C}$  be an  $[n, 4, d]_q$  code with

$$\mathcal{M}_{\mathcal{C}} = \mathcal{H}_1 + \mathcal{H}_2 + \ell + qQ_0 - (2l_1 + l_2 + 2m_1 + m_2).$$

Then,  $n = 2(q + 1)^2 + \theta_1 + q - 6\theta_1 = 2q^2 - 3$ . One can check that  $m(\Delta_{11}) = 2(2q + 1) + \theta_1 + q - (4\theta_1 + 2) = 2q - 3$ ,  $m(\Delta_{12}) = m(\Delta_{21}) = 2(2q + 1) + 1 + q - (3\theta_1 + 3) = 2q - 3$  and  $m(\Delta_{22}) = 2(2q + 1) + \theta_1 - (2\theta_1 + 4) = 3q - 3$ . Let  $\delta$  be a plane in  $\Sigma$ . If  $\delta$  is a tangent plane not containing any of  $l_1, l_2, m_1, m_2$ , then  $m(\delta) = 2q + 1 + \theta_1 + 1 - 6 = 3q - 3$ . If  $\delta$  is a tangent plane containing  $l_1$  or  $m_1$ , then  $m(\delta) = 2(2q + 1) + 1 + q - (2\theta_1 + 4) = 3q - 3$ . If  $\delta$  is not a tangent plane containing  $l_2$  or  $m_2$ , then  $m(\delta) = 2(2q + 1) + 1 - (\theta_1 + 5) = 3q - 3$ . If  $\delta$  is not a tangent plane containing  $\ell$ , then  $m(\delta) = 2\theta_1 + \theta_1 - 6 = 3q - 3$ . If  $\delta$  is not a



tangent plane not containing  $\ell$  nor  $Q_0$ , then  $m(\delta) = 2\theta_1 + 1 - 6 = 2q - 3$ . If  $\delta$  is not a tangent plane not containing  $Q_0$ , then  $m(\delta) = 2\theta_1 + 1 + q - 6 = 3q - 3$ . Hence, the spectrum of  $\mathcal{C}$  is  $(a_{2q-3}, a_{3q-3}) = (q^3 - 2q^2 + q + 3, 3q^2 - 2)$  since  $a_{3q-3} = 1 + 2(q-1)^2 + 4(q-1) + q - 1 + \theta_2 - 3 - 2(q-1) = 3q^2 - 2$ , and we have  $d = n - (3q - 3) = 2q^2 - 3q$ . Hence,  $\mathcal{C}$  is  $q$ -divisible. Let  $\mathcal{C}^*$  be a projective dual of  $\mathcal{C}$  with parameters  $[n^*, 4, d^*]_q$ . Then, we have the spectrum of  $\mathcal{C}^*$  as stated since  $m(Q) = 2 + q - 4 = q - 2$  and  $m(P) = 0$  or  $1$  for any other point  $P$ .  $\square$

In the next section, we show that a  $[g_q(4, d), 4, d = q^3 - 3q^2 + 3q]_q$  code does not exist for  $7 \leq q \leq 13$ . Hence, the code  $\mathcal{C}^*$  in Lemma 3.6 is optimal for  $7 \leq q \leq 13$ . But the optimality for  $q \geq 16$  is still open.

We note that the optimality of the following code is unknown for  $q \geq 7$ .

**Lemma 3.7.** *There exists a  $[g_q(4, d^*) + 1 = 2q^3 - 3q^2 + q + 3, 4, d^* = 2q^3 - 5q^2 + 4q]_q$  code  $\mathcal{C}^*$  with spectrum  $(a_{q^2-2q+3}, a_{q^2-q+3}, a_{2q^2-4q+3}, a_{2q^2-3q+3}) = (2, 1, 2q^2 - 2q, q^3 - q^2 + 3q - 2)$ .*

*Proof.* Let  $\mathcal{C}$  be an  $[n, 4, d]_q$  code with

$$\mathcal{M}_{\mathcal{C}} = \mathcal{H}_1 + \mathcal{H}_2 + l + q(Q_0 + Q_1 + Q_3) - (2l_1 + 2l_2 + m_1 + 2m_2).$$

Then,  $n = 2(q+1)^2 + \theta_1 + 3q - 7\theta_1 = 2q^2 + q - 4$ . One can check that  $m(\Delta_{11}) = 2(2q+1) + \theta_1 + 2q - (3\theta_1 + 4) = 4q - 4$  and  $m(\Delta_{12}) = 2(2q+1) + 1 + 3q - (4\theta_1 + 3) = 3q - 4 = m(\Delta_{21}) = m(\Delta_{22})$ . Let  $\delta$  be a plane in  $\Sigma$ . If  $\delta$  is a tangent plane not containing any of  $l_1, l_2, m_1, m_2$ , then  $m(\delta) = 2q + 1 + \theta_1 + 1 - 7 = 3q - 4$ . If  $\delta$  is a tangent plane containing  $l_1$  or  $m_2$ , then  $m(\delta) = 2(2q + 1) + 1 + 2q - (2\theta_1 + 5) = 4q - 4$ . If  $\delta$  is not a tangent plane and if  $\delta$  contains none of  $l_1, l_2, m_1, m_2$ , then  $m(\delta) = 2(2q + 1) + 1 + 2q - (2\theta_1 + 5) = 4q - 4$ . If  $\delta$  is a tangent plane containing  $l_2$ , then  $m(\delta) = 2(2q + 1) + 1 + q - (2\theta_1 + 5) = 3q - 4$ . If  $\delta$  is a tangent plane containing  $m_1$ , then  $m(\delta) = 2(2q + 1) + 1 + q - (\theta_1 + 6) = 4q - 4$ . If  $\delta$  is not a tangent plane and if  $\delta$  contains none of  $l_1, l_2, m_1, m_2$ , then  $m(\delta) = 2\theta_1 + \theta_1 + q - 7 = 4q - 4$ . If  $\delta$  is not a tangent plane containing  $\ell$  or  $\ell'$ , then  $m(\delta) = 2\theta_1 + 1 - 7 = 2q - 4$ . If  $\delta$  is not a tangent plane containing none of  $Q_0, Q_1, Q_3$ , then  $m(\delta) = 2\theta_1 + 1 + q - 7 = 3q - 4$ . If  $\delta$  is not a tangent plane not containing  $Q_2$ , then  $m(\delta) = 2\theta_1 + 1 - 7 = 2q - 4$ . Hence, the spectrum of  $\mathcal{C}$  is  $(a_{2q-4}, a_{3q-4}, a_{4q-4}) = (q^3 - 4q^2 + 5q - 2, 5q^2 - 9q + 7, 1 + 5(q - 1) = 5q - 4)$  since  $a_{3q-4} = 3 + 2(q-1)^2 + q - 1 + 3(\theta_2 - 3 - 3(q-1)) = 5q^2 - 4q + 7$ . And, we have  $d = n - (4q - 4) = 2q^2 - 3q$ . Thus,  $\mathcal{C}$  is  $q$ -divisible. Let  $\mathcal{C}^*$  be a projective dual of  $\mathcal{C}$  with parameters  $[n^*, 4, d^*]_q$ . Then, we have the spectrum of  $\mathcal{C}^*$  as stated since  $m(Q_0) = m(Q_3) = q - 1$ ,  $m(Q_1) = q - 2$  and  $m(P) = 0$  or  $1$  for any other point  $P$  and since  $a_{2q^2-4q+3} = n - 2(q - 1) - (q - 2)$ .  $\square$

#### 4. Optimality of the code in Lemma 3.6

In this section, we prove the non-existence of  $[g_q(4, d), 4, d]_q$  codes for  $d = q^3 - 3q^2 + 3q$  for  $3 \leq q \leq 13$ , which gives Theorem 1.4.

Let  $\mathcal{C}$  be an  $[n, 4, d]_q$  code with  $d = q^3 - 3q^2 + 3q$ ,  $n = g_q(4, d) = q^3 - 2q^2 + q + 2$ ,  $q \geq 3$ . Then,  $\mathcal{C}$  is projective from (2), and the set of 1-points in an  $(n - d)$ -plane forms a  $(q^2 - 2q + 2, q - 1)$ -arc. But such an arc does not exist for  $3 \leq q \leq 8$  (see [1]), a contradiction. For  $q = 9$ ,  $\mathcal{C}$  is a  $[578, 4, 513]_9$  code, which does not exist, see [20].

Now, assume  $q = 11$  or  $13$ . Since  $q$  is prime,  $\mathcal{C}$  is  $q$ -divisible by the following.

**Theorem 4.1** ([29]). *Let  $\mathcal{C}$  be a Griesmer  $[n, k, d]_p$  code with a prime  $p$ . If  $p^e$  divides  $d$ , then  $\mathcal{C}$  is  $p^e$ -divisible.*

It follows from (3) that an  $i$ -plane satisfies  $i \equiv 2 \pmod{q}$ . Assume that an  $i$ -plane  $\delta$  contains an  $r$ -line  $\ell$ . Since  $\gamma_2 = n - d$ , considering the planes through  $\ell$ , we get  $n \leq (n - d - r)q + i$ . Thus, an  $i$ -plane gives an  $(i, r)$ -arc with  $r \leq (i + q(n - d) - n)/q$ . Hence, from the possible sizes of arcs [1], we have  $a_i = 0$  for all  $i \notin \{90, 101\}$  when  $q = 11$  and that  $a_i = 0$  for all  $i \notin \{119, 132, 145\}$  when  $q = 13$ . For  $q = 11$ , we obtain  $(a_{90}, a_{101}) = (118, 1346)$  from (4) and (5), which contradicts (6). For  $q = 13$ , we obtain  $(a_{122}, a_{132}, a_{145}) = (122, -78, 2336)$  from the three equalities (4)-(6), a contradiction again. Hence, we get the following.

**Lemma 4.2.** *There exists no  $[g_q(4, d), 4, d]_q$  codes for  $d = q^3 - 3q^2 + 3q$  with  $3 \leq q \leq 13$ .*

*Remark 3.* The complement of a  $(q^2 - 2q + 2, q - 1)$ -arc in  $\text{PG}(2, q)$  is a double blocking set of size  $3q - 1$ , which does exist for  $q = 13, 16, 19, 25, 27, 31, 37, 43$  [8]. But the existence of such an arc is still unknown for  $q = 11, 17$  [1].

**5. Construction of new linear codes over  $\mathbb{F}_q$  with  $q = 7, 8$**

In this section, we prove Theorems 1.5 and 1.6. We first give a method to construct good codes by some orbits of a given projectivity in  $\text{PG}(k - 1, q)$ . For a non-zero element  $\alpha \in \mathbb{F}_q$ , let  $R = \mathbb{F}_q[x]/(x^N - \alpha)$  be the ring of polynomials over  $\mathbb{F}_q$  modulo  $x^N - \alpha$ . We associate the vector  $(a_0, a_1, \dots, a_{N-1}) \in \mathbb{F}_q^N$  with the polynomial  $a(x) = \sum_{i=0}^{N-1} a_i x^i \in R$ . For  $\mathbf{g} = (g_1(x), \dots, g_m(x)) \in R^m$ ,

$$C_{\mathbf{g}} = \{(r(x)g_1(x), \dots, r(x)g_m(x)) \mid r(x) \in R\}$$

is called the 1-generator quasi-twisted (QT) code with generator  $\mathbf{g}$ .  $C_{\mathbf{g}}$  is usually called quasi-cyclic (QC) when  $\alpha = 1$ . When  $m = 1$ ,  $C_{\mathbf{g}}$  is called  $\alpha$ -cyclic or pseudo-cyclic (PC) or constacyclic. All of these codes are generalizations of cyclic codes ( $\alpha = 1, m = 1$ ). Take a monic polynomial  $g(x) = x^k - \sum_{i=0}^{k-1} a_i x^i$  in  $\mathbb{F}_q[x]$  dividing  $x^N - \alpha$  with non-zero  $\alpha \in \mathbb{F}_q$ , and let  $T$  be the companion matrix of  $g(x)$ . Let  $\tau$  be the projectivity of  $\text{PG}(k - 1, q)$  defined by  $T$ . We denote by  $[g^n]$  or by  $[a_0 a_1 \dots a_{k-1}^n]$  the  $k \times n$  matrix  $[P, TP, T^2P, \dots, T^{n-1}P]$ , where  $P$  is the column vector  $(1, 0, 0, \dots, 0)^T$  ( $h^T$  stands for the transpose of a row vector  $h$ ). Then  $[g^N]$  generates an  $\alpha^{-1}$ -cyclic code. Hence one can construct a cyclic or PC code from an orbit of  $\tau$ . For non-zero vectors  $P_2^T, \dots, P_m^T \in \mathbb{F}_q^k$ ,

we denote the matrix

$$[P, TP, T^2P, \dots, T^{n_1-1}P; P_2, TP_2, \dots, T^{n_2-1}P_2; \dots; P_m, TP_m, \dots, T^{n_m-1}P_m]$$

by  $[g^{n_1} + P_2^{n_2} + \dots + P_m^{n_m} \cdot P_j^{n_j}]$  is denoted by  $P_j$  if  $n_j = 1$ . Then, the matrix  $[g^N + P_2^N + \dots + P_m^N]$  defined from  $m$  orbits of  $\tau$  of length  $N$  generates a QC or QT code, see [27]. It is shown in [27] that many good codes can be constructed from orbits of projectivities. It sometimes happens that QC or QT codes are divisible or can be extended to divisible codes. For example, Let  $g(x) = x^4 - 4x^3 - 2x^2 - x - 1 \in \mathbb{F}_7[x]$  and let  $\tau$  be the projectivity of  $\text{PG}(k-1, q)$  defined by the companion matrix of  $g(x)$ . Let  $c_t$  be the number of orbits of  $\tau$  with length  $t$ . Then, we have  $(c_1, c_2, c_4, c_6, c_{12}) = (2, 4, 12, 1, 28)$ . By the search of various combinations of orbits for 7-divisible codes using a computer, we have found a  $[182, 4, 154]_7$  code as in Lemma 5.1. In this section, the weight distribution with  $(A_0, A_d, \dots) = (1, \alpha, \dots)$  is expressed as  $0^1 d^\alpha \dots$ .

**Lemma 5.1.** *There exists a 7-divisible  $[g_7(4, d) + 1, 4, d]_7$  code for  $d = 154$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$\begin{aligned} & [1124^{12}] + 0011^{12} + 0041^{12} + 0051^{12} + 0401^{12} + 0301^{12} + 0551^{12} \\ & + 0661^{12} + 0411^{12} + 0361^{12} + 0561^{12} + 0351^{12} + 0161^{12} + 4001^{12} \\ & + 0151^4 + 1051^4 + 4021^2 + 5101^2 + 5261 + 5261. \end{aligned}$$

Then  $\mathcal{C}$  is a 7-divisible  $[182, 4, 154]_7$  code with weight distribution

$$0^1 154^{1746} 161^{600} 168^{54}. \quad \square$$

**Lemma 5.2.** *There exists a 7-divisible  $[g_7(4, d) + 1, 4, d]_7$  code for  $d = 105$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$\begin{aligned} & [1304^8] + 0001^8 + 0041^8 + 0441^8 + 0551^8 + 0661^8 + 0411^8 + 0531^8 + 0561^8 \\ & + 0151^8 + 0251^8 + 5051^8 + 2061^8 + 1121^8 + 6621^8 + 0601^4 + 6521. \end{aligned}$$

Then  $\mathcal{C}$  is a  $[125, 4, 105]_7$  code with weight distribution

$$0^1 105^{1698} 112^{654} 119^{48}. \quad \square$$

Note that Griesmer codes over  $\mathbb{F}_7$  are 7-divisible by Theorem 4.1.

**Lemma 5.3.** *There exists a  $[g_7(4, d), 4, d]_7$  code for  $d = 119$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$\begin{aligned} & [3246^6] + 0021^6 + 0061^6 + 0201^6 + 0201^6 + 6061^6 + 6061^6 + 1021^6 \\ & + 1021^6 + 1141 + 1141 + 3521 + 6251 + 6251 + 2561. \end{aligned}$$

Then  $\mathcal{C}$  is a 7-divisible  $[60, 4, 49]_7$  code with weight distribution

$$0^1 49^{1560} 56^{840}.$$

Applying Lemma 2.7, as the projective dual of  $\mathcal{C}$ , one can get a  $[140, 4, 119]_7$  code  $\mathcal{C}^*$  with weight distribution

$$0^1 119^{2160} 126^{120} 133^{120}. \quad \square$$

**Lemma 5.4.** *There exists a 7-divisible  $[g_7(4, d) + 1, 4, d]_7$  code for  $d = 126$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$\begin{aligned} & [1106^6] + 0041^6 + 0041^6 + 0101^6 + 0401^6 + 0401^6 + 0221^6 + 0531^6 \\ & + 0541^6 + 0161^6 + 0431^6 + 0521^6 + 4051^6 + 2461^6 + 2661^6 + 2431^3 \\ & + 2431^3 + 6001 + 1221 + 1221 + 2651 + 2651. \end{aligned}$$

Then  $\mathcal{C}$  is a 7-divisible  $[101, 4, 84]_7$  code with weight distribution

$$0^1 84^{1554} 91^{798} 98^{48}.$$

Applying Lemma 2.7, as the projective dual of  $\mathcal{C}$ , one can get a  $[149, 4, 126]_7$  code  $\mathcal{C}^*$  with weight distribution

$$0^1 126^{1896} 133^{402} 140^{102}. \quad \square$$

**Lemma 5.5.** *There exists a 7-divisible  $[g_7(4, d) + 1, 4, d]_7$  code for  $d = 133$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$\begin{aligned} & [1106^6] + 0041^6 + 0041^6 + 0051^6 + 0401^6 + 0531^6 + 0131^6 \\ & + 0321^6 + 0631^6 + 0161^6 + 1021^6 + 3661^6 + 4541^6 + 4541^6 \\ & + 0601^3 + 2061^3 + 1221 + 1221 + 2651. \end{aligned}$$

Then  $\mathcal{C}$  is a 7-divisible  $[93, 4, 77]_7$  code with weight distribution

$$0^1 77^{1464} 84^{930} 91^6.$$

Applying Lemma 2.7, as the projective dual of  $\mathcal{C}$ , one can get a  $[157, 4, 133]_7$  code  $\mathcal{C}^*$  with weight distribution

$$0^1 133^{1920} 140^{402} 147^{78}. \quad \square$$

**Lemma 5.6.** *There exist 7-divisible  $[g_7(4, d), 4, d]_7$  codes for  $d = 406, 413$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$\begin{aligned} & [3246^6] + 0021^6 + 0061^6 + 0201^6 + 0201^6 + 6061^6 + 6061^6 + 1021^6 \\ & + 1021^6 + 9P + 3521 + 6251 + 6251 + 2561, \end{aligned}$$

where  $P = 1141$ . Then  $\mathcal{C}$  is a 7-divisible  $[67, 4, 49]_7$  code with weight distribution

$$0^1 49^{300} 56^{1302} 63^{798}.$$

Applying Lemma 2.7, as the projective dual of  $\mathcal{C}$ , one can get a  $[483, 4, 413]_7$  code  $\mathcal{C}^*$  with weight distribution

$$0^1 413^{2160} 420^{120} 427^{114} 476^6.$$

Since the multiset  $\mathcal{M}_{\mathcal{C}^*}$  contains a line  $\langle 0001, 0110 \rangle$ , we can get a 7-divisible  $[475, 4, 406]_7$  code by Lemma 2.2.  $\square$

Let  $\mathbb{F}_8 = \{0, 1, \alpha, \alpha^2, \dots, \alpha^6\}$ , with  $\alpha^3 = \alpha + 1$ . For simplicity, we denote  $\alpha, \alpha^2, \dots, \alpha^6$  by  $2, 3, \dots, 7$  so that  $\mathbb{F}_8 = \{0, 1, 2, 3, \dots, 7\}$ .

**Lemma 5.7.** *There exists an 8-divisible  $[g_8(4, d) + 1, 4, d]_8$  code for  $d = 336$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$\begin{aligned} & [1040^{14}] + 1100^{14} + 1300^{14} + 1400^{14} + 1500^{14} + 1600^{14} \\ & + 1700^{14} + 1660^{14} + 1330^{14} + 1664^2 + 6P, \end{aligned}$$

where  $P = 1267$ . Then  $\mathcal{C}$  is a 8-divisible  $[134, 4, 112]_8$  code with weight distribution

$$0^1 112^{1400} 120^{2688} 128^7.$$

Applying Lemma 2.7, as the projective dual of  $\mathcal{C}$ , one can get a  $[386, 4, 336]_8$  code  $\mathcal{C}^*$  with weight distribution

$$0^1 336^{3192} 344^{896} 384^7. \quad \square$$

**Lemma 5.8.** *There exists an 8-divisible  $[g_8(4, d) + 1, 4, d]_8$  code for  $d = 352$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$\begin{aligned} & [1101^6] + 1230^6 + 1430^6 + 1454^6 + 1500^6 + 1504^6 + 1550^6 \\ & + 1553^6 + 1760^6 + 1665^2 + 1001 + 1001. \end{aligned}$$

Then  $\mathcal{C}$  is a 4-divisible  $[58, 4, 48]_8$  code with weight distribution

$$0^1 48^{1589} 52^{2184} 56^{322}.$$

Applying Lemma 2.7, as the projective dual of  $\mathcal{C}$ , one can get a  $[404, 4, 352]_8$  code  $\mathcal{C}^*$  with weight distribution

$$0^1 352^{3696} 368^{392} 384^7. \quad \square$$

**Lemma 5.9.** *There exist 8-divisible  $[g_8(4, d), 4, d]_8$  codes for  $d = 648, 656, 664, 672$ .*

*Proof.* Let  $\mathcal{C}$  be the code with generator matrix

$$[1110^7] + 0011^7 + 0471^7 + 0641^7 + 0761^7 + 2571 + 5341 + 3261.$$

Then  $\mathcal{C}$  is a 4-divisible  $[38, 4, 28]_8$  code with weight distribution

$$0^1 28^{56} 32^{2695} 36^{1344}.$$

Applying Lemma 2.7, as the projective dual of  $\mathcal{C}$ , one can get a  $[769, 4, 672]_8$  code  $\mathcal{C}^*$  with weight distribution

$$0^1 672^{3829} 688^{266}.$$

Since the multiset  $\mathcal{M}_{\mathcal{C}^*}$  contains three skew lines

$$\langle 0100, 1010 \rangle, \langle 1200, 1001 \rangle, \langle 1300, 1101 \rangle,$$

we can get  $[760, 4, 664]_8$ ,  $[751, 4, 656]_8$ ,  $[742, 4, 648]_8$  codes by Lemma 2.2.  $\square$

**Lemma 5.10.** *There exists a  $[g_8(4, d) + 1, 4, d]_8$  code for  $d = 166$ .*

*Proof.* Let  $\mathcal{C}_1$  be the 8-divisible  $[742, 4, 648]_8$  code obtained in Lemma 5.9. Applying Lemma 2.7, as the projective dual of  $\mathcal{C}_1$ , one can get a 8-divisible  $[103, 4, 80]_8$  code  $\mathcal{C}_1^*$  with weight distribution

$$0^1 80^{224} 88^{2548} 96^{1323}.$$

So, from (3), there are 189 7-planes in  $\text{PG}(3, 8)$  for  $\mathcal{C}_1^*$ . Equivalently, there are 189 2-points in  $\text{PG}(3, 8)$  for  $\mathcal{C}_1$ . Let  $\mathcal{M}_{\mathcal{C}_2}$  be the set of 2-points for  $\mathcal{C}_1$ . Then,  $\mathcal{C}_2$  is a  $[189, 4, 163]_8$  code with weight distribution

$$0^1 163^{756} 164^{1071} 165^{1582} 166^{294} 167^{126} 177^{91} 180^{154} 181^{21}.$$

$\mathcal{C}_2$  can be extended to a  $[192, 4, 166]_8$  code with weight distribution

$$0^1 166^{1176} 167^{1176} 168^{1477} 180^{98} 183^{168}$$

by adding the three points 1407, 1502, 1702 to  $\mathcal{M}_{\mathcal{C}_2}$ .  $\square$

Now, Theorems 1.5 and 1.6 follow from Lemmas 5.1-5.10 since an  $[n, k, d]_q$  code yields an  $[n - 1, k, d - 1]_q$  code by puncturing.

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