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SPECTRAL INSTABILITY OF ROLLS IN THE 2-DIMENSIONAL GENERALIZED SWIFT-HOHENBERG EQUATION

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Abstract. The aim of this paper is to investigate the spectral instability of roll waves bifurcating from an equilibrium in the 2-dimensional generalized Swift-Hohenberg equation. We characterize unstable Bloch wave vectors to prove that the rolls are spectrally unstable in the whole parameter region where the rolls exist, while they are Eckhaus stable in 1 dimension [13]. As compared to [18], showing that the stability of the rolls in the 2-dimensional Swift-Hohenberg equation without a quadratic nonlinearity is determined by Eckhaus and zigzag curves, our result says that the quadratic nonlinearity of the equation is the cause of such instability of the rolls.

1. Introduction

This paper deals with the 2-dimensional generalized Swift-Hohenberg equation (gSHE) with quadratic-cubic nonlinearity

$$(1)\quad \partial_t u = -(1+\partial_x^2+\partial_y^2)^2 u + \varepsilon^2 u + bu^2 - su^3, \quad t \geq 0,\ (x,y)\in\mathbb{R}^2,\ u\in\mathbb{R}^1,$$

where $\varepsilon \in \mathbb{R}^1$ is a bifurcation parameter, and b and s are nonzero constants. In our analysis we treat only ε as a control parameter. There are several types of Swift-Hohenberg equations depending upon the nonlinearity, and they serve models for pattern formation such as stripes, squares, and hexagons, etc. For example, see [1, 4, 6, 7, 10, 15, 16, 20–22, 24] and the references therein for numerical and analytical studies of various types of solutions in 2 space dimensions.

In the present paper we are interested in the stability of bifurcating stationary periodic patterns $u_{rolls}(t,x,y) = \tilde{u}(x)$ which are periodic in x and

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independent of y, called "rolls" or "stripes". The existence of bifurcating rolls can be considered from the instability of a constant solution $u \equiv 0$. Indeed, the linearization of (1) about $u \equiv 0$, that is, seeking a solution $v(t,x,y) = e^{\lambda(k,l)t+i(kx+ly)}$ for (1) gives $\lambda(k,l) = -(1-k^2-l^2)^2+\varepsilon^2$ which is zero at $\varepsilon = 0$ corresponding to the weakly unstable wave vectors (k,l) with $k^2+l^2 = 1$. This is often referred to as the weak instability of the homogeneous state $u \equiv 0$ [17]. Under the assumption that the rolls depend only upon x-variable, i.e., $l = 0$, one can readily expect small amplitude roll solutions of the form

$$\tilde{u}(x) \approx \varepsilon e^{i(1+\varepsilon\omega)x} + \mathcal{O}(\varepsilon^2) + c.c. \tag{2}$$

bifurcating from $u \equiv 0$ in a neighborhood $(\varepsilon, k^2) = (0,1)$. Here, *c.c.* denotes the complex conjugate, and ω defined in (5) is a parameter related to k satisfying $k^2 = 1$ at $\omega = 0$. That is, the roll solutions grow smoothly in amplitude as the control parameter ε increases from 0.

Such roll solutions (2) were constructed rigorously in [13] for $b = 1$ and $s = 2$, studying the existence and stability of the bifurcating periodic solution in the 1-dimensional gSHE. Its existence result in [13] is also valid for the 2-dimensional case (1) with $27s - 38b^2 > 0$ (see Theorem 1.1 below), but the stability is quite different because the perturbations in our case depend upon both spatial variables x and y. Our purpose of this paper is to study how perturbations including transverse direction affect the spectral stability or instability of the rolls. More precisely, observing rigorously the Bloch wave vectors $\sigma \in \mathbb{R}^2$ that generate unstable modes of the linearized operator of (1) about the rolls, we prove that for sufficiently small $\varepsilon > 0$ the rolls are spectrally unstable for all wave numbers k where the rolls exist. In addition, we will show how the spectral instability leads to linear instability of the rolls under a localized initial perturbation. However, this is contrary to the stability result for the 1-dimensional case. It has been proved in [13] that the bifurcating periodic solutions $\tilde{u}$ in the 1-dimensional gSHE are Eckhaus stable, i.e., stable for the certain range of parameters (ε, k).

Our instability result is also contrary to the one for the case $b = 0$ in (1). According to [17,18], as considered primary and foremost works for the study of stability of periodic patterns, there is a positive ε_1 such that the roll solutions with $\varepsilon \in (0, \varepsilon_1]$ and $|k^2 - 1| \leq \varepsilon$ are stable if and only if

$$\begin{aligned} &\varepsilon \geq E_E(k) := 3(k^2-1)^2 + \mathcal{O}((k^2-1)^3) \text{ and } {}^{1} \\ &k^2 - 1 \geq K_Z(\varepsilon) := -\varepsilon^4/512 + \mathcal{O}(\varepsilon^6). \end{aligned} \tag{3}$$

By the standard Floquet-Bloch theory, the spectra of the linear operator with spatially periodic coefficients are all essential spectra that are described as the continuous union of discrete eigenvalues of the Bloch operators for all wave vectors $\sigma \in \mathbb{R}^2$ (see the spectral identity (40)). In fact, if $b = 0$, the

[1] The first bound $\varepsilon \geq E_E(k)$ is called the Eckhaus criterion ([8,25]) and the second bound $k^2 - 1 \geq K_Z(\varepsilon)$ is the zigzag instability bound ([2,21]).

criteria for the stability and instability (3) were determined by observing the eigenvalues of the Bloch operator for the most unstable modes $\sigma \approx (2,0)$. Of course, even if $b \neq 0$ in our case, we can also find such criteria for the instability by investigating wave vectors $\sigma \in \mathbb{R}^2$ near $(0,0)$ which corresponds to $(2,0)$ in [18] (see Section 4). However, the nonzero quadratic nonlinearity gives other unstable wave vectors near $\sigma^* := (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ that cause the spectral instability for small $\varepsilon > 0$ and all k in which the rolls exist.

The paper is organized as follows. In the remaining part of the introduction, we state the existence result of the rolls established in [13], and provide the main result showing the spectral instability of the rolls. We also discuss the work of interest in the future. In Section 2 we briefly repeat the existence result in [13] how we can apply the Lyapunov-Schmidt reduction to construct the bifurcating roll solutions for general b and s. In Section 3 we discuss the Bloch operators and identify several regions of the Bloch wave vectors $\sigma \in \mathbb{R}^2$ that may be unstable. We follow the mathematical framework developed in [18] to verify how the quadratic nonlinearity affects the instability. In particular, we again apply the Lyapunov-Schmidt reduction to the eigenvalue problem of the Bloch operators for unstable modes σ near $\sigma^* = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. After the study of spectral instability we will construct an exponentially growing solution of the linearized equation of (1) arising from the unstable modes near σ^*. Lastly, Section 4 provides further discussion of unstable wave vectors of the Bloch operators for other regions. However, this section is not required to demonstrate our main result, so the reader may consider Section 4 as an appendix.

1.1. Main results

We first state the existence result about the roll solutions of (1). Making a coordinate change $x \to \xi := kx$, where k is a wave number, the roll solutions $u_{rolls}(t,x,y) = \tilde{u}(\xi)$ with period 2π satisfy

$$0 = -(1 + k^2\partial_\xi^2)^2 u + \varepsilon^2 u + bu^2 - su^3, \quad t \geq 0, \ u \in \mathbb{R}^1. \tag{4}$$

The periodic solutions $\tilde{u}(\xi)$ were constructed in [13] only for $b = 1$ and $s = 2$. However, its analysis is valid also for all b and s with $27s - 38b^2 > 0$ without any difficulty. We briefly repeat the proof in Section 2.

Theorem 1.1 (Existence, [13])**.** *Assume $27s - 38b^2 > 0$ and let ω be a parameter satisfying*

$$k^2 - 1 = 2\varepsilon\omega. \tag{5}$$

Then there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ there is a unique (up to translation) stationary 2π-periodic solution $\tilde{u}_{\varepsilon,\omega}(\xi) \in H^4_{per}([0,2\pi])$ of (4), which is even in ξ and bifurcating from the uniform state

$u \equiv 0$. *These periodic solutions have the following expansion*

$$
\begin{aligned}
\tilde{u}_{\varepsilon,\omega}(\xi) &= \varepsilon \frac{6\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} \cos\xi \\
&\quad + \varepsilon^2 \Big[\frac{18b(1-4\omega^2)}{27s-38b^2} - \frac{32b^2\omega\sqrt{1-4\omega^2}}{(27s-38b^2)\sqrt{27s-38b^2}} \cos\xi \\
&\qquad + \frac{2b(1-4\omega^2)}{27s-38b^2} \cos 2\xi \Big] + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{6}
$$

In particular, if $\omega = \pm\frac{1}{2}$, *then* $\tilde{u}_{\varepsilon,\omega}(\xi) \equiv 0$.

Remark 1.2. Following [13,23], the parameter ω defined in (5) is used throughout the paper rather than k. This is to organize the bifurcation equations as the asymptotic expansions in terms of ε when applying the Lyapunov-Schmidt reduction. We notice that the parameter region $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ where the periodic solutions exist can be rephrased as $k^2 \in [1-\varepsilon, 1+\varepsilon]$, which is consistent with the one of the existence result in [17, 18]. In addition, the number "2" in (5) is not necessary, but this scaling is very natural in the derivation of the formal Ginzberg-Landau amplitude equation of gSHE (see [13, Section 2.4] for the derivation of gSHE).

In order to state the result of spectral instability of the rolls $\tilde{u}_{\varepsilon,\omega}(\xi)$ in 2-dimensional space, we linearize the equation (4) about $\tilde{u}_{\varepsilon,\omega}(\xi)$ to obtain

$$
\mathcal{L}_{\varepsilon,\omega} := -(1 + (1+2\varepsilon\omega)\partial_\xi^2 + \partial_y^2)^2 + (\varepsilon^2 + 2b\tilde{u}_{\varepsilon,\omega} - 3s\tilde{u}_{\varepsilon,\omega}^2) \tag{7}
$$

acting on $L^2(\mathbb{R}^2)$ with densely defined domain $H^4(\mathbb{R}^2)$. It is well known that the $L^2(\mathbb{R}^2)$-spectra of $\mathcal{L}_{\varepsilon,\omega}$ having spatially periodic coefficients are all essential spectra. Moreover, the spectra can be characterized by the $L^2([0, 2\pi])$-eigenvalues of the Bloch operator family given by

$$
B(\varepsilon,\omega,\sigma) := -(1 + (1+2\varepsilon\omega)(\partial_\xi + i\sigma_1)^2 - \sigma_2^2)^2 + (\varepsilon^2 + 2b\tilde{u}_{\varepsilon,\omega} - 3s\tilde{u}_{\varepsilon,\omega}^2) \tag{8}
$$

for all $\sigma = (\sigma_1, \sigma_2) \in [-\frac{1}{2}, 0] \times [0, \infty)$ which are called the Bloch wave vector. A detailed discussion about the Bloch operators is given in Section 3. Throughout this paper, following [18], $\mathcal{S}_{\varepsilon,\omega}$ denotes the set of all unstable Bloch wave vectors, i.e., the set of all $\sigma \in \mathbb{R}^2$ for which $B(\varepsilon,\omega,\sigma)$ has a positive eigenvalue. We notice that $B(\varepsilon,\omega,\sigma)$ has only real-valued eigenvalues because of its self-adjointness. We are ready to state our main result.

Theorem 1.3 (Spectral instability)**.** *Assume* $27s - 38b^2 > 0$ *and* $b \neq 0$. *Let* ε_0 *be taken from the existence result of* $\tilde{u}_{\varepsilon,\omega}$ *in Theorem 1.1. Then there exists* $\tilde{\varepsilon}_0 \in (0, \varepsilon_0]$ *such that* $\tilde{u}_{\varepsilon,\omega}(\xi)$ *is spectrally unstable for all* $\varepsilon \in (0, \tilde{\varepsilon}_0]$ *and all* $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. *In particular, for any fixed* $r > 1$,

$$
\{\sigma \in \mathbb{R}^2 \mid |\sigma - \sigma^*| \le \mathcal{O}(\varepsilon^r)\} \subset \mathcal{S}_{\varepsilon,\omega} \text{ for all } \omega \in [-\tfrac{1}{2}, \tfrac{1}{2}],
$$

where $\sigma^* := (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \in \mathbb{R}^2$.

The proof of Theorem 1.3 is given in Section 3. The spectral problem $\lambda v = \mathcal{L}_{\varepsilon,\omega} v$ on $L^2(\mathbb{R}^2)$ is formally obtained by plugging $\tilde{v}(t,\xi,y) = e^{\lambda t} v(\xi,y)$ into the linear perturbation equation $\tilde{v}_t = \mathcal{L}_{\varepsilon,\omega}\tilde{v}$. However, by the standard Floquet-Bloch analysis, there is no $v(\xi,y)$ lies in $L^2(\mathbb{R}^2)$, so that the spectrum of $\mathcal{L}_{\varepsilon,\omega}$ must be entirely essential spectrum. Thus, the form of solutions $v(\xi,y)$ that grow exponentially at the linear level corresponding to the positive spectrum λ is not so explicit. For this reason, we provide an exponentially growing solution arising from the unstable mode σ^* in Section 3.3.

Remark 1.4.

1. It has been studied in [13] that the periodic solutions $\tilde{u}_{\varepsilon,\omega}(\xi)$ in one dimension, i.e., $\sigma_2 = 0$, are unstable when $\omega \in [-\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \cup (\frac{1}{2\sqrt{3}}, \frac{1}{2}]$, while they are stable when $\omega \in (-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$. It is said to be Eckhaus stable, which contains the universal factor $\sqrt{3}$ as compared to the existence result $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. Thus, our instability result for $\omega \in (-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$ means the transverse instability in the sense that the transverse direction $\sigma_2 \neq 0$ affects the instability of the rolls.

2. The unstable mode $\sigma^* = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ is determined by two small eigenvalues of $B(0,\omega,\sigma)$, saying

$$\mu_0(\sigma) = -(1 - \sigma_1^2 - \sigma_2^2)^2 \quad \text{and} \quad \mu_1(\sigma) = -(1 - (1+\sigma_1)^2 - \sigma_2^2)^2.$$

That is, both eigenvalues $\mu_0(\sigma)$ and $\mu_1(\sigma)$ are zero at σ^*. The technique of Lyapunov-Schmidt reduction allows us to investigate the eigenvalues near zero of $B(\varepsilon,\omega,\sigma)$ for $\varepsilon > 0$ sufficiently small and $\sigma \approx \sigma^*$. In particular, the quadratic nonlinearity (i.e., $b \neq 0$) causes $B(\varepsilon,\omega,\sigma)$ to have at least one positive eigenvalue even for $\omega \in (-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$. This unstable mode does not appear in [17, 18] dealing with $b = 0$, and the rolls are stable for the parameter region (3).

3. We notice that the coefficients b and s in (1) do not affect the instability result of Theorem 1.3 as long as $27s - 38b^2 > 0$ and $b \neq 0$. This is due to the fact that the result is solely based on the unstable modes near σ^*. The signs of eigenvalues of the Bloch operators $B(\varepsilon,\omega,\sigma)$ for some other modes $\sigma \in \mathbb{R}^2$ may vary depending upon b and s. For example, $\sigma = (0,1) \in \mathcal{S}_{\varepsilon,\omega}$ for all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ only if $\frac{38}{27}b^2 < s \leq \frac{70}{27}b^2$. A detailed description of $\mathcal{S}_{\varepsilon,\omega}$ will be given in Section 4 as an appendix.

1.2. Discussion

In Section 3.3 we show that the spectral instability of Theorem 1.3 leads to linear instability by constructing an exponentially growing solution of the linearized equation of (1). It would be interesting to see that the solution at the linear level truly causes nonlinear instability. In particular, Theorem 1.3 exhibits the rolls are spectrally unstable in the whole parameter range of ω extending its unstable range from $\omega \in [-\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \cup (\frac{1}{2\sqrt{3}}, \frac{1}{2}]$ to $[-\frac{1}{2}, \frac{1}{2}]$ as compared to 1-dimensional gSHE ([13, Theorem 1.2]). We address the transverse nonlinear instability of the roll solutions in the forthcoming work [3] with references [9, 11, 19]. Nonlinear modulational instability under localized perturbations around periodic traveling solutions was studied by Jin et al. [11] for several dispersive equations in one dimension. For the 2-dimensional case Rousset and Tzvekov established nonlinear instability of the line solitary waverwaves with respect to transverse perturbations [19]. The framework of [11] and [19] was partly initiated by the work of Grenier [9].

In previous work, Jung has studied the diffusive stability of bifurcating stationary periodic solutions in the 1-dimensional Brusselator model [23], a typical system for Turing instability. In fact, the present paper follows several computational techniques laid out in [23]. The study of stability or instability of the rolls in the 2-dimensional Brusselator model would also be interesting direction for future study.

2. Rolls bifurcating from a uniform state

In this section we briefly review the construction of roll solutions $\tilde{u}(\xi)$ established in [13]. As mentioned in the introduction, the problem for $\tilde{u}(\xi)$ reads

$$0 = N(\varepsilon, k, \tilde{u}) := -(1 + k^2\partial_\xi^2)^2\tilde{u} + \varepsilon^2\tilde{u} + b\tilde{u}^2 - s\tilde{u}^3, \tag{9}$$

where $N : \mathbb{R}^2 \times H^4_{per}([0, 2\pi]) \to L^2_{per}([0, 2\pi])$ is an analytic mapping. Throughout the paper, $L^2_{per}([0, 2\pi])$ (resp. $H^4_{per}([0, 2\pi])$) denotes the class of L^2 (resp. H^4) periodic functions on $[0, 2\pi]$. We notice that $N(0, \pm 1, 0) = 0$ because the instability of $u \equiv 0$ occurs at $\varepsilon = 0$ with the wave number $k^2 = 1$. Thus, it is natural to look for 2π-periodic solutions $\tilde{u}(\xi)$ satisfying (9) in a neighborhood of $(\varepsilon, k, \tilde{u}) = (0, \pm 1, 0)$. Such bifurcating periodic solutions can be constructed by the Lyapunov-Schmidt reduction.

2.1. Lyapunov-Schmidt reduction for (9) and the bifurcation equation

The advantage of the Lyapunov-Schmidt reduction is that an infinite dimensional problem (9) solving for $\tilde{u} \in H^4_{per}([0, 2\pi])$ can be reduced to an appropriate finite dimensional problem which is equivalent to (9). In order to solve the equation (9) in a neighborhood of $(0, \pm 1, 0)$, we linearize $N(\varepsilon, k, \tilde{u})$ about $(\varepsilon, k, \tilde{u}) = (0, \pm 1, 0)$ to obtain the linear operator

$$L := \partial_{\tilde{u}} N(0, \pm 1, 0) = -(1 + \partial_\xi^2)^2, \tag{10}$$

where $L : H^4_{per}([0, 2\pi]) \subset L^2_{per}([0, 2\pi]) \to L^2_{per}([0, 2\pi])$ is a Fredholm operator. Then its kernel, denoted by $ker(L)$, is spanned by

$$U_1(\xi) = \cos\xi \quad \text{and} \quad U_2(\xi) = \sin\xi, \tag{11}$$

and thus the linear operator L is not invertible. In order to apply the Lyapunov-Schmidt reduction we consider the orthogonal projection of L onto the kernel of L

$$P : L^2_{per}([0, 2\pi]) \to ker(L)$$

defined by

$$Pu := \frac{1}{\pi}\int_0^{2\pi} uU_1 \, d\xi U_1 + \frac{1}{\pi}\int_0^{2\pi} uU_2 \, d\xi U_2, \tag{12}$$

which can also be defined as a vector form

$$\tilde{P}u := \frac{1}{\pi}\Big(\int_0^{2\pi} uU_1 \, d\xi, \int_0^{2\pi} uU_2 \, d\xi\Big)^T \in \mathbb{R}^2. \tag{13}$$

Since L is self-adjoint and a Fredholm operator of index zero, $PL^2_{per}([0, 2\pi]) = ker(L)$ and $(I - P)L^2_{per}([0, 2\pi]) = ran(L)$. Here $ran(L)$ denotes the range of L.

We now solve the equation (9) by decomposing $\tilde{u} \in H^4_{per}([0, 2\pi])$ into $U + V$, where $U = P\tilde{u} = \alpha_1 U_1 + \alpha_2 U_2 \in ker(L)$ for $\alpha_1,\ \alpha_2 \in \mathbb{R}$ and $V = (I - P)\tilde{u}$. Then one can rewrite the equation (9) into two equations:

$$\begin{aligned} 0 &= \tilde{P}N(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2 + V), \\ 0 &= (I - P)N(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2 + V). \end{aligned} \tag{14}$$

We first focus on the second equation of (14). While the linear operator L on $H^4_{per}([0, 2\pi])$ is not invertible, the operator $(I - P)L = (I - P)\partial_{\tilde{u}}N(0, \pm 1, 0)$ is invertible between $(I - P)H^4_{per}([0, 2\pi])$ and $ran(L) = (I - P)L^2_{per}([0, 2\pi])$. Thus, by the Implicit Function Theorem the second equation of (14) can be uniquely solved for $V = V(\varepsilon, k, U)$ in a neighborhood of $(\varepsilon, k, U) = (0, \pm 1, 0)$.

We then substitute $V = V(\varepsilon, k, U)$ into the first equation of (14) to obtain the vector equation

$$0 = f(\varepsilon, k, \alpha_1, \alpha_2) := \tilde{P}N(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2 + V(\varepsilon, k, U)) \in \mathbb{R}^2. \tag{15}$$

This is frequently referred to as the bifurcation equation (or the reduced equation) for (9), which is the 2-dimensional problem equivalent to (9). That is, by solving the bifurcation equation (15) for small α_1 and α_2 in a neighborhood of $(\varepsilon, k) = (0, \pm 1)$ one can characterize all small solutions

$$\tilde{u}(\xi) = \alpha_1 U_1 + \alpha_2 U_2 + V(\varepsilon, k, \alpha_1 U_1 + \alpha_2 U_2) \tag{16}$$

bifurcating from $u \equiv 0$ in a neighborhood of $(\varepsilon, k, \alpha_1, \alpha_2) = (0, \pm 1, 0, 0)$. However, as discussed in [13, 17, 23], the bifurcation equation (15) has a special form

$$0 = \tilde{f}(\varepsilon, k, \alpha_1, \alpha_2)\alpha \tag{17}$$

for a scalar function $\tilde{f}(\varepsilon, k, \alpha_1, \alpha_2)$, i.e., the product of a scalar function and a vector $\alpha = (\alpha_1, \alpha_2)^T$. This is due to the fact that the original differential equation (9) is translation invariant ($\xi \mapsto \xi + \eta$) and reflection symmetric ($\xi \mapsto -\xi$). More precisely, by the identities

$$(U_1, U_2)(\xi + \eta) = (\cos\eta U_1(\xi) - \sin\eta U_2(\xi), \sin\eta U_1(\xi) + \cos\eta U_2(\xi))$$

and

$$(U_1, U_2)(-\xi) = (U_1, -U_2)(\xi),$$

the bifurcation equation is also invariant under the symmetries

$$\alpha \mapsto R(\eta)\alpha := \begin{pmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{pmatrix} \alpha \quad \text{and} \quad \alpha \mapsto S\alpha := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \alpha,$$

which lead to

$$f(\varepsilon, k, R(\eta)\alpha) = R(\eta) f(\varepsilon, k, \alpha) \quad \text{and} \quad f(\varepsilon, k, S\alpha) = S f(\varepsilon, k, \alpha).$$

These symmetries yield the form (17). In our case, for computational convenience, we set $\alpha_2 = 0$, and thus solve $\tilde{f}(\varepsilon, k, \alpha_1) = 0$ for α_1.

2.2. Solving the second equation of (14)

As a starting point, we solve the second equation of (14) with $\alpha_2 = 0$ for V in a neighborhood of $(\varepsilon, k, \alpha_1) = (0, \pm 1, 0)$. Inserting $\alpha_2 = 0$ into the second equation of (14) gives

$$\begin{aligned} 0 &= (I-P)N(\varepsilon, k, \alpha_1 U_1 + V) \\ &= (I-P)(\varepsilon^2 - (1 + k^2\partial_\xi^2)^2)(\alpha_1 U_1 + V) + b(I-P)(\alpha_1 U_1 + V)^2 \\ &\quad - s(I-P)(\alpha_1 U_1 + V)^3 \\ &= (\varepsilon^2 - (1 + k^2\partial_\xi^2)^2)V + b(I-P)(\alpha_1 U_1 + V)^2 \\ &\quad - s(I-P)(\alpha_1 U_1 + V)^3. \end{aligned} \tag{18}$$

Here, we have used the fact that both $PL^2_{per}([0, 2\pi])$ and $(I-P)L^2_{per}([0, 2\pi])$ are invariant under the linear operator $\varepsilon^2 - (1 + k^2\partial_\xi^2)^2$. In what follows we will find the asymptotic expansion of V with respect to α_1. First, we see immediately from (18) that

$$V = V(\varepsilon, k, \alpha_1) = \mathcal{O}(\alpha_1^2)$$

for $\alpha_1 \to 0$. Indeed, recalling that $\tilde{u}(\xi)$ bifurcates from $u \equiv 0$, we can verify by direct calculation that

$$V(\varepsilon, k, 0) = 0 \quad \text{and} \quad \partial_{\alpha_1} V(\varepsilon, k, 0) = 0. \tag{19}$$

In order to obtain the leading order term, we take the second derivative of (18) with respect to α_1 and put $\alpha_1 = 0$. Using (19) we obtain

$$\begin{aligned} 0 &= (I-P)(\varepsilon^2 - (1 + k^2\partial_\xi^2)^2)\partial_{\alpha_1}^2 V|_{\alpha_1=0} + 2b(I-P)U_1^2 \\ &= (\varepsilon^2 - (1 + k^2\partial_\xi^2)^2)\partial_{\alpha_1}^2 V|_{\alpha_1=0} + b(1 + \cos 2\xi). \end{aligned} \tag{20}$$

Since $(I-P)L^2_{per}([0, 2\pi])$ is invariant under the linear operator $\varepsilon^2 - (1 + k^2\partial_\xi^2)^2$, the leading order term takes the form

$$\partial_{\alpha_1}^2 V|_{\alpha_1=0} = c_1 + c_2 \cos 2\xi \tag{21}$$

for some $c_1(\varepsilon,k)$ and $c_2(\varepsilon,k)\in\mathbb{R}$. The constants c_1 and c_2 are determined by inserting (21) into (20)

$$\begin{aligned}-b-b\cos 2\xi &= (\varepsilon^2-(1+k^2\partial_\xi^2)^2)(c_1+c_2\cos 2\xi)\\ &= (\varepsilon^2-1)c_1+(\varepsilon^2-(1-4k^2)^2)c_2\cos 2\xi;\end{aligned} \tag{22}$$

thus we arrive at

$$c_1=\frac{b}{1-\varepsilon^2}\quad\text{and}\quad c_2=\frac{b}{(1-4k^2)^2-\varepsilon^2}. \tag{23}$$

Next, we take the third derivative of (18) and plug $\alpha_1=0$ to obtain $\partial_{\alpha_1}^3 V|_{\alpha_1=0}$. Due to (19) and (21),

$$\begin{aligned}0 &= (I-P)(\varepsilon^2-(1+k^2\partial_\xi^2)^2)\partial_{\alpha_1}^3 V|_{\alpha_1=0}+6b(I-P)[U_1(c_1+c_2\cos 2\xi)]\\ &\quad -6s(I-P)[U_1^3]\\ &= (I-P)(\varepsilon^2-(1+k^2\partial_\xi^2)^2)\partial_{\alpha_1}^3 V|_{\alpha_1=0}+(3bc_2-\frac{3}{2}s)\cos 3\xi.\end{aligned} \tag{24}$$

It leads to the form

$$\partial_{\alpha_1}^3 V|_{\alpha_1=0}=c_3\cos 3\xi \tag{25}$$

for some $c_3(\varepsilon,k)\in\mathbb{R}$. We can also determine c_3 by plugging (25) into (24), but it is not necessary to obtain the bifurcation equation.

Consequently, the equation (18) can be solved locally for V:

$$V(\varepsilon,k,\alpha_1)=\frac{1}{2}\alpha_1^2(c_1+c_2\cos 2\xi)+\frac{1}{6}\alpha_1^3c_3\cos 3\xi+\mathcal{O}(\alpha_1^4) \tag{26}$$

in a neighborhood of $(\varepsilon,k,\alpha_1)=(0,\pm1,0)$.

2.3. The bifurcation equation for (9)

We now find the bifurcation equation (17) by solving the first equation of (14). Under the projection P it follows that

$$\begin{aligned}&PN(\varepsilon,k,\alpha_1U_1+V)\\ &=(\varepsilon^2-(1-k^2)^2)\alpha_1U_1+bP(\alpha_1U_1+V)^2-sP(\alpha_1U_1+V)^3.\end{aligned} \tag{27}$$

By inserting (26) into (27), the last two terms on the right-hand side of (27) are handled as

$$bP(\alpha_1U_1+V)^2=b\alpha_1^2P[U_1^2]+2b\alpha_1P[U_1V]+bP[V^2]=b(c_1+\frac{1}{2}c_2)\alpha_1^3U_1+\mathcal{O}(\alpha_1^5)$$

and

$$\begin{aligned}-sP[(\alpha_1U_1+V)^3] &= -s\alpha_1^3P[U_1^3]-\frac{3}{2}s\alpha_1^2P[\alpha_1^2(c_1+c_2\cos 2\xi)\cos^2\xi]+\mathcal{O}(\alpha_1^5)\\ &= -\frac{3}{4}s\alpha_1^3U_1+\mathcal{O}(\alpha_1^5).\end{aligned}$$

We conclude that the bifurcation equation with $\alpha_2=0$ is

$$0=\tilde{f}(\varepsilon,k,\alpha_1)\alpha_1, \tag{28}$$

where

$$\tilde{f}(\varepsilon, k, \alpha_1) = \varepsilon^2 - (1-k^2)^2 + \Big(b(c_1 + \frac{1}{2}c_2) - \frac{3}{4}s\Big)\alpha_1^2 + O(\alpha_1^4). \tag{29}$$

Indeed, we readily see from direct computation that the bifurcation equation (17) in α-vector form can be written as replacing α_1 in (29) by $|\alpha|$.

Proof of Theorem 1.1. We now solve (29) for $\alpha_1 \to 0$ with respect to ε and k in a neighborhood of $(\varepsilon, k) = (0, \pm 1)$. First, we set

$$\mathcal{A} = \frac{\varepsilon^2 - (1-k^2)^2}{\frac{3}{4}s - b(c_1 + \frac{1}{2}c_2)} = \frac{\varepsilon^2 - (1-k^2)^2}{\frac{3}{4}s - b(\frac{b}{1-\varepsilon^2} + \frac{b}{2(1-4k^2)^2 - 2\varepsilon^2})}, \tag{30}$$

and then the equation $\tilde{f}(\varepsilon, k, \alpha_1) = 0$ becomes

$$\mathcal{A} - \alpha_1^2 + O(\alpha_1^4) = 0. \tag{31}$$

It follows immediately that (31) is solvable for small α_1 if and only if $\mathcal{A} \geq 0$. Indeed, by plugging $\alpha_1 = \sqrt{\mathcal{A}}\mathcal{B}$ into (31), we see that a positive solution of (31) has the form

$$\alpha_1 = \sqrt{\mathcal{A}} + \mathcal{O}(|\mathcal{A}|^{3/2}).$$

Let us introduce a new parameter ω defined by

$$k^2 - 1 = 2\omega\varepsilon. \tag{32}$$

This scaling is very natural from the Ginzburg-Landau derivation of (1) (see [13, Section 2.4]). Using this scaling (32) we arrive at

$$\mathcal{A} = \frac{36}{27s - 38b^2}\varepsilon^2(1 - 4\omega^2)\Big[1 - \frac{32b^2}{3(27s - 38b^2)}\omega\varepsilon + \mathcal{O}(\varepsilon^2)\Big] \tag{33}$$

for small $\varepsilon > 0$. This leads to the assumption of b and s for the existence of the roll solutions

$$27s - 38b^2 > 0, \quad \text{i.e.,} \quad s > \frac{38}{27}b^2, \tag{34}$$

so that $\mathcal{A} \geq 0$ if and only if $4\omega^2 \leq 1$, i.e., $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. Here we notice that $\mathcal{A} = 0$ when $\omega^2 = \frac{1}{4}$. In conclusion, there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ a unique positive small solution of (31) exists as

$$\alpha_1 = \frac{6}{\sqrt{27s - 38b^2}}\varepsilon\sqrt{1 - 4\omega^2} - \frac{32b^2}{(27s - 38b^2)\sqrt{27s - 38b^2}}\omega\varepsilon^2\sqrt{1 - 4\omega^2} + \mathcal{O}(\varepsilon^3).$$

Plugging this expansion into (26) yields

$$V(\varepsilon, \omega, \alpha_1) = \frac{18b}{27s - 38b^2}(1 + \frac{1}{9}\cos 2\xi)\varepsilon^2(1 - 4\omega^2) + \mathcal{O}(\varepsilon^3).$$

Consequently, it follows from (16) that all small solutions $\tilde{u}_{\varepsilon,\omega}(\xi)$ (up to translation) of (9) bifurcating from $u \equiv 0$ have the expansion (6) for all $\varepsilon \in (0, \varepsilon_0]$ and all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. In particular, $\tilde{u}_{\varepsilon,\omega}(\xi) \equiv 0$ when $\omega = \pm\frac{1}{2}$ because $\mathcal{A} = 0$. □

3. Instability of the roll solutions

In this section we study the spectral instability of the rolls $\tilde{u}_{\varepsilon,\omega}(\xi)$. Recalling $\xi = kx$ and (32), the generalized Swift-Hohenberg equation (1) can be written as

$$\partial_t u = -(1 + (1 + 2\varepsilon\omega)\partial_\xi^2 + \partial_y^2)^2 u + \varepsilon^2 u + bu^2 - su^3. \tag{35}$$

The main purpose of this section is to prove that $\tilde{u}_{\varepsilon,\omega}(\xi)$ is spectrally unstable under the perturbations which depend upon both spatial variables ξ and y (i.e., the transverse direction). To begin, we linearize (35) about $\tilde{u}_{\varepsilon,\omega}(\xi)$ described in (6)

$$\partial_t \tilde{v} = \mathcal{L}_{\varepsilon,\omega}\tilde{v} := -(1+(1+2\varepsilon\omega)\partial_\xi^2+\partial_y^2)^2\tilde{v}+\mathcal{F}(\tilde{u}_{\varepsilon,\omega})\tilde{v}, \quad \tilde{v} = \tilde{v}(t,\xi,y) \in \mathbb{C} \tag{36}$$

acting on $L^2(\mathbb{R}^2;\mathbb{C})$ with densely defined domain $H^4(\mathbb{R}^2;\mathbb{C})$, where

$$\begin{aligned}
&\mathcal{F}(\tilde{u}_{\varepsilon,\omega}) \\
&= \varepsilon\frac{12b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}\cos\xi \\
&\quad + \varepsilon^2\Big[1 - \frac{54s-36b^2}{27s-38b^2}(1-4\omega^2) - \frac{64b^3\omega\sqrt{1-4\omega^2}}{(27s-38b^2)\sqrt{27s-38b^2}}\cos\xi \\
&\qquad - \frac{54s-4b^2}{27s-38b^2}(1-4\omega^2)\cos 2\xi\Big] \\
&\quad + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{37}$$

Here, we emphasize that the linear operator $\mathcal{L}_{\varepsilon,\omega}$ is defined on the whole space $\mathbb{R}^2$, not on the periodic domain. In order to study of the spectrum of $\mathcal{L}_{\varepsilon,\omega}$, our starting point is to discuss the family of Bloch operators which is necessary for studying spectral problems of linear differential operators with spatially periodic coefficients. We can definitely extend the Bloch operator family defined in one dimension [13, 23] to the multi-dimensional case discussed in [12, 18].

3.1. Bloch operators

Since the roll solutions $\tilde{u}_{\varepsilon,\omega}(\xi)$ are 2π-periodic in ξ and independent of y, every coefficient of the linear operator $\mathcal{L}_{\varepsilon,\omega}$ is also 2π-periodic in ξ, while it has constant coefficients in terms of y. By the standard Floquet-Bloch theory ([12, 14, 18]), any bounded eigenfunction $v(\xi, y)$ of $\mathcal{L}_{\varepsilon,\omega}$ takes the form

$$v(\xi, y) = e^{i(\sigma_1\xi+\sigma_2 y)}W(\xi,\sigma,\lambda), \quad W(\xi+2\pi,\sigma,\lambda) = W(\xi,\sigma,\lambda), \tag{38}$$

where $\sigma := (\sigma_1, \sigma_2) \in \mathbb{R}^2$ are Floquet exponents, which we will refer to as Bloch wave vectors throughout this paper. Here we remark that W depends only upon the ξ-variable and is 2π-periodic. Thus, we readily from (38) see that the linear operator $\mathcal{L}_{\varepsilon,\omega}$ has no point spectrum, that is, the spectrum must be entirely essential spectrum. Moreover, it is a well-known fact that these continuous spectra can be described as the closure of continuous union of all eigenvalues of the Bloch operators defined in (39). This fact motivates the use of the Bloch operators to the study of stability of spatially periodic or roll solutions. The interested reader can consult [14, Section 3.3] and [18, Section 2] for the general theory of Bloch analysis.

By substituting (38) into the right-hand side of (36) we define the σ-dependent operators $B(\varepsilon,\omega,\sigma)$, called the Bloch operator family: for $\sigma=(\sigma_1,\sigma_2)\in\mathbb{R}^2$,

$$B(\varepsilon,\omega,\sigma)W := -(1+(1+2\varepsilon\omega)(\partial_\xi+i\sigma_1)^2-\sigma_2^2)^2W+\mathcal{F}(\tilde{u}_{\varepsilon,\omega})W \tag{39}$$

acting on $L^2_{per}([0,2\pi])$ with densely defined domain $H^4_{per}([0,2\pi])$. We notice that for each $\sigma\in\mathbb{R}^2$, the $L^2_{per}([0,2\pi])$-spectrum of $B(\varepsilon,\omega,\sigma)$ on the compact domain $[0,2\pi]$ is entirely discrete, i.e., point spectrum. As mentioned in the above paragraph, the $L^2(\mathbb{R}^2)$-spectra of $\mathcal{L}_{\varepsilon,\omega}$ can be characterized by

$$Spec_{L^2(\mathbb{R}^2)}(\mathcal{L}_{\varepsilon,\omega}) = closure\Big(\bigcup_{\sigma\in\mathbb{R}^2} Spec_{L^2_{per}([0,2\pi])}(B(\varepsilon,\omega,\sigma))\Big) \tag{40}$$

(see [18, Section 2] for the proof of this spectral identity). However, we do not need to consider the Bloch operators for all $\sigma\in\mathbb{R}^2$ in (40). First, $B(\varepsilon,\omega,\sigma)$ is even in $\sigma_2\in\mathbb{R}$ and any $\sigma_1\in\mathbb{R}$ can be written as $\sigma_1=\sigma_*+m$ for some $\sigma_*\in[-\frac{1}{2},\frac{1}{2})$ and $m\in\mathbb{Z}$; so that $W(\xi,\lambda)$ in (38) can be replaced by $e^{im\xi}W(\xi,\lambda)$ which is also 2π-periodic. In addition, the Bloch operators $B(\varepsilon,\omega,\sigma)$ have the following two reflection symmetries

$$(R_1W)(\xi)=W(-\xi) \quad\text{and}\quad (R_2W)(\xi)=\overline{W}(\xi) \tag{41}$$

satisfying $B(\varepsilon,\omega,\sigma_1,\sigma_2)=R_jB(\varepsilon,\omega,-\sigma_1,\sigma_2)R_j$ for $j=1,2$. Here, the overbar denotes complex conjugation. Consequently, in the spectral identity (40), it is enough to investigate the eigenvalue problems of $B(\varepsilon,\omega,\sigma)$ on $L^2_{per}([0,2\pi])$

$$0=[B(\varepsilon,\omega,\sigma)-\lambda]W \tag{42}$$

only for all $\sigma\in[-\frac{1}{2},0]\times[0,\infty)$ rather than all $\sigma\in\mathbb{R}^2$. For our purpose in this section, noting that $B(\varepsilon,\omega,\sigma)$ is a self-adjoint operator, we will verify that there is a positive eigenvalue λ of $B(\varepsilon,\omega,\sigma)$ for some $\sigma\in[-\frac{1}{2},0]\times[0,\infty)$ which are referred to as unstable Bloch wave vectors. We denote the collection of unstable Bloch wave vectors by $\mathcal{S}_{\varepsilon,\omega}$.

To characterize such unstable Bloch wave vectors $\sigma\in\mathcal{S}_{\varepsilon,\omega}\subset[-\frac{1}{2},0]\times[0,\infty)$, let us first solve the eigenvalue problem (42) at the critical point $\varepsilon=0$. Since the bifurcation parameter ε is sufficiently small bifurcating from 0, the operators $B(\varepsilon,\omega,\sigma)$ can be considered as small perturbations of

$$B(0,\omega,\sigma)=-(1+(\partial_\xi+i\sigma_1)^2-\sigma_2^2)^2. \tag{43}$$

Since it has constant coefficients, its eigenvalue problem can be solved immediately as

$$B(0,\omega,\sigma)\phi_m=\mu_m(\sigma)\phi_m, \tag{44}$$

in which $\phi_m(\xi) := e^{im\xi}$ with $m \in \mathbb{Z}$ and the eigenvalues $\mu_m(\sigma)$ are given by

$$\mu_m(\sigma) = -(1 - (m + \sigma_1)^2 - \sigma_2^2)^2 \leq 0.$$

It follows that $\mu_m(\sigma) = 0$ if and only if $(m+\sigma_1)^2+\sigma_2^2 = 1$. Therefore, unstable wave vectors σ, if they exist, will appear when $(m+\sigma_1)^2+\sigma_2^2 \approx 1$. Otherwise, that is, if $(m + \sigma_1)^2 + \sigma_2^2$ is away from 1, the eigenvalues $\mu_m(\sigma)$ will have negative upper bound, and thus the eigenvalues of $B(\varepsilon, \omega, \sigma)$, considered as small perturbations of $B(0, \omega, \sigma)$, will also have negative upper bound as long as ε is sufficiently small.

Recalling $m \in \mathbb{Z}$ and $\sigma \in [-\frac{1}{2}, 0] \times [0, \infty)$ and solving $(m+\sigma_1)^2+\sigma_2^2 = 1$, we identify the set of all wave vectors σ that we need to consider in the eigenvalue problems (42). Let us set

$$\begin{aligned}\mathcal{S}_0 := \{\sigma \in [-\frac{1}{2}, 0] \times [0, \infty) \mid C_0 : \sigma_1^2 + \sigma_2^2 = 1,\ C_1 : (\sigma_1 + 1)^2 + \sigma_2^2 = 1,\\ C_{-1} : (\sigma_1 - 1)^2 + \sigma_2^2 = 1\},\end{aligned}$$

and distinguish the following four regions around the set $\mathcal{S}_0$

$$\begin{aligned}\mathcal{R}_1 :&= \{\sigma \in [-\frac{1}{2}, 0] \times [0, \infty) \mid dist(\sigma, \sigma^*) \leq \delta\},\\ \mathcal{R}_2 :&= \{\sigma \in [-\frac{1}{2}, 0] \times [0, \infty) \mid dist(\sigma, (0, 0)) \leq \delta\},\\ \mathcal{R}_3 :&= \{\sigma \in [-\frac{1}{2}, 0] \times [0, \infty) \mid dist(\sigma, \mathcal{C}_0) \leq \delta,\ dist(\sigma, \sigma^*) \geq \delta\},\\ \mathcal{R}_4 :&= \{\sigma \in [-\frac{1}{2}, 0] \times [0, \infty) \mid dist(\sigma, \mathcal{C}_1) \leq \delta,\ dist(\sigma, \sigma^*) \geq \delta,\\ &\qquad dist(\sigma, (0, 0)) \geq \delta\}.\end{aligned}$$

Here $\sigma^* = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and we assume $\delta > 0$ is sufficiently small. Later, we will determine an appropriate size of δ with respect to ε to characterize the unstable wave vectors. The above distinction follows from the operator $B(0, \omega, \sigma)$. Indeed, the two circles C_0 and C_1 (resp. C_1 and C_{-1}) intersect at the point $\sigma^* = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ (resp. $(0, 0)$) (see Figure 1). For each region of $\mathcal{R}_1$ and $\mathcal{R}_2$, the operator $B(0, \omega, \sigma)$ has two small eigenvalues, while there is only one small eigenvalue for each region of $\mathcal{R}_3$ and $\mathcal{R}_4$. We will investigate the eigenvalue problem (42) for $\mathcal{R}_1$ in this section and for other regions in the next section.

If $b = 0$, as discussed in [18], $\mathcal{F}_{\varepsilon,\omega}$ in (37) can also be π-periodic even if $\tilde{u}_{\varepsilon,\omega}$ is 2π-periodic. It follows that the frequency m of the eigenfunctions $\phi_m(\xi) = e^{im\xi}$ is even, i.e., $m \in 2\mathbb{Z}$ in (44), in which case there are only two regions of $\sigma \in [-1, 0] \times [0, \infty)$ around the set $\mathcal{S}_0$ because $S_0 \subset [-1, 0] \times [0, \infty)$ consists of two circles $\sigma_1^2 + \sigma_2^2 = 1$ and $(\sigma_1 + 2)^2 + \sigma_2^2 = 1$.

Our goal for this section is to provide the proof of our main result, Theorem 1.3. We emphasize that solving an eigenvalue problem of the Bloch operator $B(\varepsilon, \omega, \sigma)$ only for $\sigma \in \mathcal{R}_1$ is enough to prove Theorem 1.3 because $\sigma^* = (-\frac{1}{2}, \frac{3}{2}) \in \mathcal{S}_{\varepsilon,\omega}$ for all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ and any $b \neq 0$ and s with $27s - 38b^2 > 0$. In

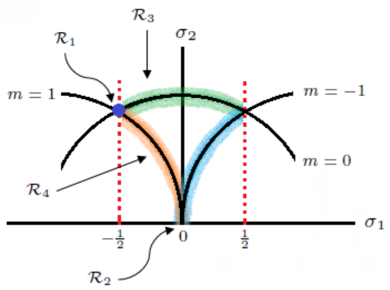

FIGURE 1. Four regions of the Bloch wave vectors. Due to the reflection symmetry R_1 in (41), it is enough to consider the eigenvalue problems (42) only for $\sigma_1 \in [-\frac{1}{2}, 0]$. The blue dot indicates σ^*.

fact, we will prove that not only σ^* but also all $\sigma \in \mathcal{R}_1$ with $|\sigma-\sigma^*| \leq \delta = \mathcal{O}(\varepsilon^r)$ for $r > 1$ are included in $\mathcal{S}_{\varepsilon,\omega}$.

3.2. An eigenvalue problem of $B(\varepsilon,\omega,\sigma)$ for $\sigma \in \mathcal{R}_1$

We consider the eigenvalue problem (42) for $\sigma \in \mathcal{R}_1$, in which region there are two critical eigenfunctions

$$\phi_0(\xi) \equiv 1 \quad \text{and} \quad \phi_1(\xi) = e^{i\xi}$$

of $B(0,\omega,\sigma)$ associated to small eigenvalues

$$\mu_0(\sigma) = -(1-\sigma_1^2-\sigma_2^2)^2 \quad \text{and} \quad \mu_1(\sigma) = -(1-(1+\sigma_1)^2-\sigma_2^2)^2.$$

Since for small $\varepsilon > 0$ the eigenfunctions of $B(\varepsilon,\omega,\sigma)$ for $\sigma \in \mathcal{R}_1$ can be considered as small perturbations of $\phi_0(\xi)$ and $\phi_1(\xi)$, it is very natural to reduce the eigenvalue problem (42) for $\sigma \in \mathcal{R}_1$ to the 2×2 eigenvalue problem by the Lyapunov-Schmidt reduction.

Define the orthogonal projection in $L^2_{per}([0,2\pi])$ onto span$\{\phi_0,\phi_1\}$:

$$PW = \frac{1}{2\pi}\int_0^{2\pi} W\overline{\phi}_0\, d\xi\phi_0 + \frac{1}{2\pi}\int_0^{2\pi} W\overline{\phi}_1\, d\xi\phi_1, \tag{45}$$

or equivalently as a vector form

$$\tilde{P}W = \frac{1}{2\pi}\Big(\int_0^{2\pi} W\overline{\phi}_0\, d\xi, \int_0^{2\pi} W\overline{\phi}_1\, d\xi\Big)^T \in \mathbb{C}^2. \tag{46}$$

We now decompose $W \in L^2_{per}([0,2\pi])$ into $W = \beta_0\phi_0 + \beta_1\phi_1 + \mathcal{V}$ with $\beta = (\beta_0,\beta_1) \in \mathbb{C}^2$ and $P\mathcal{V} = 0$, and then rewrite (42) as

$$\begin{aligned} 0 &= \tilde{P}[B(\varepsilon,\omega,\sigma)-\lambda](\beta_0\phi_0+\beta_1\phi_1+\mathcal{V}),\\ 0 &= (I-P)[B(\varepsilon,\omega,\sigma)-\lambda](\beta_0\phi_0+\beta_1\phi_1+\mathcal{V}). \end{aligned} \tag{47}$$

As discussed in Section 2, the second equation of (47) can be solved locally for $\mathcal{V} = \mathcal{V}(\varepsilon, \omega, \sigma, \lambda, \beta)$ in the vicinity of $(\varepsilon, \sigma, \lambda, \beta) = (0, \sigma^*, 0, (0, 0))$. Since $B - \lambda$ is linear and invertible on $ran(I - P)$ for small ε and λ, we can readily check that $\mathcal{V}$ has the form

$$\mathcal{V} = \mathcal{V}_0(\varepsilon, \omega, \sigma, \lambda)\beta_0 + \mathcal{V}_1(\varepsilon, \omega, \sigma, \lambda)\beta_1.$$

In what follows we will obtain the asymptotic expansion of $\mathcal{V}_j$, $j = 0, 1$, with respect to the bifurcation parameter ε

$$\mathcal{V}_j(\varepsilon, \omega, \sigma, \lambda) = \mathcal{V}_j|_{\varepsilon=0} + \partial_\varepsilon \mathcal{V}_j|_{\varepsilon=0}\varepsilon + \mathcal{O}(\varepsilon^2), \quad j = 0, 1.$$

To find $\mathcal{V}_j$ we first differentiate the second equation of (47) with respect to β_j for $j = 0, 1$, respectively,

$$0 = (I - P)[B(\varepsilon, \omega, \sigma) - \lambda](\phi_j + \mathcal{V}_j), \quad j = 0, 1. \tag{48}$$

Plugging $\varepsilon = 0$ into this form gives

$$(I - P)[B(0, \omega, \sigma) - \lambda]\mathcal{V}_j|_{\varepsilon=0} = -(I - P)[B(0, \omega, \sigma) - \lambda]\phi_j = 0.$$

Since the operator $B(0, \omega, \sigma)$ on $ran(I - P)$ is bijective, we have

$$\mathcal{V}_j|_{\varepsilon=0} = 0, \quad j = 0, 1. \tag{49}$$

Next, take the derivative of (48) with respect to ε

$$0 = (I - P)[\partial_\varepsilon B(\varepsilon, \omega, \sigma)](\phi_j + \mathcal{V}_j) + (I - P)[B(\varepsilon, \omega, \sigma) - \lambda]\partial_\varepsilon \mathcal{V}_j.$$

Inserting $\varepsilon = 0$ into this form and then using the identities (49) yields that for each $j = 0, 1$,

$$\begin{aligned}
&(I - P)[B(0, \omega, \sigma) - \lambda]\partial_\varepsilon \mathcal{V}_j|_{\varepsilon=0} \\
= &\ -(I - P)[\partial_\varepsilon B(\varepsilon, \omega, \sigma)|_{\varepsilon=0}]\phi_j \\
= &\ -(I - P)\Big[-4\omega(\partial_\xi + i\sigma_1)^2\Big(1 + (\partial_\xi + i\sigma_1)^2 - \sigma_2^2\Big) + \frac{12b\sqrt{1 - 4\omega^2}}{\sqrt{27s - 38b^2}}\cos\xi\Big]\phi_j \\
= &\ -\frac{12b\sqrt{1 - 4\omega^2}}{\sqrt{27s - 38b^2}}(I - P)[\phi_j \cos\xi].
\end{aligned}$$

By direct calculation, $(I - P)[\phi_0 \cos\xi] = \frac{1}{2}e^{-i\xi}$ and $(I - P)[\phi_1 \cos\xi] = \frac{1}{2}e^{i2\xi}$, so we arrive at

$$(I - P)[B(0, \omega, \sigma) - \lambda]\partial_\varepsilon \mathcal{V}_0|_{\varepsilon=0} = -\frac{6b\sqrt{1 - 4\omega^2}}{\sqrt{27s - 38b^2}}e^{-i\xi}, \tag{50}$$

and

$$(I - P)[B(0, \omega, \sigma) - \lambda]\partial_\varepsilon \mathcal{V}_1|_{\varepsilon=0} = -\frac{6b\sqrt{1 - 4\omega^2}}{\sqrt{27s - 38b^2}}e^{i2\xi}. \tag{51}$$

Since both span$\{e^{-i\xi}\}$ and span$\{e^{2i\xi}\}$ are invariant under the invertible operator $(I - P)[B(0, \omega, \sigma) - \lambda]$, each $\partial_\varepsilon \mathcal{V}_j|_{\varepsilon=0}$ has the form

$$\partial_\varepsilon \mathcal{V}_0|_{\varepsilon=0} = h_0 e^{-i\xi} \quad \text{and} \quad \partial_\varepsilon \mathcal{V}_1|_{\varepsilon=0} = h_1 e^{i2\xi}$$

for some complex functions $h_0(\varepsilon,\omega,\sigma,\lambda)$ and $h_1(\varepsilon,\omega,\sigma,\lambda)$. Inserting these forms into (50) and (51), respectively, it is straightforward to check that

$$(I-P)[B(0,\omega,\sigma)-\lambda]h_0e^{-i\xi} = [-(1-(\sigma_1-1)^2-\sigma_2^2)^2-\lambda]h_0e^{-i\xi},$$

and

$$(I-P)[B(0,\omega,\sigma)-\lambda]h_1e^{i2\xi} = [-(1-(\sigma_1+2)^2-\sigma_2^2)^2-\lambda]h_1e^{i2\xi}.$$

Consequently, from (50)-(51) and recalling $(\sigma,\lambda)\approx(\sigma^*,0)$ for $\sigma\in\mathcal{R}_1$, h_0 and h_1 are determined by

$$\begin{aligned} h_0 &= \frac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}[(1-(\sigma_1-1)^2-\sigma_2^2)^2+\lambda]} \\ &= \frac{3b\sqrt{1-4\omega^2}}{2\sqrt{27s-38b^2}} + \mathcal{O}\Big(|\sigma_1+\frac{1}{2}|+|\sigma_2-\frac{\sqrt{3}}{2}|+|\lambda|\Big), \end{aligned} \tag{52}$$

and

$$\begin{aligned} h_1 &= \frac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}[(1-(\sigma_1+2)^2-\sigma_2^2)^2+\lambda]} \\ &= \frac{3b\sqrt{1-4\omega^2}}{2\sqrt{27s-38b^2}} + \mathcal{O}\Big(|\sigma_1+\frac{1}{2}|+|\sigma_2-\frac{\sqrt{3}}{2}|+|\lambda|\Big). \end{aligned} \tag{53}$$

So far we have shown that the second equation (47) can be uniquely solved for

$$\mathcal{V}(\varepsilon,\omega,\sigma,\lambda,\beta) = [h_0e^{-i\xi}\varepsilon+\mathcal{O}(\varepsilon^2)]\beta_0 + [h_1e^{2i\xi}\varepsilon+\mathcal{O}(\varepsilon^2)]\beta_1 \tag{54}$$

in a neighborhood of $(\varepsilon,\sigma,\lambda,\beta)=(0,\sigma^*,0,(0,0))$.

Let us find the 2×2 reduced eigenvalue problem of (42) by solving the first equation of (47). Recalling (37) and (39), and using

$$P[\phi_0\cos\xi]=\frac{1}{2}\phi_1,\quad P[\phi_1\cos\xi]=\frac{1}{2}\phi_0,\quad \text{and}\quad P[\phi_0\cos 2\xi]=0=P[\phi_1\cos 2\xi],$$

we first compute $P[B(\varepsilon,\omega,\sigma)-\lambda](\beta_0\phi_0+\beta_1\phi_1)$ as

$$\begin{aligned} &P[-(1+(1+2\varepsilon\omega)(\partial_\xi+i\sigma_1)^2-\sigma_2^2)^2-\lambda+\mathcal{F}(\tilde{u}_{\varepsilon,\omega})](\beta_0\phi_0+\beta_1\phi_1) \\ &= [-(1-(1+2\varepsilon\omega)\sigma_1^2-\sigma_2^2)^2-\lambda]\beta_0\phi_0 \\ &\quad + [-(1-(1+2\varepsilon\omega)(1+\sigma_1)^2-\sigma_2^2)^2-\lambda]\beta_1\phi_1 \\ &\quad + \varepsilon\frac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}(\beta_0\phi_1+\beta_1\phi_0) \\ &\quad + \varepsilon^2\Big[\Big(1-\frac{54s-36b^2}{27s-38b^2}(1-4\omega^2)\Big)(\beta_0\phi_0+\beta_1\phi_1) \\ &\qquad\qquad - \frac{32b^3\omega\sqrt{1-4\omega^2}}{(27s-38b^2)\sqrt{27s-38b^2}}(\beta_0\phi_1+\beta_1\phi_0)\Big] \\ &\quad + \mathcal{O}(\varepsilon^3)(\beta_0+\beta_1)(\phi_0+\phi_1). \end{aligned} \tag{55}$$

Similarly, by inserting (54) into $P[B(\varepsilon,\omega,\sigma)-\lambda]\mathcal{V}$ and using

$$P[e^{-i\xi}\cos\xi]=\frac{1}{2}\phi_0 \quad \text{and} \quad P[e^{i2\xi}\cos\xi]=\frac{1}{2}\phi_1,$$

we deduce that

$$\begin{aligned}
&P[B(\varepsilon,\omega,\sigma)-\lambda]\mathcal{V}=P[\mathcal{F}(\tilde{u}_{\varepsilon,\omega})]\mathcal{V}\\
&=P\Big[\varepsilon\frac{12b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}\cos\xi+\mathcal{O}(\varepsilon^2)\Big]\\
&\quad\times[(h_0e^{-i\xi}\varepsilon+\mathcal{O}(\varepsilon^2))\beta_0+(h_1e^{i2\xi}\varepsilon+\mathcal{O}(\varepsilon^2))\beta_1]\\
&=\varepsilon^2\frac{12b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}(h_0\beta_0P[e^{-i\xi}\cos\xi]+h_1\beta_1P[e^{i2\xi}\cos\xi])\\
&\quad+\mathcal{O}(\varepsilon^3)(\beta_0+\beta_1)(\phi_0+\phi_1)\\
&=\varepsilon^2\frac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}(h_0\beta_0\phi_0+h_1\beta_1\phi_1)+\mathcal{O}(\varepsilon^3)(\beta_0+\beta_1)(\phi_0+\phi_1).
\end{aligned}\tag{56}$$

Recalling (46) and applying (55)-(56) to the first equation of (47), we obtain the 2×2 matrix $\mathcal{M}_1(\varepsilon,\omega,\sigma,\lambda)$ satisfying

$$\begin{pmatrix}0\\0\end{pmatrix}=\mathcal{M}_1(\varepsilon,\omega,\sigma,\lambda)\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix}:=\begin{pmatrix}m_{11}&m_{12}\\m_{21}&m_{22}\end{pmatrix}\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix},\tag{57}$$

in which each entry of $\mathcal{M}_1$ is

$$\begin{aligned}
m_{11}&=-(1-(1+2\varepsilon\omega)\sigma_1^2-\sigma_2^2)^2-\lambda\\
&\quad+\varepsilon^2\Big(1-\tfrac{54s-36b^2}{27s-38b^2}(1-4\omega^2)+\tfrac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}h_0\Big)+\mathcal{O}(\varepsilon^3),\\
m_{12}&=m_{21}=\varepsilon\tfrac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}-\varepsilon^2\tfrac{32b^3\omega\sqrt{1-4\omega^2}}{(27s-38b^2)\sqrt{27s-38b^2}}+\mathcal{O}(\varepsilon^3),\\
m_{22}&=-(1-(1+2\varepsilon\omega)(1+\sigma_1)^2-\sigma_2^2)^2-\lambda\\
&\quad+\varepsilon^2\Big(1-\tfrac{54s-36b^2}{27s-38b^2}(1-4\omega^2)+\tfrac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}h_1\Big)+\mathcal{O}(\varepsilon^3).
\end{aligned}\tag{58}$$

Here, h_0 and h_1 can be found in (52)-(53). For computational convenience, let us rewrite m_{11} and m_{22} in terms of $\hat{\sigma}:=\sigma-\sigma^*=(\sigma_1+\frac{1}{2},\sigma_2-\frac{\sqrt{3}}{2})$. For sufficiently small $|\hat{\sigma}|<\delta$,

$$\begin{aligned}
m_{11}=&-(\hat{\sigma}_1-\sqrt{3}\hat{\sigma}_2)^2+\varepsilon\omega(\hat{\sigma}_1-\sqrt{3}\hat{\sigma}_2)+\varepsilon^2(1-\tfrac{1}{4}\omega^2)\\
&-\varepsilon^2(1-4\omega^2)\tfrac{54s-45b^2}{27s-38b^2}\\
&-\lambda+\mathcal{O}(\varepsilon^2|\hat{\sigma}|+\varepsilon^2|\lambda|+\varepsilon|\hat{\sigma}|^2+|\hat{\sigma}|^3+\varepsilon^3),\\
m_{22}=&-(\hat{\sigma}_1+\sqrt{3}\hat{\sigma}_2)^2-\varepsilon\omega(\hat{\sigma}_1+\sqrt{3}\hat{\sigma}_2)+\varepsilon^2(1-\tfrac{1}{4}\omega^2)\\
&-\varepsilon^2(1-4\omega^2)\tfrac{54s-45b^2}{27s-38b^2}\\
&-\lambda+\mathcal{O}(\varepsilon^2|\hat{\sigma}|+\varepsilon^2|\lambda|+\varepsilon|\hat{\sigma}|^2+|\hat{\sigma}|^3+\varepsilon^3).
\end{aligned}\tag{59}$$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We fix any $b \neq 0$ and s with $27s - 38b^2 > 0$. In order to investigate the spectral instability of $\tilde{u}_{\varepsilon,\omega}(\xi)$ for all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$, we will prove that for each $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ there is a positive λ satisfying $\det \mathcal{M}_1(\varepsilon, \omega, \hat{\sigma}, \lambda) = 0$ for some wave vectors $\sigma \in \mathcal{R}_1$. We again notice that the matrix $\mathcal{M}_1$ is Hermitian due to the self-adjointness of $B(\varepsilon, \omega, \sigma)$, that is, every eigenvalue of $\mathcal{M}_1$ is real-valued.

According to the Weierstrass Preparation Theorem ([5, Section 2.6]), there is an analytic function $q(\varepsilon, \hat{\sigma}, \lambda)$ in a neighborhood of $(\varepsilon, \hat{\sigma}, \lambda) = (0, (0, 0), 0)$ such that $q(0, (0, 0), 0) = 1$ and

$$q(\varepsilon, \hat{\sigma}, \lambda) \det \mathcal{M}_1(\varepsilon, \omega, \hat{\sigma}, \lambda) = \lambda^2 + a_1\lambda + a_0 = 0, \tag{60}$$

in which $a_0(\varepsilon, \hat{\sigma})$ and $a_1(\varepsilon, \hat{\sigma})$ are also analytic functions in a neighborhood of $(\varepsilon, \hat{\sigma}) = (0, (0, 0))$. Indeed, a_0 and a_1 are determined by

$$\begin{aligned} a_0(\varepsilon, \hat{\sigma}) &= q(\varepsilon, \hat{\sigma}, 0) \det \mathcal{M}_1(\varepsilon, \omega, \hat{\sigma}, 0) \\ &= \det \mathcal{M}_1(\varepsilon, \omega, \hat{\sigma}, 0) + \mathcal{O}(\varepsilon + |\hat{\sigma}|) \det \mathcal{M}_1(\varepsilon, \omega, \hat{\sigma}, 0), \end{aligned} \tag{61}$$

and

$$\begin{aligned} a_1(\varepsilon, \hat{\sigma}) &= q_\lambda(\varepsilon, \hat{\sigma}, 0) \det \mathcal{M}_1(\varepsilon, \omega, \hat{\sigma}, 0) + q(\varepsilon, \hat{\sigma}, 0) \det \mathcal{M}_{1\lambda}(\varepsilon, \omega, \hat{\sigma}, 0) \\ &= \det \mathcal{M}_{1\lambda}(\varepsilon, \omega, \hat{\sigma}, 0) + \mathcal{O}(1) \det \mathcal{M}_1(\varepsilon, \omega, \hat{\sigma}, 0) \\ &\quad + \mathcal{O}(\varepsilon + |\hat{\sigma}|)\ (\det \mathcal{M}_1(\varepsilon, \omega, \hat{\sigma}, 0) + \det \mathcal{M}_{1\lambda}(\varepsilon, \omega, \hat{\sigma}, 0)). \end{aligned} \tag{62}$$

Here $\cdot_\lambda$ means the derivative of $\cdot$ with respect to λ. Since both eigenvalues $\lambda_1(\varepsilon, \omega, \hat{\sigma}) = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}$ and $\lambda_2(\varepsilon, \omega, \hat{\sigma}) = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}$ are real, $a_1^2 - 4a_0$ is nonnegative. It follows that both eigenvalues are all nonpositive if and only if $a_0 \geq 0$ and $a_1 \geq 0$. However, at $|\hat{\sigma}| = 0$, i.e., $\sigma = \sigma^* = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, we have by direct calculation that

$$a_0(\varepsilon, (0, 0)) = -\frac{36b^2(1 - 4\omega^2)}{27s - 38b^2}\varepsilon^2 + \mathcal{O}(\varepsilon^3) \tag{63}$$

and

$$\begin{aligned} a_1(\varepsilon, (0, 0)) = &\ -2\Big[1 - \tfrac{1}{4}\omega^2 - (1 - 4\omega^2)\tfrac{54s - 45b^2}{27s - 38b^2}\Big]\varepsilon^2 + \mathcal{O}(1)\tfrac{36b^2(1 - 4\omega^2)}{27s - 38b^2}\varepsilon^2 \\ &\ + \mathcal{O}(\varepsilon^3) \\ \leq &\ -2\Big[\tfrac{15}{16} - (1 - 4\omega^2)\tfrac{54s - (45 - C)b^2}{27s - 38b^2}\Big]\varepsilon^2 + \mathcal{O}(\varepsilon^3) \end{aligned} \tag{64}$$

for some positive number $C > 0$.

We recall that $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ and $27s - 38b^2 > 0$. If $1 - 4\omega^2$ is bounded away from 0, we see immediately that

$$a_0(\varepsilon, (0, 0)) < 0$$

for sufficiently small $\varepsilon > 0$, in which case λ_2 is positive. Noting that $54s - (45 - C)b^2 > 0$, we now assume $1 - 4\omega^2$ is sufficiently small so that

$$0 \leq 1 - 4\omega^2 \leq \frac{1}{2} \cdot \frac{27s - 38b^2}{54s - (45 - C)b^2}.$$

Then we conclude that

$$a_1(\varepsilon,(0,0)) \le -\frac{7}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3) < 0$$

for sufficiently small $\varepsilon > 0$, in which case λ_1 is positive. In particular, for $\omega = \pm\frac{1}{2}$ the Bloch operator (39) has constant coefficients, so that the eigenvalues $\lambda_{\pm,j}$, $j = 1,2$ for $\sigma \in \mathcal{R}_1$ are obtained as follows:

$$\begin{aligned} \lambda_{\pm,1} = & -(\hat\sigma_1 - \sqrt{3}\hat\sigma_2)^2 \pm \frac{1}{2}\varepsilon(\hat\sigma_1 - \sqrt{3}\hat\sigma_2) + \frac{15}{16}\varepsilon^2 \\ & + \mathcal{O}(\varepsilon^2|\hat\sigma| + \varepsilon|\hat\sigma|^2 + |\hat\sigma|^3), \\ \lambda_{\pm,2} = & -(\hat\sigma_1 + \sqrt{3}\hat\sigma_2)^2 \mp \frac{1}{2}\varepsilon(\hat\sigma_1 + \sqrt{3}\hat\sigma_2) + \frac{15}{16}\varepsilon^2 \\ & + \mathcal{O}(\varepsilon^2|\hat\sigma| + \varepsilon|\hat\sigma|^2 + |\hat\sigma|^3). \end{aligned} \tag{65}$$

Furthermore, if $|\hat\sigma| \le \delta = \mathcal{O}(\varepsilon^r)$ for any fixed $r > 1$, we readily see from the leading order terms of ε and $\hat\sigma$ in (59) that the identities (63) and (64) remain valid by replacing $\mathcal{O}(\varepsilon^3)$ by $\mathcal{O}(\varepsilon^{2r})$. It completes the proof of Theorem 1.3. □

3.3. Linear instability of the roll solutions

In this section we observe that the spectral instability of Theorem 1.3 leads to linear instability in the sense that there exists an exponentially growing solution to the linearized equation (36).

In Section 3.2 we have found that there exists $\tilde\varepsilon_0 > 0$ such that $B(\varepsilon,\omega,\sigma)$ has a positive eigenvalue $\lambda(\varepsilon,\omega,\sigma)$ for $(\varepsilon,\omega) \in (0,\tilde\varepsilon_0) \times [-\frac{1}{2},\frac{1}{2}]$, where σ is sufficiently close to $\sigma^* = (-\frac{1}{2},\frac{\sqrt{3}}{2})$ by $O(\varepsilon^r)$ for some $r > 1$. Moreover, we notice that $\lambda(\sigma^*)$ is a simple eigenvalue from (63)-(64) [2] when $\omega \in (-\frac{1}{2},\frac{1}{2})$, thus $\lambda(\varepsilon,\omega,\sigma)$ can be defined as a positive smooth mapping in the vicinity of σ^* for fixed ε and ω. Let us denote the square of unstable wave vectors by $I \times J$ in $\mathcal{R}_1$,

$$I \times J := [-1/2, -1/2 + O(\varepsilon^r)] \times [\sqrt{3}/2, \sqrt{3}/2 + O(\varepsilon^r)] \subset \mathcal{S}_{\varepsilon,\omega}$$

for some $O(\varepsilon^r) > 0$.

Proposition 3.1. *Let s, b and $\tilde\varepsilon_0$ be the same as in Theorem 1.3 and fix any $\varepsilon \in (0,\tilde\varepsilon_0]$ and $\omega \in (-\frac{1}{2},\frac{1}{2})$. Let $W(\xi,\sigma)$ be the continuous family of the eigenfunctions of $B(\varepsilon,\omega,\sigma)$ corresponding to the positive eigenvalue $\lambda(\varepsilon,\omega,\sigma)$ for $\sigma \in I \times J$. Then for the function*

$$U(\xi,y) = \int_{I\times J} e^{i(\sigma_1\xi+\sigma_2 y)} W(\xi,\sigma)\, d\sigma_1 d\sigma_2, \tag{66}$$

it holds that for all $t \ge 0$,

$$\|e^{\mathcal{L}_{\varepsilon,\omega}t}U\|_{L^2(\mathbb{R}^2)} \ge C\frac{e^{\lambda(\varepsilon,\omega,\sigma^*)t}}{(1+t)^{\frac{1}{l}}}$$

[2]Such $a_0(\varepsilon,(0,0))$ and $a_1(\varepsilon,(0,0))$ do not allow an equal root to (60).

for some positive constants C and l.

Proof. In what follows we suppress ε, ω-dependencies of λ and B for simplicity. We note that

$$V(t,\xi,y) := \int_{I\times J} e^{\lambda(\sigma)t} e^{i(\sigma_1\xi+\sigma_2 y)} W(\xi,\sigma) d\sigma_1 d\sigma_2$$

solves the linearized equation (36) with the initial data U. Indeed, $V(0,\xi,y) = U(\xi,y)$,

$$\partial_t V = \int_{I\times J} \lambda(\sigma) e^{\lambda(\sigma)t} e^{i(\sigma_1\xi+\sigma_2 y)} W(\xi,\sigma) d\sigma_1 d\sigma_2,$$

and from (42),

$$\begin{aligned}\mathcal{L}_{\varepsilon,\omega} V &= \int_{I\times J} e^{\lambda(\sigma)t} e^{i(\sigma_1\xi+\sigma_2 y)} B(\sigma) W(\xi,\sigma) d\sigma_1 d\sigma_2 \\ &= \int_{I\times J} \lambda(\sigma) e^{\lambda(\sigma)t} e^{i(\sigma_1\xi+\sigma_2 y)} W(\xi,\sigma) d\sigma_1 d\sigma_2,\end{aligned}$$

thus $e^{\mathcal{L}_{\varepsilon,\omega}t} U = V$. Rewriting V as

$$V(t,\xi,y) = \int_{\mathbb{R}} e^{i\sigma_2 y} \left(\int_I \chi_J(\sigma_2) e^{\lambda(\sigma)t} e^{i\sigma_1\xi} W(\xi,\sigma) d\sigma_1 \right) d\sigma_2,$$

it follows from the Plancherel theorem that

$$\|V(t,\xi,\cdot)\|^2_{L^2_y(\mathbb{R})} = \left\| \int_I \chi_J(\sigma_2) e^{\lambda(\sigma)t} e^{i\sigma_1\xi} W(\xi,\sigma) d\sigma_1 \right\|^2_{L^2_{\sigma_2}(\mathbb{R})}.$$

Here $\chi_J(\sigma_2)$ is the characteristic function on J. Integrating in ξ over $\mathbb{R}$, we have that for all $t \geq 0$,

$$\begin{aligned}\|V(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^2)} &= \int \left\| \int_I \chi_J(\sigma_2) e^{\lambda(\sigma)t} e^{i\sigma_1\xi} W(\xi,\sigma) d\sigma_1 \right\|^2_{L^2_\xi(\mathbb{R})} d\sigma_2 \\ &= \int_J \left\| \int_I e^{\lambda(\sigma)t} e^{i\sigma_1\xi} W(\xi,\sigma) d\sigma_1 \right\|^2_{L^2_\xi(\mathbb{R})} d\sigma_2 \\ &\simeq \int_J \int_I \left\| e^{\lambda(\sigma)t} W(\xi,\sigma) \right\|^2_{L^2_{per}([0,2\pi])} d\sigma_1 d\sigma_2,\end{aligned} \tag{67}$$

where we used Fubini and the fact that the following equivalence holds for some constants C_1 and C_2:

$$\begin{aligned}C_1 \int_I \left\| e^{\lambda(\sigma)t} W(\xi,\sigma) \right\|^2_{L^2_{per}([0,2\pi])} d\sigma_1 &\leq \left\| \int_I e^{\lambda(\sigma)t} e^{i\sigma_1\xi} W(\xi,\sigma) d\sigma_1 \right\|^2_{L^2_\xi(\mathbb{R})} \\ &\leq C_2 \int_I \left\| e^{\lambda(\sigma)t} W(\xi,\sigma) \right\|^2_{L^2_{per}([0,2\pi])} d\sigma_1.\end{aligned}$$

We refer to [11, Lemma 3.3] for the above inequalities. By the continuity of the map $\sigma \in I \times J \mapsto W(\cdot, \sigma) \in L^2_{per}(\mathbb{R})$, the right end of (67) is bounded for each t, and thus we obtain that

$$C_1 \int_J \int_I e^{2\lambda(\sigma)t} d\sigma_1 d\sigma_2 \leq \|V(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^2)} \leq C_2 \int_J \int_I e^{2\lambda(\sigma)t} d\sigma_1 d\sigma_2$$

for constants C_1 and C_2 depending on ε, ω and σ^*. Since $\lambda(\sigma)$ is smooth on $I \times J$, if σ^* has the multiplicity of $l \geq 1$, we can write

$$\lambda(\sigma) - \lambda(\sigma^*) = \sum_{k=0}^{l} a_k (\sigma_1 - \sigma_1^*)^{l-k} (\sigma_2 - \sigma_2^*)^k + o(|\sigma - \sigma^*|^l),$$

where not all a_k are zero. Substituting the above expansion into $\lambda(\sigma)$ with letting $\sigma_i - \sigma_i^* = k_i$ yields that

$$\begin{aligned} \int_J \int_I e^{2\lambda(\sigma)t} dk_1 dk_2 &= e^{2\lambda(\sigma^*)t} \int_0^{O(\varepsilon^r)} \int_0^{O(\varepsilon^r)} e^{\left(\sum_{k=0}^{l} a_k {k_1}^{l-k} {k_2}^{k} + o(|k|^l)\right)t} dk_1 dk_2 \\ &\geq e^{2\lambda(\sigma^*)t} \int_0^{O(\varepsilon^r)} \int_0^{O(\varepsilon^r)} e^{-C(k_1^l + k_2^l)t} dk_1 dk_2 \end{aligned}$$

for some constant $C > 0$. Similarly as in [11], we estimate the integral bounded below by $O(\varepsilon^{2r}) e^{-C\varepsilon^{rl}}$ when $0 \leq t \leq 1$, and by $Ct^{-\frac{2}{l}}$ when $t > 1$. In particular, for the case $t > 1$ we change variables $p_i = k_i^l t$ $(i = 1, 2)$ such that

$$\int_0^{O(\varepsilon^r)} e^{-Ck_1^l t} dk_1 = t^{-\frac{1}{l}} \int_0^{O(\varepsilon^r)^l t} \frac{p_1^{\frac{1}{l}-1} e^{-Cp_1}}{l} dp_1.$$

We arrive at

$$C \frac{e^{\lambda(\sigma^*)t}}{(1+t)^{\frac{1}{l}}} \leq \|V(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^2)}$$

for some constant C depending on ε, ω and σ^*. □

4. Discussion about unstable Bloch wave vectors

This section is devoted to providing more information about unstable wave vectors of the Bloch operators. Although the proof of Theorem 1.3 was done by investigating the unstable modes in the region $\mathcal{R}_1$, we try to examine what unstable modes the Bloch operators $B(\varepsilon, \omega, \sigma)$ have in other regions (see Figure 1 in the previous section).

4.1. An eigenvalue problem of $B(\varepsilon, \omega, \sigma)$ for $\sigma \in \mathcal{R}_2$

In this region we focus on how the transverse direction, i.e., $\sigma_2 \neq 0$, affects the unstable modes of the Bloch operator because the case of $\sigma_2 = 0$ was already derived in [13] for $b = 1$ and $s = 2$. Without the transverse direction (i.e., $\sigma_2 = 0$), the following set

$$\Gamma := \{\sigma_1 \in \mathbb{R} \mid |\sigma_1| < \delta\}$$

for small $\delta > 0$ is only the region we need to consider. It has been proved in [13] that the periodic solutions $\tilde{u}_{\varepsilon,\omega}$ are Eckhaus-unstable, that is, for each $\omega \in [-\frac{1}{2}, \frac{1}{2\sqrt{3}}) \cup (\frac{1}{2\sqrt{3}}, \frac{1}{2}]$ there is a positive eigenvalue of $B(\varepsilon, \omega, \sigma_1)$ for $\sigma_1 \in \Gamma \setminus \{0\}$. In what follows we prove that this result is valid also for the region $\mathcal{R}_2$ unless $\sigma_1 = 0$. In addition, we will see how the unstable modes are determined if $\sigma_1 = 0$ and $\sigma_2 \neq 0$. We first state the observation of the unstable wave vectors for the region $\mathcal{R}_2$.

Theorem 4.1. *Fix any $r > 1$. Then for sufficiently small $\varepsilon > 0$, $\mathcal{R}_2 \setminus \{(0,0)\} \subset \mathcal{S}_{\varepsilon,\omega}$ with $\delta = \mathcal{O}(\varepsilon^r)$ if*

$$\omega \in [-\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \cup (\frac{1}{2\sqrt{3}}, \frac{1}{2}] \quad \text{or} \quad \omega < \frac{-2b^2\varepsilon}{3(27s - 38b^2)} + \mathcal{O}(\varepsilon^3). \tag{68}$$

Remark 4.2. The sufficient conditions (68) for instability are the Eckhaus criterion and the zigzag instability, respectively, corresponding to (3) in [18]. For the case $b = 0$ studied in [18], the coefficients of $\mathcal{L}_{\varepsilon,\omega}$ can also be π-periodic, so that our region $\mathcal{R}_2$ corresponds to $\sigma \approx (2, 0)$ there.

For $\sigma \in \mathcal{R}_2$, there are two critical eigenfunctions $\phi_1(\xi) = e^{i\xi}$ and $\phi_{-1}(\xi) = e^{-i\xi}$ of $B(0, \omega, \sigma)$. However, we take the basis functions $U_1 := \cos\xi = \frac{1}{2}(\phi_1(\xi) + \phi_{-1}(\xi))$ and $U_2 := \sin\xi = \frac{1}{2i}(\phi_1(\xi) - \phi_{-1}(\xi))$ to be compatible with the calculations in [13], and we then define the orthogonal projection in $L^2_{per}([0, 2\pi])$ onto $\text{span}\{U_1, U_2\}$:

$$PW = \frac{1}{\pi}\int_0^{2\pi} WU_1 \ d\xi U_1 + \frac{1}{\pi}\int_0^{2\pi} WU_2 \ d\xi U_2,$$

or equivalently as a vector form

$$\tilde{P}W = \frac{1}{\pi}\Big(\int_0^{2\pi} WU_1 \ d\xi, \int_0^{2\pi} WU_2 \ d\xi\Big)^T \in \mathbb{C}^2. \tag{69}$$

Decomposing $W \in L^2_{per}([0, 2\pi])$ into $W = \beta_1 U_1 + \beta_2 U_2 + \mathcal{V}$ with $\beta = (\beta_1, \beta_2) \in \mathbb{C}^2$ and $P\mathcal{V} = 0$, we rewrite (42) for $\sigma \in \mathcal{R}_2$ as two equations

$$\begin{aligned} 0 &= \tilde{P}[B(\varepsilon, \omega, \sigma) - \lambda](\beta_1 U_1 + \beta_2 U_2 + \mathcal{V}), \\ 0 &= (I - P)[B(\varepsilon, \omega, \sigma) - \lambda](\beta_1 U_1 + \beta_2 U_2 + \mathcal{V}). \end{aligned} \tag{70}$$

Repeating the procedure that we have done for $\sigma \in \mathcal{R}_1$ (or see [13] for a detailed calculation with $\sigma_2 = 0$), we can solve the second equation of (70) locally for $\mathcal{V}$

$$\begin{aligned} \mathcal{V} = {} & [(h_1 + h_2\cos 2\xi + h_3 \sin 2\xi)\varepsilon + \mathcal{O}(\varepsilon^2)]\beta_1 \\ & + [(r_1 \cos 2\xi + r_2 \sin 2\xi)\varepsilon + \mathcal{O}(\varepsilon^2)]\beta_2, \end{aligned} \tag{71}$$

in a neighborhood $(\varepsilon, \sigma, \lambda, \beta) \approx (0, (0,0), 0, (0,0))$. Here, the functions h_j and r_j are determined by

$$
\begin{aligned}
h_1 =& \tfrac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} \cdot \tfrac{1}{(1-\sigma_1^2-\sigma_2^2)^2+\lambda)} \\
=& \tfrac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} + \mathcal{O}(\sigma_1^2+\sigma_2^2+|\lambda|), \\
h_2 = r_2 =& \tfrac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} \cdot \tfrac{(3+\sigma_1^2+\sigma_2^2)^2+16\sigma_1^2+\lambda}{[(3+\sigma_1^2+\sigma_2^2)^2+16\sigma_1^2+\lambda]^2-64\sigma_1^2(3+\sigma_1^2+\sigma_2^2)^2} \\
=& \tfrac{2b\sqrt{1-4\omega^2}}{3\sqrt{27s-38b^2}} + \mathcal{O}(\sigma_1^2+\sigma_2^2+|\lambda|), \\
h_3 = -r_1 =& \tfrac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} \cdot \tfrac{-8i\sigma_1(3+\sigma_1^2+\sigma_2^2)}{[(3+\sigma_1^2+\sigma_2^2)^2+16\sigma_1^2+\lambda]^2-64\sigma_1^2(3+\sigma_1^2+\sigma_2^2)^2} \\
=& \mathcal{O}(|\sigma_1|(1+\sigma_2^2+|\lambda|)).
\end{aligned}
\tag{72}
$$

By substituting (71)-(72) into the first equation of (70), we obtain the following 2×2 reduced eigenvalue equation of (42) for $\sigma \in \mathcal{R}_2$,

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathcal{M}_2(\varepsilon, \omega, \sigma, \lambda) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} := \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \tag{73}
$$

in which all entries of $\mathcal{M}_2$ are given by

$$
\begin{aligned}
m_{11} =& -(2\varepsilon\omega + (1+2\varepsilon\omega)\sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2(1+2\varepsilon\omega)^2 - \lambda + \varepsilon^2 \\
& - \varepsilon^2 \tfrac{81s-38b^2}{27s-38b^2}(1-4\omega^2) + \tfrac{12b\varepsilon^2\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}(h_1 + \tfrac{1}{2}h_2) + \mathcal{O}(\varepsilon^3) \\
=& -2\varepsilon^2(1-4\omega^2) + \mathcal{O}(\varepsilon^3) - 4\sigma_1^2 - \lambda + \mathcal{O}(\varepsilon|\sigma|^2 + \varepsilon^2|\lambda| + |\sigma|^4), \\
m_{12} = \bar{m}_{21} =& 4i\sigma_1(1+2\varepsilon\omega)(2\varepsilon\omega + (1+2\varepsilon\omega)\sigma_1^2 + \sigma_2^2) + \tfrac{6b\varepsilon^2\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} r_1 \\
& + \mathcal{O}(\varepsilon^3) \\
=& 8i\sigma_1\varepsilon\omega + \mathcal{O}(\varepsilon^2|\sigma|(1+|\lambda|) + |\sigma|^3) + \mathcal{O}(\varepsilon^3), \\
m_{22} =& -(2\varepsilon\omega + (1+2\varepsilon\omega)\sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2(1+2\varepsilon\omega)^2 - \lambda + \varepsilon^2 \\
& - \varepsilon^2 \tfrac{27s-34b^2}{27s-38b^2}(1-4\omega^2) + \tfrac{6b\varepsilon^2\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} r_2 + \mathcal{O}(\varepsilon^3) \\
=& -4\sigma_1^2 - \lambda + \mathcal{O}(\varepsilon|\sigma|^2 + \varepsilon^2|\lambda| + |\sigma|^4) + \mathcal{O}(\varepsilon^3).
\end{aligned}
\tag{74}
$$

Here we emphasize that the coefficients b and s in (1) do not affect the estimates of all entries of $\mathcal{M}_2$ as long as $27s - 38b^2 > 0$. We see from (74) that the terms including b and s all disappear or are absorbed into the error terms. This is the reason that the stability and instability results in one dimension can be obtained regardless of the values of b and s as long as $27s - 38b^2 > 0$. In addition, if we compare our reduced matrix $\mathcal{M}_2$ to the one of [13], nothing has changed except that the term $|\sigma_1|$ in [13] has been replaced by $|\sigma|$ here. Thus, the treatment of the reduced eigenvalue problem (73) follows exactly the one of [13].

Proof of Theorem 4.1. We first notice that the error terms $\mathcal{O}(\varepsilon^3)$ that appear in m_{12}, m_{21} and m_{22} can be absorbed into other error terms $\mathcal{O}(\varepsilon^2|\sigma|(1+|\lambda|))$

and $\mathcal{O}(\varepsilon\sigma^2+\varepsilon^2|\lambda|)$, respectively. Consequently, the reduced matrix $\mathcal{M}_2$ can be rewritten as

$$\begin{aligned}&\mathcal{M}_2(\varepsilon,\omega,\sigma,\lambda)\\ &=\begin{pmatrix} c(\varepsilon,\omega)-4\sigma_1^2-\lambda & 8i\sigma_1\omega\varepsilon\\ -8i\sigma_1\omega\varepsilon & -4\sigma_1^2-\lambda\end{pmatrix}\\ &+\begin{pmatrix}\mathcal{O}(\varepsilon|\sigma|^2+\varepsilon^2|\lambda|+|\sigma|^4) & \mathcal{O}(\varepsilon^2|\sigma|(1+|\lambda|)+|\sigma|^3)\\ \mathcal{O}(\varepsilon^2|\sigma|(1+|\lambda|)+|\sigma|^3) & \mathcal{O}(\varepsilon|\sigma|^2+\varepsilon^2|\lambda|+|\sigma|^4)\end{pmatrix},\end{aligned}\tag{75}$$

where $c(\varepsilon,\omega):=-2\varepsilon^2(1-4\omega^2)+\mathcal{O}(\varepsilon^3)$. This is due to the reflection symmetries defined in (41) and the fact that the matrix $\mathcal{M}_2$ at $\sigma=(0,0)$ and $\lambda=0$ becomes

$$\mathcal{M}_2(\varepsilon,\omega,(0,0),0)=\begin{pmatrix} c(\varepsilon,\omega)+\mathcal{O}(\varepsilon^3) & 0\\ 0 & 0\end{pmatrix}\tag{76}$$

(see [13, Lemma 3.1] for the proof). We now solve $\det\mathcal{M}_2(\varepsilon,\omega,\sigma,\lambda)=0$ for λ. A direct computation from (75) gives

$$\begin{aligned}\det\mathcal{M}_2(\varepsilon,\omega,\sigma,\lambda)&=\lambda^2-\lambda(c(\varepsilon,\omega)-8\sigma_1^2)-(c(\varepsilon,\omega)-4\sigma_1^2)4\sigma_1^2-64\sigma_1^2\omega^2\varepsilon^2\\ &\quad+F(\varepsilon,\omega,\sigma,\lambda),\end{aligned}$$

where

$$\begin{aligned}F(\varepsilon,\omega,\sigma,\lambda)=\mathcal{O}\Big(&|\lambda|(\varepsilon|\sigma|^2+\varepsilon^2|\lambda|+|\sigma|^4)+(\varepsilon^2+|\sigma|^2)(\varepsilon|\sigma|^2+\varepsilon^2|\lambda|+|\sigma|^4)\\ &+(\varepsilon|\sigma|^2+\varepsilon^2|\lambda|+|\sigma|^4)^2+\varepsilon|\sigma|(\varepsilon^2|\sigma|(1+|\lambda|)+\sigma^3)\\ &+(\varepsilon^2|\sigma|(1+|\lambda|)+|\sigma|^3)^2\Big).\end{aligned}$$

Again using the Weierstrass Preparation Theorem, there is an analytic function $q(\varepsilon,\sigma,\lambda)$ in a neighborhood of $(\varepsilon,\sigma,\lambda)=(0,(0,0),0)$ such that $q(0,(0,0),0)=1$ and

$$q(\varepsilon,\sigma,\lambda)\det\mathcal{M}_2(\varepsilon,\omega,\sigma,\lambda)=\lambda^2+a_1\lambda+a_0=0.$$

Here, similarly as in (61)-(62), the analytic functions $a_0(\varepsilon,\sigma)$ and $a_1(\varepsilon,\sigma)$ are determined by

$$\begin{aligned}a_0(\varepsilon,\sigma)&=8\sigma_1^2(\varepsilon^2(1-12\omega^2)+2\sigma_1^2)+\mathcal{O}(|\sigma|^2(\varepsilon+|\sigma|)^3),\\ a_1(\varepsilon,\sigma)&=2\varepsilon^2(1-4\omega^2)+8\sigma_1^2+\mathcal{O}((\varepsilon+|\sigma|)^3).\end{aligned}\tag{77}$$

As mentioned in the proof of Theorem 1.3, two eigenvalues are real and these are all nonpositive if and only if $a_0\geq 0$ and $a_1\geq 0$.

(i) In the case $\sigma=(0,0)\in\mathcal{R}_2$, $a_0=0$ and $a_1=2\varepsilon^2(1-4\omega^2)+\mathcal{O}(\varepsilon^3)>0$ for small $\varepsilon>0$ unless $\omega=\pm\frac{1}{2}$, while $a_0=a_1=0$ if $\omega=\pm\frac{1}{2}$ because $\tilde{u}_{\varepsilon,\omega}\equiv 0$. It implies that $(0,0)\notin\mathcal{S}_{\varepsilon,\omega}$ for all $\omega\in[-\frac{1}{2},\frac{1}{2}]$, that is, the rolls are stable under the perturbation with the same period of $\tilde{u}_{\varepsilon,\omega}$, called co-periodic stable.

(ii) If $\sigma\in\mathcal{R}_2$ with $\sigma_1\neq 0$, then the expansion of a_0 in (77) immediately leads to the spectral instability when $1-12\omega^2<0$. That is, there is small

$\varepsilon > 0$ such that for any fixed $r > 1$,

$$\{\sigma \in \mathcal{R}_2 \mid \sigma_1 \neq 0,\ |\sigma| \leq \mathcal{O}(\varepsilon^r)\} \subset \mathcal{S}_{\varepsilon,\omega}$$

for all $\omega \in [-\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \cup (\frac{1}{2\sqrt{3}}, \frac{1}{2}]$.

(iii) We now consider the eigenvalue equation (42) for $\sigma \in \mathcal{R}_2$ with $\sigma_1 = 0$, in which case the expression of $\mathcal{M}_2(\varepsilon, \omega, \sigma, \lambda)$ in (75) is not enough to investigate the signs of eigenvalues. We need refined estimates of σ_2 absorbed in the error terms of (75). However, as shown in the second case (ii), the instability is readily determined by the coefficient of σ_2^2 of $\det \mathcal{M}_2(\varepsilon, \omega, \sigma, 0)$; hence we first consider each entry of $\mathcal{M}_2(\varepsilon, \omega, \sigma, 0)$ with $\sigma_1 = 0$ as follows:

$$\begin{aligned}
m_{11}|_{\sigma_1=0} = &\ -(2\varepsilon\omega + \sigma_2^2)^2 + \varepsilon^2 - \varepsilon^2 \tfrac{81s-38b^2}{27s-38b^2}(1-4\omega^2) \\
&+ \tfrac{12b\varepsilon^2\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}(h_1 + \tfrac{1}{2}h_2) + \mathcal{O}(\varepsilon^3), \\
= &\ -2\varepsilon^2(1-4\omega^2) + \mathcal{O}(\varepsilon^3) + \mathcal{O}(\varepsilon\sigma_2^2 + \sigma_2^4), \\
m_{12}|_{\sigma_1=0} = &\ m_{21}|_{\sigma_1=0} = 0, \\
m_{22}|_{\sigma_1=0} = &\ -(2\varepsilon\omega + \sigma_2^2)^2 + \varepsilon^2 - \varepsilon^2 \tfrac{27s-34b^2}{27s-38b^2}(1-4\omega^2) \\
&+ \tfrac{6b\varepsilon^2\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} r_2 + \mathcal{O}(\varepsilon^3) \\
= &\ -4\varepsilon\omega\sigma_2^2 - \tfrac{8b^2\varepsilon^2(1-4\omega^2)}{3(27s-38b^2)}\sigma_2^2 + \mathcal{O}(\sigma_2^4 + \varepsilon^3\sigma_2^2).
\end{aligned}$$

Then the coefficient of σ_2^2 of $\det \mathcal{M}_2(\varepsilon, \omega, \sigma, 0)$ is

$$\Big(-2\varepsilon^2(1-4\omega^2) + \mathcal{O}(\varepsilon^3)\Big)\Big(-4\varepsilon\omega - \frac{8b^2\varepsilon^2(1-4\omega^2)}{3(27s-38b^2)} + \mathcal{O}(\varepsilon^3)\Big). \tag{78}$$

In order to have $\mathcal{O}(\varepsilon^5)$ as error terms in this expansion, we should find the coefficient of the term of order ε^3 in $m_{11}|_{\sigma_1=0}$. However, following the proof of (76) in [13], we see that

$$\alpha_1 \tilde{f}_{\alpha_1} = -2\varepsilon^2(1-4\omega^2) + \mathcal{O}(\varepsilon^3),$$

where $\tilde{f} = \tilde{f}(\varepsilon, \omega, \alpha_1)$ can be found in (29). From determining the term of order ε^3, we arrive at

$$\alpha_1 \tilde{f}_{\alpha_1} = -2\varepsilon^2(1-4\omega^2) + \frac{64b^2\omega(1-4\omega^2)}{3(27s-38b^2)}\varepsilon^3 + \mathcal{O}(\varepsilon^4). \tag{79}$$

Replacing the first term of (78) by the right-hand side of (79), we have the expansion

$$\begin{aligned}
&\det \mathcal{M}_2(\varepsilon, \omega, \sigma, 0)|_{\sigma_1=0} \\
= &\Big(-2\varepsilon^2(1-4\omega^2) + \tfrac{64b^2\omega(1-4\omega^2)}{3(27s-38b^2)}\varepsilon^3 + \mathcal{O}(\varepsilon^4)\Big) \\
&\times \Big(-4\varepsilon\omega - \tfrac{8b^2\varepsilon^2(1-4\omega^2)}{3(27s-38b^2)} + \mathcal{O}(\varepsilon^3)\Big)\sigma_2^2 \\
&+ \mathcal{O}(\varepsilon^2(\sigma_2^4 + \varepsilon^3\sigma_2^2) + \varepsilon\sigma_2^2(\varepsilon\sigma_2^2 + \sigma_2^4) + (\varepsilon\sigma_2^2 + \sigma_2^4)(\sigma_2^4 + \varepsilon^3\sigma_2^2))
\end{aligned}$$

$$= \Big[8\varepsilon^3\omega(1-4\omega^2) - \tfrac{256b^2\omega^2(1-4\omega^2)}{3(27s-38b^2)}\varepsilon^4 + \tfrac{16b^2(1-4\omega^2)^2}{3(27s-38b^2)}\varepsilon^4\Big]\sigma_2^2 + \mathcal{O}(\varepsilon\sigma_2^6 + \varepsilon^2\sigma_2^4 + \varepsilon^5\sigma_2^2 + \sigma_2^8).$$

Consequently, if $|\sigma_2| \le \delta = \mathcal{O}(\varepsilon^r)$ for any fixed $r > 1$, the sufficient condition of instability is determined by

$$8\varepsilon^3\omega(1-4\omega^2) - \frac{256b^2\omega^2(1-4\omega^2)}{3(27s-38b^2)}\varepsilon^4 + \frac{16b^2(1-4\omega^2)^2}{3(27s-38b^2)}\varepsilon^4 < 0. \tag{80}$$

In the case that $\sigma_1 \neq 0$ and $1-4\omega^2 = 0$, we have already proved that $\sigma \in \mathcal{S}_{\varepsilon,\omega}$, so we may assume $1-4\omega^2 \neq 0$. Therefore, the inequality (80) can be solved as

$$\omega < \frac{-2b^2\varepsilon}{3(27s-38b^2)} + \mathcal{O}(\varepsilon^3).$$

This completes the proof of Theorem 4.1. □

4.2. An eigenvalue problem of $B(\varepsilon,\omega,\sigma)$ for $\sigma \in \mathcal{R}_3$

In this region the constant function $\phi_0(\xi) \equiv 1$ is only the eigenfunction of $B(0,\omega,\sigma)$ corresponding to a small eigenvalue $\mu_0(\sigma) = -(1-\sigma_1^2-\sigma_2^2)^2$. That is, the eigenvectors of $B(\varepsilon,\omega,\sigma)$ for $\sigma \in \mathcal{R}_3$ can be considered as small perturbations of $\phi_0 \equiv 1$, so we will solve the eigenvalue problem (42) for $\sigma \in \mathcal{R}_3$ in one dimension by defining the orthogonal projection in $L^2_{per}([0,2\pi])$ onto span$\{\phi_0 \equiv 1\}$:

$$PW = \frac{1}{2\pi}\int_0^{2\pi} W\phi_0 \, d\xi\phi_0 = \frac{1}{2\pi}\int_0^{2\pi} W \, d\xi.$$

Decomposing $W \in L^2_{per}([0,2\pi])$ into $W = \beta\phi_0 + \mathcal{V}$ with $\beta \in \mathbb{C}$ and $P\mathcal{V} = 0$, we rewrite (42) as

$$0 = P[B(\varepsilon,\omega,\sigma)-\lambda](\beta\phi_0+\mathcal{V}) \quad \text{and} \quad 0 = (I-P)[B(\varepsilon,\omega,\sigma)-\lambda](\beta\phi_0+\mathcal{V}). \tag{81}$$

By recalling $\sigma_1^2 + \sigma_2^2 \approx 1$ in $\mathcal{R}_3$, the second equation of (81) can be solved locally for $\mathcal{V}$:

$$\mathcal{V}(\varepsilon,\sigma,\lambda,\beta) = \beta\varepsilon(h_1 e^{i\xi} + h_2 e^{-i\xi}) + \beta\mathcal{O}(\varepsilon^2) \tag{82}$$

in a neighborhood of $(\varepsilon,\lambda,\beta) = (0,0,(0,0))$ and $\sigma_1^2+\sigma_2^2 \approx 1$, where

$$h_1 = \frac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} \cdot \frac{1}{(2\sigma_1+\sigma_1^2+\sigma_2^2)^2+\lambda},$$
$$h_2 = \frac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}} \cdot \frac{1}{(2\sigma_1-\sigma_1^2-\sigma_2^2)^2+\lambda}.$$

Recalling the Bloch operator (39), for computational convenience in $\mathcal{R}_3$, we use polar coordinates

$$(\sigma_1,\sigma_2) = \sqrt{1+d}\Big(\frac{1}{\sqrt{1+2\varepsilon\omega}}\cos\theta, \sin\theta\Big), \tag{83}$$

where $|d| \le \delta$ and $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3} - \delta]$ for small $\delta > 0$, so that h_1 and h_2 can be rewritten as

$$h_1 = \frac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}\Big[\frac{1}{(2\cos\theta+1)^2} - \frac{\lambda}{(2\cos\theta+1)^4} + \mathcal{O}((|d|+|\varepsilon|)(1+|\lambda|)+|\lambda|^2)\Big] \tag{84}$$

and

$$h_2 = \frac{6b\sqrt{1-4\omega^2}}{\sqrt{27s-38b^2}}\Big[\frac{1}{(2\cos\theta-1)^2} - \frac{\lambda}{(2\cos\theta-1)^4} + \mathcal{O}((|d|+|\varepsilon|)(1+|\lambda|)+|\lambda|^2)\Big]. \tag{85}$$

Here, we notice that the denominator $2\cos\theta+1$ of h_1 is away from zero because $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3} - \delta]$.

By plugging (82)-(85) into the first equation of (81), we obtain the following scalar bifurcation equation in polar coordinates

$$\begin{aligned} 0 &= \varepsilon^2 - d^2 - \tfrac{54s-36b^2}{27s-38b^2}\varepsilon^2(1-4\omega^2) - \lambda \\ &\quad + \tfrac{36b^2\varepsilon^2(1-4\omega^2)}{27s-38b^2}\Big[\tfrac{1}{(2\cos\theta+1)^2} - \tfrac{\lambda}{(2\cos\theta+1)^4} + \tfrac{1}{(2\cos\theta-1)^2} - \tfrac{\lambda}{(2\cos\theta-1)^4}\Big] \\ &\quad + \mathcal{O}(\varepsilon^2(|d|(1+|\lambda|)+|\lambda|^2)+\varepsilon^3). \end{aligned} \tag{86}$$

In order to discuss the sign of λ, we arrange (86) as

$$\begin{aligned} &\Big[1 + \tfrac{36b^2\varepsilon^2(1-4\omega^2)}{27s-38b^2}\Big(\tfrac{1}{(2\cos\theta+1)^4} + \tfrac{1}{(2\cos\theta-1)^4}\Big)\Big]\lambda \\ &= \varepsilon^2 - d^2 + \Big[36b^2\Big(\tfrac{1}{(2\cos\theta+1)^2} + \tfrac{1}{(2\cos\theta-1)^2}\Big) - (54s-36b^2)\Big]\tfrac{\varepsilon^2(1-4\omega^2)}{27s-38b^2} \\ &\quad + \mathcal{O}(\varepsilon^2(|d|(1+|\lambda|)+|\lambda|^2)+\varepsilon^3). \end{aligned} \tag{87}$$

We see immediately from the facts $1-4\omega^2 \ge 0$ and $27s-38b^2 > 0$ that $\lambda > 0$ if and only if the right-hand side of (87) is positive.

We will focus on the sufficient conditions, uniformly for $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3} - \delta]$ and for $\omega \in [-\frac{1}{2}, \frac{1}{2}]$, of instability. Let us fix any $r > 1$ and substitute $\delta = \mathcal{O}(\varepsilon^r)$ into the right-hand side of (87) to derive conditions under which

$$\begin{aligned} \mathcal{D} &:= 1 - \mathcal{O}(\varepsilon^{2r-2}) \\ &\quad + \Big[36b^2\Big(\tfrac{1}{(2\cos\theta+1)^2} + \tfrac{1}{(2\cos\theta-1)^2}\Big) - (54s-36b^2)\Big]\tfrac{(1-4\omega^2)}{27s-38b^2} \\ &> 0. \end{aligned} \tag{88}$$

Upon setting

$$\mathcal{G} := 36b^2\Big(\frac{1}{(2\cos\theta+1)^2} + \frac{1}{(2\cos\theta-1)^2}\Big) - (54s-36b^2),$$

we notice that $\frac{1}{(2\cos\theta+1)^2} + \frac{1}{(2\cos\theta-1)^2} \ge 2$ for any $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3} - \delta]$ which implies that

$$\mathcal{D} = 1 - \mathcal{O}(\varepsilon^{2r-2}) + \mathcal{G}\frac{(1-4\omega^2)}{27s-38b^2} \ge 1 - \mathcal{O}(\varepsilon^{2r-2}) + (108b^2 - 54s)\frac{(1-4\omega^2)}{27s-38b^2}$$

for all $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3} - \delta]$. Therefore, if $\frac{38}{27}b^2 < s \le 2b^2$, then $\mathcal{D} > 0$ for small $\varepsilon > 0$ and all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. That is, in this case, there is a band of unstable $\sigma \in \mathcal{R}_3$ of width $\mathcal{O}(\varepsilon^r)$ uniformly for θ.

For the case $s > 2b^2$, it follows that the sufficient conditions of instability depend upon θ and ω. In particular, at $\theta = \frac{\pi}{2}$ and $d = 0$, i.e., at $\sigma = (0, 1)$,

$$\mathcal{D} = 1 + (108b^2 - 54s)\frac{1 - 4\omega^2}{27s - 38b^2}.$$

This gives that $\mathcal{D} > 0$ for all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ if $\frac{38}{27}b^2 < s \leq \frac{70}{27}b^2$, while if $s > \frac{70}{27}b^2$, $\mathcal{D} > 0$ for all $\omega \in [-\frac{1}{2}, -\frac{1}{2}\sqrt{\frac{70b^2-27s}{108b^2-54s}}) \cup (\frac{1}{2}\sqrt{\frac{70b^2-27s}{108b^2-54s}}, \frac{1}{2}]$. We state these observations in Theorem 4.3 below.

4.3. An eigenvalue problem of $B(\varepsilon, \omega, \sigma)$ for $\sigma \in \mathcal{R}_4$

In this region, similarly as in $\mathcal{R}_3$, the function $\phi_1(\xi) \equiv e^{i\xi}$ is only the critical eigenfunction of $B(0, \omega, \sigma)$ corresponding to a small eigenvalue $\mu_1(\sigma) = -(1 - (\sigma_1 + 1)^2 - \sigma_2^2)^2$. In order to obtain a scalar bifurcation equation, we define the orthogonal projection in $L^2_{per}([0, 2\pi])$ onto span$\{\phi_1(\xi)\}$

$$PW = \frac{1}{2\pi}\int_0^{2\pi} W\bar{\phi}_1 \, d\xi \phi_1 = \frac{1}{2\pi}\int_0^{2\pi} We^{-i\xi} \, d\xi e^{i\xi},$$

and we rewrite (42) as

$$0 = P[B(\varepsilon, \omega, \sigma) - \lambda](\beta\phi_1 + \mathcal{V}) \quad \text{and} \quad 0 = (I - P)[B(\varepsilon, \omega, \sigma) - \lambda](\beta\phi_1 + \mathcal{V}), \tag{89}$$

where $\beta \in \mathbb{C}$ and $P\mathcal{V} = 0$. Noting that $(\sigma_1 + 1)^2 + \sigma_2^2 \approx 1$ for $\sigma \in \mathcal{R}_4$, we again use polar coordinates

$$(\sigma_1 + 1, \sigma_2) = \sqrt{1 + d}\left(\frac{1}{\sqrt{1 + 2\varepsilon\omega}}\cos\theta, \sin\theta\right), \tag{90}$$

where $|d| \leq \delta$ and $\theta \in [\delta, \frac{\pi}{3} - \delta]$ for some small $\delta > 0$. Under the polar coordinates we repeat the procedure of $\mathcal{R}_3$ for this region to obtain the following bifurcation equation

$$\begin{aligned} &\left[1 + \tfrac{36b^2\varepsilon^2(1-4\omega^2)}{27s-38b^2}\left(\tfrac{1}{(2\cos\theta+1)^4} + \tfrac{1}{(2\cos\theta-1)^4}\right)\right]\lambda \\ &= \varepsilon^2 - d^2 + \left[36b^2\left(\tfrac{1}{(2\cos\theta+1)^2} + \tfrac{1}{(2\cos\theta-1)^2}\right) - (54s - 36b^2)\right]\tfrac{\varepsilon^2(1-4\omega^2)}{27s-38b^2} \\ &\quad + \mathcal{O}(\varepsilon^2(|d|(1 + |\lambda|) + |\lambda|^2) + \varepsilon^3) \end{aligned} \tag{91}$$

which is exactly the same as the bifurcation equation (87) for $\mathcal{R}_3$ except the range of θ. In this region, recalling (88), the sufficient conditions under which $\mathcal{D} > 0$ depend on $\theta \in [\delta, \frac{\pi}{3} - \delta]$ and $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. Indeed, for any $b \neq 0$ and s with $27s - 38b^2 > 0$ and any $\theta \in [\delta, \frac{\pi}{3} - \delta]$,

$$\mathcal{D} > 1 - \mathcal{O}(\varepsilon^{2r-2}) + (76b^2 - 54s)\frac{1 - 4\omega^2}{27s - 38b^2} = 1 - \mathcal{O}(\varepsilon^{2r-2}) - 2(1 - 4\omega^2) > 0$$

if $\omega^2 > \frac{1}{2\sqrt{2}} + \mathcal{O}(\varepsilon^{2r-2})$. This is consistent with the result (68) for small $\varepsilon > 0$ because $\sigma \in \mathcal{R}_2$ as $\theta \to 0$.

We summarize the observations of $\mathcal{S}_{\varepsilon,\omega}$ for two regions $\mathcal{R}_3$ and $\mathcal{R}_4$ in the following theorem.

Theorem 4.3. *Fix any $r > 1$. Then for sufficiently small $\varepsilon > 0$, the set of unstable wave vectors $\mathcal{S}_{\varepsilon,\omega}$ has the following properties.*

(a) *If $\frac{38}{27}b^2 < s \le 2b^2$, $\mathcal{R}_3 \subset \mathcal{S}_{\varepsilon,\omega}$ with $\delta = \mathcal{O}(\varepsilon^r)$ for all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$.*

(b) *$\sigma = (0,1) \in \mathcal{S}_{\varepsilon,\omega}$ for all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$ if $\frac{38}{27}b^2 < s \le \frac{70}{27}b^2$. Otherwise, $\sigma = (0,1) \in \mathcal{S}_{\varepsilon,\omega}$ for all $\omega \in [-\frac{1}{2}, -\frac{1}{2}\sqrt{\frac{70b^2-27s}{108b^2-54s}}) \cup (\frac{1}{2}\sqrt{\frac{70b^2-27s}{108b^2-54s}}, \frac{1}{2}]$.*

(c) *For any $b \neq 0$ and s with $27s - 38b^2 > 0$, $\mathcal{R}_4 \subset \mathcal{S}_{\varepsilon,\omega}$ with $\delta = \mathcal{O}(\varepsilon^r)$ for all $\omega^2 > \frac{1}{2\sqrt{2}} + \mathcal{O}(\varepsilon^{2r-2})$.*

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