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SEQUENTIAL OPTIMALITY THEOREMS FOR SECOND-ORDER CONE LINEAR FRACTIONAL VECTOR OPTIMIZATION PROBLEMS

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Abstract. In this paper, we consider a second-order cone linear fractional vector optimization problems (FVP), and obtain sequential optimality theorems for (FVP) which hold without any constraint qualification and which are expressed by sequences.

1. Preliminary Results and Introduction

Jeyakumar, Lee and Dinh [9] proved optimality theorems for convex optimization problem, which held without any constraint qualification and which were expressed in terms of sequences. We call the theorems as sequential optimality ones. Such optimality theorems have been studied for many kinds of convex optimization problems. In particular, Kim and Lee [11] studied sequential optimality conditions for efficient solutions of semidefinite linear fractional vector optimization problems. Kim, Kim and Lee [12] investigated sequential optimality theorems for weakly efficient solutions for semidefinite linear fractional vector optimization problems. Linear fractional vector optimization and pseudo linear fractional optimization were studied in [3, 4].

In this paper, we consider a second-order cone linear fractional vector optimization problem (FVP) and establish sequential optimality theorems for efficient solutions and properly efficient solutions which hold without any constraint qualification and which are expressed by sequences.

Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For a subset $D \subset X$, the closure of D, induced by the norm topology on X, is denoted by cl$D$.

Let C be a closed convex cone in X. Then the positive dual cone of C is defined by

$$C^* := \{z \in X \ : \ \langle x, z \rangle \geqq 0 \ \ \forall x \in C\}.$$

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The indicator function $\delta_A : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a function. The conjugate function of h, $h^* : X \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$h^*(v) := \sup\{\langle v, x\rangle - h(x) \ : \ x \in \text{dom}h\}$$

where $\text{dom}h := \{x \in X \mid h(x) < +\infty\}$.

The function h is said to be proper if h does not take the value $-\infty$ and $\text{dom}h \neq \emptyset$. The epigraph of the function h is defined by

$$\text{epi}h := \{(x, r) \in X \times \mathbb{R} \ : \ h(x) \leqq r\}.$$

Moreover if $\liminf_{y \to x} h(y) \geqq h(x)$ for all $x \in X$, we say that h is lower semicontinuous. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $\lambda \in [0, 1]$,

$$h(\lambda x + (1-\lambda)y) \leqq \lambda h(x) + (1-\lambda)h(y) \ \text{ for all } x, y \in X.$$

Lemma 1.1. [1] *Let* $h_1, h_2 : X \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semicontinuous convex functions. Then* $\text{epi}(h_1 + h_2)^* = \text{cl}(\text{epi } h_1^* + \text{epi } h_2^*)$. *If, in addition, one of* h_1 *and* h_2 *is continuous at some* $x_0 \in \text{dom } h_1 \cap \text{dom } h_2$, *then*

$$\text{epi}(h_1 + h_2)^* = \text{epi } h_1^* + \text{epi } h_2^*.$$

Lemma 1.2. [13] *Let* I *be an arbitrary index set and let* $h_i : X \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semicontinuous convex functions. Suppose that there exists* $x_0 \in X$ *such that* $\sup_{i \in I} h_i(x_0) < \infty$. *Then*

$$\text{epi}(\sup_{i \in I} h_i)^* = \text{clco} \bigcup_{i \in I} \text{epi } h_i^*$$

where $\sup_{i \in I} h_i : X \to \mathbb{R} \cup \{+\infty\}$ *is defined by* $(\sup_{i \in I} h_i)(x) = \sup_{i \in I} h_i(x)$ *for all* $x \in X$.

In this paper, we consider a second-order cone fractional vector optimization problem:

$$\begin{array}{lll} \text{(FVP)} \qquad & \text{Minimize} & \left(\dfrac{c_1^T x + \alpha_1}{d_1^T x + \beta_1}, \cdots, \dfrac{c_p^T x + \alpha_p}{d_p^T x + \beta_p}\right) \\ & \text{subject to} & x \in \{x \in \mathbb{R}^n \ : \ \|Hx + e\| \leqq a^T x + b\}, \end{array}$$

where $c_i, d_i \in \mathbb{R}^n,\ i = 1, \cdots, p$, $\alpha_i, \beta_i,\ i = 1, \cdots, p$, are given real numbers, H is an $(m-1) \times n$ matrix, $e \in \mathbb{R}^{m-1}$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ and $\|z\| = \sqrt{z^T z},\ z \in \mathbb{R}^{m-1}$.

Let $K = \{(y, t)^T \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|y\| \leqq t\}$, that is, K is a second-order cone in $\mathbb{R}^m$. Then K is self-dual, that is, $K^* = K$.

Define $\triangle := \{x \in \mathbb{R}^n \ : \ \|Hx+e\| \leqq a^Tx+b\}$. Then $\triangle = \{x \in \mathbb{R}^n \ : \left(\begin{array}{c} Hx+e \\ a^Tx+b \end{array}\right)^T \in K\}$. We assume that $\triangle \neq \emptyset$. Assume that $d_i^Tx+\beta_i > 0, \ i = 1, \cdots, p$, for any $x \in \triangle$. Let $q_i(x) := \frac{c_i^Tx+\alpha_i}{d_i^Tx+\beta_i}, \ i = 1, \cdots, p$, for any $x \in \triangle$.

Definition 1. (1) $\bar{x} \in \triangle$ is said to be an efficient solution for (FVP) if there exists no other feasible $x \in \triangle$ such that

$$\Big(\frac{c_1^Tx+\alpha_1}{d_1^Tx+\beta_1}, \cdots, \frac{c_p^Tx+\alpha_p}{d_p^Tx+\beta_p}\Big) \leq \Big(\frac{c_1^T\bar{x}+\alpha_1}{d_1^T\bar{x}+\beta_1}, \cdots, \frac{c_p^T\bar{x}+\alpha_p}{d_p^T\bar{x}+\beta_p}\Big).$$

(2) $\bar{x} \in \triangle$ is said to be a properly efficient solution for (FVP) if $\bar{x}$ is an efficient solution of (FVP) and there exists a constant $M > 0$ such that for each $i = 1, \cdots, p$, we have

$$\frac{\frac{c_i^T\bar{x}+\alpha_i}{d_i^T\bar{x}+\beta_i} - \frac{c_i^Tx+\alpha_i}{d_i^Tx+\beta_i}}{\frac{c_j^Tx+\alpha_j}{d_j^Tx+\beta_j} - \frac{c_j^T\bar{x}+\alpha_j}{d_j^T\bar{x}+\beta_j}} \leqq M$$

for some j such that $\frac{c_j^Tx+\alpha_j}{d_j^Tx+\beta_j} > \frac{c_j^T\bar{x}+\alpha_j}{d_j^T\bar{x}+\beta_j}$ whenever $x \in \triangle$ and $\frac{c_i^Tx+\alpha_i}{d_i^Tx+\beta_i} < \frac{c_i^T\bar{x}+\alpha_i}{d_i^T\bar{x}+\beta_i}$.

2. Sequential Optimality Theorems

Now we give the following necessary optimality theorem for the efficient solution and the properly efficient solution of (FVP):

Theorem 2.1. *Let $\lambda \in \mathbb{R}^p$ be such that $\lambda_i > 0, \ i = 1, \cdots, p$. Let $\bar{x} \in \triangle$. Then the following are equivalent:*

(i) $\bar{x}$ is an efficient solution of (FVP)*;*

(ii) there exist $\mu \in K, \ y_j \geqq 0, j = 1, \cdots, p,$

$$\begin{aligned}(\mathbf{0}, 0) \quad \in \quad & \Big\{\sum_{i=1}^p \lambda_i(c_i - q_i(\bar{x})d_i, -\alpha_i + q_i(\bar{x})\beta_i)\Big\} + \{0\} \times \mathbb{R}^+ \\ & + \mathrm{cl}\Big(\bigcup_{\mu \in K}\Big\{\Big(-\left(\begin{array}{c} H \\ a^T \end{array}\right)^T \mu, \left(\begin{array}{c} e \\ b \end{array}\right)^T \mu\Big)\Big\} \\ & + \bigcup_{y_j \geqq 0}\{\sum_{j=1}^p y_j(c_j - q_j(\bar{x})d_j, -\alpha_j + q_j(\bar{x})\beta_j)\} + \{0\} \times \mathbb{R}^+\Big).\end{aligned}$$

(iii) there exist $\mu^l \in K$ and $y_j^l \geqq 0, \ j = 1, \cdots, p$, such that

$$\sum_{i=1}^p \lambda_i(c_i - q_i(\bar{x})d_i) + \lim_{l \to \infty}\Big[-\left(\begin{array}{c} H \\ a^T \end{array}\right)^T \mu^l + \sum_{j=1}^p y_j^l(c_j - q_j(\bar{x})d_j)\Big] = \mathbf{0}$$

and $\lim_{l \to \infty} \left(\begin{array}{c} H\bar{x}+e \\ a^T\bar{x}+b \end{array}\right)^T \mu^l = 0.$

Proof. $((i) \Rightarrow (ii))$ Let $\bar{x}$ be an efficient solution of (FVP). Then $\bar{x}$ is an efficient solution of the following vector optimization problem:

$$\begin{array}{ll} \text{Minimize} & \Big(c_1^T x + \alpha_1 - q_1(\bar{x})(d_1^T x + \beta_1), \cdots, c_p^T x + \alpha_p - q_p(\bar{x})(d_p^T x + \beta_p)\Big) \\ \text{subject to} & x \in \{x \in \mathbb{R}^n \ : \ \|Hx + e\| \leqq a^T x + b\}. \end{array}$$

Then we can check that $\bar{x}$ is an optimal solution of the problem (P):

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^{p} \lambda_i \Big[c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i)\Big] \\ \text{subject to} & x \in \{x \in \mathbb{R}^n \ |\|Hx + e\| \leqq a^T x + b\}, \\ & c_j^T x + \alpha_j - q_j(\bar{x})(d_j^T x + \beta_j) \leqq 0, \quad j = 1, \cdots, p. \end{array}$$

Let $\triangle = \{x \in \mathbb{R}^n \ : \ \|Hx + e\| \leqq a^T x + b\}$ and $A = \{x \in \mathbb{R}^n \ : \ c_j^T x + \alpha_j - q_j(\bar{x})(d_j^T x + \beta_j) \leqq 0, \ j = 1, \cdots, p\}$. Let $\widetilde{\triangle} = \triangle \cap A$. Then $\delta_{\widetilde{\triangle}}(x) = \delta_{\triangle}(x) + \delta_A(x)$. By Lemma 1.1,

$$\text{epi}\delta^*_{\widetilde{\triangle}} = \text{cl}(\text{epi}\delta^*_{\triangle} + \text{epi}\delta^*_A).$$

Let $F(x) := \sum_{i=1}^{p} \lambda_i \Big[c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i)\Big]$. Since $\bar{x}$ is an optimal solution of (P), $F(x) + \delta_{\widetilde{\triangle}}(x) \geqq F(\bar{x}) + \delta_{\widetilde{\triangle}}(\bar{x})$ for any $x \in \triangle$. Thus $0 \cdot x - [F(x) + \delta_{\widetilde{\triangle}}(x)] \leqq 0$ for any $x \in \triangle$. Thus $(F + \delta_{\widetilde{\triangle}})^*(0) \leqq 0$,

$$(\mathbf{0}, 0) \in epi(F + \delta_{\widetilde{\triangle}})^* = epiF^* + epi\delta^*_{\widetilde{\triangle}}.$$

Using Lemma 1.2 and Theorem 4.1 in [10], we can check that

$$\begin{aligned} \text{epi}F^* &= \Big\{\sum_{i=1}^{m} \lambda_i (c_i - q_i(\bar{x}) d_i, -\alpha_i + q_i(\bar{x})\beta_i)\Big\} + \{0\} \times \mathbb{R}^+, \\ \text{epi}\delta^*_{\triangle} &= \text{cl}\Big(\bigcup_{\mu \in K} \Big\{\Big(-\begin{pmatrix} H \\ a^T \end{pmatrix}^T \mu, \begin{pmatrix} e \\ b \end{pmatrix}^T \mu\Big)\Big\} + \{0\} \times \mathbb{R}^+\Big) \\ \text{epi}\delta^*_A &= \text{cl}\Big[\bigcup_{y_j \geqq 0} \{\sum_{j=1}^{p} y_j (c_j - q_j(\bar{x}) d_j, -\alpha_j + q_i(\bar{x})\beta_j)\} + \{0\} \times \mathbb{R}^+\Big]. \end{aligned}$$

Thus (ii) holds.

($(ii) \Rightarrow (iii)$) Suppose that (ii) holds. Then there exist $\mu^l \in K$, $r \in \mathbb{R}^+$ and $r^l \in \mathbb{R}^+$ such that

$$\sum_{i=1}^{p} \lambda_i (c_i - q_i(\bar{x}) d_i) + \lim_{l\to\infty} \Big[- \begin{pmatrix} H \\ a^T \end{pmatrix}^T \mu^l + \sum_{j=1}^{p} y_j^l (c_j - q_j(\bar{x}) d_j) \Big] = \mathbf{0} \quad (1)$$

$$\sum_{i=1}^{p} \lambda_i (-\alpha_i + q_i(\bar{x}) \beta_i) + r \quad (2)$$
$$+ \lim_{l\to\infty} \Big[- \begin{pmatrix} e \\ b \end{pmatrix}^T \mu^l + \sum_{j=1}^{p} y_j^l (-\alpha_j + q_j(\bar{x}) \beta_j) + r^l \Big] = 0.$$

From (2.1),

$$0 = \sum_{i=1}^{p} \lambda_i (c_i - q_i(\bar{x}) d_i) \bar{x} \quad (3)$$
$$+ \lim_{l\to\infty} \Big[- \begin{pmatrix} H \\ a^T \end{pmatrix}^T \mu^l + \sum_{j=1}^{p} y_j^l (c_j - q_j(\bar{x}) d_j) \Big]^T \bar{x}$$

From (2.2) and (2.3),

$$0 = -r - \lim_{l\to\infty} \Big[\begin{pmatrix} H\bar{x} + e \\ a^T \bar{x} + b \end{pmatrix}^T \mu^l + r^l \Big].$$

Since $r^l \geqq 0$ and $r \geqq 0$, we have

$$r = 0, \ \lim_{l\to\infty} \begin{pmatrix} H\bar{x} + e \\ a^T \bar{x} + b \end{pmatrix}^T \mu^l = 0 \text{ and } \lim_{l\to\infty} r^l = 0.$$

Thus (iii) holds.

($(iii) \Rightarrow (i)$) Suppose that (iii) holds. Then for any $x \in \triangle$,

$$\sum_{i=1}^{p} \lambda_i \Big[c_i^T (x - \bar{x}) - q_i(\bar{x}) d_i^T (x - \bar{x}) \Big]$$
$$+ \lim_{l\to\infty} \Big[- \begin{pmatrix} H(x - \bar{x}) \\ a^T (x - \bar{x}) \end{pmatrix}^T \mu^l + \sum_{j=1}^{p} y_j^l \Big(c_j^T (x - \bar{x}) - q_i(\bar{x}) d_j^T (x - \bar{x}) \Big) \Big] = 0.$$

So, for any $x \in \triangle$,

$$\begin{aligned}&\sum_{i=1}^{p} \lambda_i(c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i)) + \lim_{l\to\infty}\Big[-\begin{pmatrix} Hx+e \\ a^T x + b\end{pmatrix}^T \mu^l \\ &\quad + \sum_{j=1}^{p} y_j^l (c_j^T x + \alpha_j - q_j(\bar{x})(d_j^T x + \beta_j))\Big] - \sum_{i=1}^{p} \lambda_i(c_i^T \bar{x} + \alpha_i - q_i(\bar{x})(d_i^T \bar{x} + \beta_i)) \\ &\quad - \lim_{l\to\infty}\Big[-\begin{pmatrix} H\bar{x}+e \\ a^T \bar{x} + b\end{pmatrix}^T \mu^l + \sum_{j=1}^{p} y_j^l (c_j^T \bar{x} + \alpha_j - q_j(\bar{x})(d_j^T \bar{x} + \beta_j))\Big] \\ &\quad = 0.\end{aligned}$$

Since $c_j^T \bar{x} + \alpha_j - q_j(\bar{x})(d_j^T \bar{x} + \beta_j) + \lim_{l\to\infty}\Big[-\begin{pmatrix} H\bar{x}+e \\ a^T \bar{x} + b\end{pmatrix}^T \mu^l\Big] = 0$ and $c_j^T x + \alpha_j - q_j(\bar{x})(d_j^T x + \beta_j) + \lim_{l\to\infty}\Big[-\begin{pmatrix} Hx+e \\ a^T x + b\end{pmatrix}^T \mu^l\Big] \leqq 0$ for any $x \in \widetilde{\triangle}$, we have, for any $x \in \widetilde{\triangle}$,

$$\sum_{i=1}^{p} \lambda_i \Big[c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T \bar{x} + \beta_i)\Big] \geqq \sum_{i=1}^{p} \lambda_i \Big[c_i^T \bar{x} + \alpha_i - q_i(\bar{x})(d_i^T \bar{x} + \beta_i)\Big].$$

Thus $\bar{x}$ is an optimal solution of the problem (P). Then we can check that $\bar{x}$ is an efficient solution of (FVP). Hence (i) holds. □

We give an assumption for (FVP):

Assumption C: $\Big\{\frac{d_i^T x+\beta_i}{d_j^T x+\beta_j} \mid i \neq j,\ x \in \triangle\Big\}$ is bounded above.

Following Proposition 2.3 in [11], we can obtain the following proposition:

Proposition 2.2. *The Assumption C is satisfied. Let $\bar{x} \in \triangle$. Then the following are equivalent:*

(i) $\bar{x}$ is a properly efficient solution of (FVP);

(ii) $\bar{x}$ is a properly efficient solution of the following linear vector optimization problem:

(LVP)

Minimize	$\big(c_1^T x + \alpha_1 - q_1(\bar{x})(d_1^T x + \beta_1), \cdots, c_p^T x + \alpha_p - q_p(\bar{x})(d_p^T x + \beta_p)\big)$
subject to	$x \in \triangle$.

Proof. Let $\bar{x} \in \triangle$. Then the following are equivalent:

(i) $\bar{x}$ is a properly efficient solution of (FVP).

(ii) (1) there does not exist $x \in \triangle$ such that $\frac{c_i^T x+\alpha_i}{d_i^T x+\beta_i} \leqq \frac{c_i^T \bar{x}+\alpha_i}{d_i^T \bar{x}+\beta_i}$ for all $i = 1, \cdots, p$ and $\frac{c_j^T x+\alpha_j}{d_j^T x+\beta_j} < \frac{c_j^T \bar{x}+\alpha_j}{d_j^T \bar{x}+\beta_j}$ for some $j \in \{1, \cdots, p\}$.

(2) there exists $M > 0$ such that, for each $i = 1, \cdots, p$ and each $x \in \triangle$ satisfying $\frac{c_i^T x+\alpha_i}{d_i^T x+\beta_i} < \frac{c_i^T \bar{x}+\alpha_i}{d_i^T \bar{x}+\beta_i}$, there exists for each $j \in \{1, \cdots, p\}$ such that

$\frac{c_j^T\bar{x}+\alpha_j}{d_j^T\bar{x}+\beta_j} < \frac{c_j^T x+\alpha_j}{d_j^T x+\beta_j}$ and

$$\frac{\frac{c_i^T\bar{x}+\alpha_i}{d_i^T\bar{x}+\beta_i} - \frac{c_i^T x+\alpha_i}{d_i^T x+\beta_i}}{\frac{c_j^T x+\alpha_j}{d_j^T x+\beta_j} - \frac{c_j^T\bar{x}+\alpha_j}{d_j^T\bar{x}+\beta_j}} \leqq M.$$

(iii) (1) there does not exist $x \in \triangle$ such that

$$c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i) \leqq 0 = c_i^T \bar{x} + \alpha_i - q_i(\bar{x})(d_i^T \bar{x} + \beta_i)$$

for all $i = 1, \cdots, p$ and $c_j^T x+\alpha_j-q_j(\bar{x})(d_j^T x+\beta_j) < 0 = c_j^T\bar{x}+\alpha_j-q_j(\bar{x})(d_j^T\bar{x}+\beta_j)$, for some $j \in \{1, \cdots, p\}$.

(2) there exists $M > 0$ such that for each i and each $x \in \triangle$ satisfying $c_i x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i) < c_i^T\bar{x} + \alpha_i - q_i(\bar{x})(d_i^T\bar{x} + \beta_i)$, there exists for each $j \in \{1, \cdots, p\}$ such that $c_j^T\bar{x}+\alpha_j-q_j(\bar{x})(d_j^T\bar{x}+\beta_j) < c_j^T x+\alpha_j-q_j(\bar{x})(d_j^T x+\beta_j)$ and

$$\frac{(c_i^T\bar{x} + \alpha_i - q_i(\bar{x})(d_i^T\bar{x} + \beta_i)) - (c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i))}{(c_j^T x + \alpha_j - q_j(\bar{x})(d_j^T x + \beta_j)) - (c_j^T\bar{x} + \alpha_j - q_j(\bar{x})(d_j^T\bar{x} + \beta_j))} \leqq \frac{d_i^T x + \beta_i}{d_j^T x + \beta_j} M.$$

Thus the result holds. □

Following the proof of Theorem 2 in [7], and using Proposition 2.2, we can obtain the following proposition:

Proposition 2.3. *The Assumption C is satisfied. Let $\bar{x} \in \triangle$. Then the following are equivalent:*

(i) $\bar{x}$ is a properly efficient solution of (FVP);

(ii) there exist $\lambda_i > 0,\ i = 1, \cdots, p$ such that $\bar{x}$ is an optimal solution of the following linear optimization problem:

$$\text{(LP)} \qquad \text{Minimize} \quad \sum_{i=1}^{p} \lambda_i \left(c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i)\right)$$
$$\text{subject to} \quad x \in \triangle.$$

Using Proposition 2.3 and following the proof of Theorem 2.1, we can obtain the following optimality theorem for properly efficient solution of (FVP):

Theorem 2.4. *The Assumption C is satisfied. Let $\bar{x} \in \triangle$. Then the following are equivalent:*

(i) $\bar{x}$ is a properly efficient solution of (FVP);

(ii) there exist $\lambda_i > 0, i = 1, \cdots, p, (\sum_{i=1}^{p} \lambda_i = 1)$, $\mu \in K$ such that

$$(\mathbf{0}, 0) \in \Big\{\sum_{i=1}^{p} \lambda_i(c_i - q_i(\bar{x})d_i, -\alpha_i + q_i(\bar{x})\beta_i)\Big\} + \{0\} \times \mathbb{R}^+ + \mathrm{cl}\Big(\bigcup_{\mu\in K} \Big\{\Big(-\begin{pmatrix} H \\ a^T \end{pmatrix}^T \mu, \begin{pmatrix} e \\ b \end{pmatrix}^T \mu\Big)\Big\} + \{0\} \times \mathbb{R}^+\Big).$$

(iii) there exist $\lambda_i > 0, i = 1, \cdots, p, (\sum_{i=1}^{p} \lambda_i = 1)$*,* $\mu^l \in K$ *such that*

$$\sum_{i=1}^{p} \lambda_i (c_i - q_i(\bar{x}) d_i) + \lim_{l \to \infty} \Big[- \begin{pmatrix} H \\ a^T \end{pmatrix}^T \mu^l \Big] = \mathbf{0}$$

$$\textit{and } \lim_{l \to \infty} \begin{pmatrix} H\bar{x} + e \\ a^T \bar{x} + b \end{pmatrix}^T \mu^l = 0.$$

The following theorem, which is second-order cone programming version of the Isermann's result [6, 8, 11, 14], gives a sufficient condition that an efficient solution of (FVP) can be properly efficient solution of (FVP).

Theorem 2.5. *The Assumption C is satisfied. Let* $\bar{x} \in \triangle$*. Assume that*

$$\bigcup_{\mu \in K} \left\{ \left(- \begin{pmatrix} H \\ a^T \end{pmatrix}^T \mu, \begin{pmatrix} e \\ b \end{pmatrix}^T \mu \right) \right\}$$
$$+ \bigcup_{y_j \geqq 0} \{ \sum_{j=1}^{p} y_j (c_j - q_j(\bar{x}) d_j, -\alpha_j + q_j(\bar{x}) \beta_j) \} + \{0\} \times \mathbb{R}_+$$

is closed. If $\bar{x}$ *is an efficient solution of (FVP), then* $\bar{x}$ *is a properly efficient solution of (FVP).*

Proof. Let $\lambda_i > 0, i = 1, \cdots, p, (\sum_{i=1}^{p} \lambda_i = 1)$. Let $\bar{x}$ is an efficient solution of (FVP). By Theorem 2.1,

$$\begin{aligned} (\mathbf{0}, 0) \quad \in \quad & \{ \sum_{i=1}^{p} \lambda_i (c_i - q_i(\bar{x}) d_i, -\alpha_i + q_i(\bar{x}) \beta_i) \} + \{0\} \times \mathbb{R}^+ \\ & + \mathrm{cl} \Big[\bigcup_{\mu \in K} \left\{ \left(- \begin{pmatrix} H \\ a^T \end{pmatrix}^T \mu, \begin{pmatrix} e \\ b \end{pmatrix}^T \mu \right) \right\} \\ & + \bigcup_{y_j \geqq 0} \{ \sum_{j=1}^{p} y_j (c_j - q_j(\bar{x}) d_j, -\alpha_j + q_j(\bar{x}) \beta_j) \} + \{0\} \times \mathbb{R}_+ \Big]. \end{aligned}$$

By assumption,

$$\begin{aligned} (\mathbf{0}, 0) \quad \in \quad & \sum_{i=1}^{p} \lambda_i (c_i - q_i(\bar{x}) d_i, -\alpha_i + q_i(\bar{x}) \beta_i) \\ & + \bigcup_{\mu \in K} \left\{ \left(- \begin{pmatrix} H \\ a^T \end{pmatrix}^T \mu, \begin{pmatrix} e \\ b \end{pmatrix}^T \mu \right) \right\} \\ & + \bigcup_{y_j \geqq 0} \{ \sum_{j=1}^{p} y_j (c_j - q_j(\bar{x}) d_j, -\alpha_j + q_j(\bar{x}) \beta_j) \} + \{0\} \times \mathbb{R}_+. \end{aligned}$$

So, there exist $\bar{\mu} \in K$ and $\bar{y}_j \geqq 0,\ j = 1, \cdots, p$, and $\bar{r} \geqq 0$ such that

$$\mathbf{0} = \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(c_i - q_i(\bar{x})d_i) - \begin{pmatrix} H \\ a^T \end{pmatrix}^T \bar{\mu} \tag{4}$$

$$\text{and} \quad 0 = \sum_{i=1}^{m} (\lambda_i + \bar{y}_i)(-\alpha_i + q_i(\bar{x})\beta_i) + \begin{pmatrix} e \\ b \end{pmatrix}^T \bar{\mu} + \bar{r}. \tag{5}$$

From (2.4), for any $x \in \triangle$,

$$0 = \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(c_i - q_i(\bar{x})d_i)^T x - \begin{pmatrix} Hx \\ a^T x \end{pmatrix}^T \bar{\mu}. \tag{6}$$

From (2.5) and (2.6),

$$\begin{aligned} \bar{r} &= \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i)) - \begin{pmatrix} Hx + e \\ a^T x + b \end{pmatrix}^T \bar{\mu} \\ &\leqq \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i)). \end{aligned}$$

Since $\bar{r} \geqq 0$, for any $x \in \triangle$,

$$\begin{aligned} &\sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(c_i^T x + \alpha_i - q_i(\bar{x})(d_i^T x + \beta_i)) \\ &\geqq 0 = \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(c_i^T \bar{x} + \alpha_i - q_i(\bar{x})(d_i^T \bar{x} + \beta_i)). \end{aligned}$$

Since $\lambda_i + \bar{y}_i > 0$, by Proposition 2.3, $\bar{x}$ is a properly efficient solution of (FVP). □

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