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NOTE ON THE UNIQUENESS OF A UNIVERSAL SEP-COVER

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Abstract. In the case that the set of sorts is countable, we will prove that any sorted profinite group has the unique SEP-cover using the notion of prime model in model theory.

We say that a profinite group G has the *embedding property* (EP) if for finite quotient groups A and B of G and for epimorphisms $f : G \to A$ and $g : B \to A$, there is an epimorphism $h : G \to B$ such that $f = g \circ h$. Given a profinite group G, an *embedding cover* of G is an epimorphism $f : H \to G$ with H having EP and a *universal embedding cover* of G is an embedding cover $f : G' \to G$ such that given an embedding cover $g : H \to G$, there is an epimorphism $h : H \to G'$ with $g = f \circ h$. In [1], Chatzidakis showed that the theory of the complete system of a profinite group having EP is ω-categorical and ω-stable. Most of all, using the prime model of a ω-stable theory, Chatzidakis showed that any profinite group has the unique universal embedding cover under the assumption that any finite group has the unique universal embedding cover, and the later assumption was already proved by Haran and Lubotzky in [3]. To my knowledge, Chatzidakis' proof is the only known proof of the uniqueness for the embedding cover of an arbitrary profinite group.

In [2], we suggested an idea of sorted profinite group to model the Galois group of a first order structure. The *sorted profinite group* is a profinite group equipped with some additional data, called the *sorting data*, on each open normal subgroup. In the case of the Galois group of a first order structure, the sorting data encodes the locations of generators of finite Galois extensions of the given first order structure (c.f. [4, Subsection 3.1]). In [5], we introduced the notion of sorted embedding property (SEP) for sorted profinite groups analogous to EP for profinite groups. Most of all, we characterized sorted profintie groups having SEP explicitly and using this characterization, we showed that

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any sorted profinite group has the unique universal SEP-cover (see [5, Theorem 2.22, Theorem 3.17]).

In this article, we aim to give a model theoretic proof of the uniqueness of sorted embedding cover along the idea of Chatzidakis when the set of sorts is countable.

1. Preliminaries

1.1. Sorted profinite groups

We write $\mathcal{N}(G)$ for the set of open normal subgroups of a profinite group G and we write $\mathbb{N}$ for the set of positive integers. **Fix a set $\mathcal{J}$, called a set of *sorts*.**

Definition 1. (1) For $n \in \mathbb{N}$, let $\mathcal{J}^n$ be the set of n-tuples of elements in $\mathcal{J}$, and let $\mathcal{J}^{<\mathbb{N}} := \bigcup_{n\in\mathbb{N}} \mathcal{J}^n$.
(2) For $J, J' \in \mathcal{J}^{<\mathbb{N}}$, we write $J^\frown J'$ for the concatenation of J and J'.
(3) For $J = (j_1, \ldots, j_n) \in \mathcal{J}^n$, $\|J\| := \{j_1, \ldots, j_n\}$.
(4) For $J \in \mathcal{J}^{<\mathbb{N}}$, $\sqrt{J} := \{J' \in \mathcal{J}^{<\mathbb{N}} : \|J\| \subseteq \|J'\|\}$.

Fix two functions

- $J^*_{\subseteq} : \mathbb{N} \times \mathcal{J}^{<\mathbb{N}} \to \mathcal{J}^{<\mathbb{N}}$**; and**
- $J^*_{\cap} : \mathcal{J}^{<\mathbb{N}} \times \mathcal{J}^{<\mathbb{N}} \to \mathcal{J}^{<\mathbb{N}}, (J, J') \mapsto J^\frown J'$**.**

Note the function $J^*_{\subseteq}$ can be arbitrary.

Definition 2. For a profinite group G, we associate a non-empty subset $F(N)$ of $\mathcal{J}^{<\mathbb{N}}$ for each $N \in \mathcal{N}(G)$ and consider an indexed family $F := \{F(N) : N \in \mathcal{N}(G)\}$. We say that the indexed family F is a *sorting data* of G if the following hold: For $N, N_1, N_2 \in \mathcal{N}(G)$,

(1) $F(G) = \mathcal{J}^{<\mathbb{N}}$.
(2) $J \in F(N) \Leftrightarrow \sqrt{J} \subseteq F(N)$;
(3) Suppose $N_1 \subseteq N_2$ and $[G : N_1] \leq k$. For $J \in \mathcal{J}^{<\mathbb{N}}$,

$$J \in F(N_1) \Rightarrow J^*_{\subseteq}(k, J) \in F(N_2).$$

(4) For $J_1 \in F(N_1)$ and $J_2 \in F(N_2)$, $J^*_{\cap}(J_1, J_2) \in F(N_1 \cap N_2)$.

We call the pair (G, F) a *sorted profinite group* and we say that the sorting data F comes from $\mathcal{J}$.

From now on, we consider only sorted profinite groups whose sorting data come from $\mathcal{J}$. Given two sorted profinite group (G_1, F_1) and (G_2, F_2), a *morphism* from (G_1, F_1) to (G_2, F_2) is a surjective continuous homomorphism $f : G_1 \to G_2$ such that for each $N \in \mathcal{N}(G_2)$,

$$F_2(N) \subseteq F_1(f^{-1}[N]).$$

1.2. Sorted embedding property

For a profinite group G, a subset $\mathcal{B}$ of $\mathcal{N}(G)$ is called a *base* if for any $N \in \mathcal{N}(G)$, there is $N' \in \mathcal{B}$ such that $N' \subset N$, and a subset X of $\mathcal{N}(G)$ *generates a base* if the set $\mathcal{B}(X) := \{N_1 \cap \cdots \cap N_k : N_i \in X\}$ forms a base.

We say that a sorted profinite group (G, F) is *finitely sorted* if the sorting data F on G is *finitely generated*, that is, there is a finite subset X of $\mathcal{N}(G)$ generating a base and there is a finite subset $F_X(N)$ of $\mathcal{J}^{<\mathbb{N}}$ for each $N \in X$ such that F is the smallest sorting data on G with $F_X(N) \subseteq F(N)$ for each $N \in X$.

For a sorted profinite group (G, F) and for a pair $((A, F_A), (B, F_B))$ of quotients of (G, F), the *sorted embedding condition* $\mathrm{Emb}_{(G,F)}((A, F_A), (B, F_B))$ is defined as follows: For every pair of morphisms $\Pi : (A, F_A) \to (B, F_B)$ and $\varphi : (G, F) \to (B, F_B)$, there is a morphism $\psi : (G, F) \to (A, F_A)$ such that $\Pi \circ \psi = \varphi$. Let $\mathrm{SIm}(G, F)$ be the family of all sorted finite groups which are quotients of (G, F), and let $\mathrm{FSIm}(G, F)$ be the family of all sorted finite groups which are quotients of (G, F) with the sorting data finitely generated.

Definition 3. A sorted profinite group (G, F) has the *(finitely) sorted embedding property*, in short, (F)SEP, if for all $(A, F_A), (B, F_B) \in \mathrm{SIm}(G, F)$ $(\in \mathrm{FSIm}(G, F))$, the condition $\mathrm{Emb}_{(G,F)}((A, F_A), (B, F_B))$ holds.

Remark 1. [5, Theorem 2.22] For a sorted profinite group (G, F), the following are equivalent.

(1) (G, F) has SEP.
(2) (G, F) has FSEP.
(3) G has EP and for all $N, N' \in \mathcal{N}(G)$ with $G/N \cong G/N'$, $F(N) = F(N')$.

For a sorted profinite group (G, F), a morphism $p : (H, F_H) \to (G, F)$ is called a *SEP-cover* if the sorted profinite group (H, F_H) has SEP. A SEP-cover $p : (H, F_H) \to (G, F)$ is *universal* if for any SEP-cover $r : (K, F_K) \to (G, F)$, there is a morphism $q : (K, F_K) \to (H, F_H)$ such that $p \circ q = r$. For a SEP-cover $p : (H, F_H) \to (G, F)$, the domain (H, F_H) is also called a SEP-cover of (G, F). In [5], we showed that any sorted profinite group has the unique universal SEP-cover and using Remark 1, we could describe the sorting data of the universal SEP-cover explicitly.

Remark 2. [5, Theorem 3.17] A sorted profinite group (G, F) has the unique universal SEP-cover $p : (E(G), E(F)) \to (G, F)$ up to isomorphism, and if G is finite, then so is $E(G)$. Also, the universal SEP-cover $p : (E(G), E(F)) \to (G, F)$ satisfies the following:

- $p : E(G) \to G$ is the unique universal EP-cover of G, and

- the sorting data $E(F)$ is given as follows: For each $(A, F_A) \in \mathrm{FSIm}(G, F)$, let $(E(A), E(F_A))$ be the unique universal SEP-cover, and let

$$\Gamma := \bigcup_{(A,F_A)\in \mathrm{FSIm}(G,F)} \mathrm{FSIm}\left(E(A), E(F_A)\right).$$

Then, for each $N \in \mathcal{N}(G)$,

$$E(F)(N) = \bigcup_{(B,F_B)\in\Gamma, B\cong G/N} F_B(\{e_B\}),$$

where e_B is the identity of B.

1.3. Sorted complete system

For a sorted profinite group, we associate a dual object called a *sorted complete system.* The sorted complete system is a first order structure to encode the inverse system of finite quotients of the sorted profinite group by its open normal subgroups.

Consider the following first order language $\mathcal{L}_{SCS}(\mathcal{J})$ with the sorts $m(k, J)$ for each $(k, J) \in \mathbb{N} \times \mathcal{J}^{<\mathbb{N}}$ together with

- a family of binary relations $\leq_{k,k',J,J'}$ and $C_{k,k',J,J'}$; and
- a family of ternary relations $P_{k,J}$.

For a sorted profinite group (G, F), the sorted complete system $\mathcal{S}(G, F)$ is a $\mathcal{L}_{SCS}(\mathcal{J})$-structure given as follows:

- For $(k, J) \in \mathbb{N} \times \mathcal{J}^{<\mathbb{N}}$,

$$m(k, J) := \bigcup_{N\in\mathcal{N}(G),[G:N]\leq k,J\in F(N)} G/N \times \{k\}.$$

- For $(k, J), (k', J') \in \mathbb{N} \times \mathcal{J}^{<\mathbb{N}}$,

$$\leq_{k,k',J,J'} := \{((gN, k), (g'N', k')) \in m(k, J) \times m(k', J') : k \geq k', N \subseteq N'\}.$$

- For $(k, J), (k', J') \in \mathbb{N} \times \mathcal{J}^{<\mathbb{N}}$,

$$C_{k,k',J,J'} := \{((gN, k), (g'N', k')) \in m(k, J) \times m(k', J') : k \geq k', gN \subseteq g'N'\}.$$

- For $(k, J) \in \mathbb{N} \times \mathcal{J}^{<\mathbb{N}}$,

$$P_{k,J} = \{((g_1N, k), (g_2N, k), (g_3N, k)) \in m(k, J)^3 : g_3N = g_1g_2N\}.$$

If there is no confusion, we omit the subscripts and write $\leq$,C, and P. We also write gN for (gN, k). Sorted complete systems are axiomatized by a $\mathcal{L}_{SCS}(\mathcal{J})$-theory, SCS (c.f. [4, Definition 3.7]).

Conversely, any model S of SCS is a sorted complete system of a sorted profinite group, denoted by $\mathcal{G}(S) := (G(S), F(S))$. Let $\sim$ be the equivalence relation on S given as follows: For $a, b \in S$,

$$a \sim b \Leftrightarrow a \leq b \wedge b \leq a.$$

For $a \in S$, let $[a]$ be the $\sim$-class of a. Then, for each $a \in m(k, J)$, $[a] \cap m(k, J)$ forms a group whose group operation is induced from P. The profinite group

$G(S)$ is the inverse limit of the group $[a] \cap m(k, J)$ with the transition maps induced from C. Note that for each $N \in \mathcal{N}(G(S))$, there is $a \in m(k, J)$ such that N is the kernel of the projection from $G(S)$ to $[a] \cap m(k, J)$. In this case, we denote N by N_a. We now associate the sorting data $F(S)$ on $G(S)$ as follows: For $N \in \mathcal{N}(G(S))$ and $J \in \mathcal{J}^{<\mathbb{N}}$,

$$J \in F(S)(N) \Leftrightarrow \exists a \in m(k, J)(N = N_a).$$

Then, the sorted complete system of $(G(S), F(S))$ is naturally isomorphic to S.

Most of all, the associations $\mathcal{S}$ and $\mathcal{G}$ define contravariant functors to make the category of sorted profinite groups and the category of sorted complete systems whose morphisms are $\mathcal{L}_{SCS}(\mathcal{J})$-embeddings equivalent (see [4, Section 3.2]).

Now, we recall the notion of co-sorted embedding property (co-SEP) for the sorted complete system which is dual to SEP for the sorted profinite group. A substructure of a sorted complete system is called a *subsystem* if it is a model of SCS, that is, it is also a sorted complete system.

Definition 4. [5, Definition 4.8] We say that a sorted complete system S has *co-sorted embedding property* (co-SEP) if for any finitely generated subsystems $S_1, S_2 \subseteq S$, and for any embeddings $\Pi : S_2 \to S_1$ and $\Phi : S_2 \to S_2$, there is an embedding $\Psi : S_1 \to S$ to make the following diagram commutative

$$\begin{array}{ccc} S_2 & \xrightarrow{\Pi} & S_1 \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ S_2 & \xrightarrow[\iota]{} & S \end{array}$$

where ι is the inclusion.

Remark 3. [5, Remark 4.9] The class of sorted complete system having co-SEP forms an elementary class and write SCS_{SEP} for the $\mathcal{L}_{SCS}(\mathcal{J})$-theory of sorted complete systems having co-SEP.

Given a sorted complete system S, we write $\mathrm{coFSIm}(S)$ for the family of all sorted complete systems isomorphic to finitely generated subsystems of S. Then, if S has co-SEP, then the first order theory of S is determined by $\mathrm{coFSIm}(S)$.

Remark 4. [5, Theorem 4.13, Theorem 4.15]

(1) Given two sorted complete systems S_1 and S_2 having co-SEP,

$$S_1 \equiv S_2 \Leftrightarrow \mathrm{coFSIm}(S_1) = \mathrm{coFSIm}(S_2).$$

(2) Given a sorted complete system S having co-SEP, $\mathrm{Th}(S)$ is ω-stable if the set $\mathcal{J}$ of sorts is countable.

2. The uniqueness of a universal SEP-cover

In this section, we aim to give a proof for the uniqueness of a universal SEP-cover of a sorted profinite group in Fact 2 under the following conditions:

- The set $\mathcal{J}$ of sorts is countable. $(\dagger_1)$
- Any finitely sorted finite group has the unique universal SEP-cover. $(\dagger_2)$
- Any sorted profinite group has a universal SEP-cover. $(\dagger_3)$

Note that we proved the condition $(\dagger_3)$ in [5, Theorem 3.13] without using any model theoretic arguments. We first recall several notions and facts on subsystems from [5, Section 4].

Definition 5. Let S be a sorted complete system.

(1) For a subset X of S, there is the smallest subsystem S_X containing X and it is called the subsystem generated by X.
(2) A subsystem S' of S is finitely generated if $S' = S_X$ for a finite subset X of S.
(3) A subset X of S is called a *presystem* if the following hold:
   - For each $x \in X$ and for each $(k, J) \in \mathbb{N} \times \mathcal{J}^{<\mathbb{N}}$,
   $$x \in m(k, J) \Rightarrow [x] \cap m(k, J) \subseteq X.$$
   - For each $s \in S_X$, there is $x \in X$ with $x \leq s$.

Remark 5. [5, Remark 4.3, Remark 4.5, Lemma 4.11]

(1) Let S' be a subsystem generated by a subset X of a sorted complete system S. Then, we can take X as a presystem. Also, if X is finite, we can take X as a finite presystem.
(2) For a finite presystem X of a sorted complete system S, $S_X \subseteq \mathrm{dcl}(X)$ and there is a quantifier-free formula $\theta_X(\bar{x})$ such that
$$SCS \models \theta_X(\bar{x}) \to \mathrm{qftp}(X)$$
so that for a sorted complete system S' and for a finite subset X' of S', if $S' \models \theta_X(X')$, then X' is a finite presystem of S' and $S_{X'} \cong S_X$.
(3) A sorted complete system S is finitely generated if and only if the sorted profinite group $\mathcal{G}(S)$ is a finitely sorted finite group.
(4) Let S and S' be countable sorted complete systems having co-SEP with $\mathrm{coFSIm}(S) = \mathrm{coFSIm}(S')$. Any isomorphism f between finitely generated subsystems of S and S' can be extended into an isomorphism between S and S'.

Remark 6. Let S and S' be sorted complete systems having co-SEP with $\mathrm{coFSIm}(S) = \mathrm{coFSIm}(S')$. Any embedding from S to S' is an elementary embedding.

Proof. Let $f : S \to S'$ be an embedding. To show f is an elementary embedding, it is enough to show that for a formula $\psi(\bar{x})$ and for a finite tuple $\bar{a}$ in S,
$$S \models \psi(\bar{a}) \Leftrightarrow S' \models \psi(f(\bar{a})).$$

Let S_0 and S_0' be elementary countable substructures of S and S' containing $S_{\bar{a}}$ and $f[S_{\bar{a}}]$ respectively. Then, we have that

$$\mathrm{coFSIm}(S_0) = \mathrm{coFSIm}(S) = \mathrm{coFSIm}(S') = \mathrm{coFSIm}(S_0')$$

by Remark 4(1). Then, by Remark 5(4), the isomorphism $f\restriction_{S_{\bar{a}}}$ is extended into an isomorphism f_0 between S_0 and S_0'. Via the isomorphism f_0, we have

$$S \models \psi(\bar{a}) \Leftrightarrow S_0 \models \psi(\bar{a}) \Leftrightarrow S_0' \models \psi(f(\bar{a})) \Leftrightarrow S' \models \psi(f(\bar{a})).$$

□

Theorem 2.1. *Any sorted profinite group (H, F_H) has a universal SEP-cover (G, F) which is unique up to isomorphism over (H, F_H).*

Proof. Following the proof scheme of [1, Theorem 2.7], we will first identify the first order theory of $\mathcal{S}(G, F)$. Recall by ($\dagger_2$), each $(A, F_A) \in \mathrm{FSIm}(H, F_H)$ has the unique universal SEP-cover. Let Γ be the class of all finitely sorted finite groups which are images of the universal SEP-cover of (A, F_A) for some $(A, F_A) \in \mathrm{FSIm}(H, F_H)$. Let T be the theory given by

$$\begin{aligned} T = {} & SCS_{SEP} \cup \mathit{Diag}\,(\mathcal{S}(H, F_H)) \\ & \cup \{\exists X (S_X \cong \mathcal{S}(A, F_A)) : (A, F_A) \in \Gamma\} \\ & \cup \{\forall X (S_X \ncong \mathcal{S}(B, F_B) : (B, F_B) \notin \Gamma\}, \end{aligned}$$

which can be written as $\mathcal{L}_{SCS}(\mathcal{J})(\mathcal{S}(H, F_H))$-sentences by Remark 5(1) and (2).

Claim 1. The theory T is consistent and complete.

Proof. Let Y be a finite subset of $\mathit{Diag}(\mathcal{S}(H, F_H))$, and let $(A_1, F_1), \ldots, (A_n, F_n) \in \Gamma$ and $(B_1, F_1'), \ldots, (B_m, F_m') \notin \Gamma$. By Remark 5(1), there are finite subsets $X_1, \ldots, X_n$ of $S(H)$ such that (A_i, F_i) is an image of the universal SEP-cover of $\mathcal{G}(S_{X_i})$ for each i. Let $S' := S_{(\bigcup_{i\leq n} X_i)\cup Y}$ be a finitely generated subsystem of $S(H)$. Then, $\mathcal{G}(S')$ is in $\mathrm{FSIm}(H, F_H)$, and for the universal SEP-cover $(E', F_{E'})$ of $\mathcal{G}(S')$, which exists by ($\dagger_2$), $(A_1, F_1), \ldots, (A_n, F_n) \in \mathrm{FSIm}(E', F_{E'}) (\subseteq \Gamma)$ and $(B_1, F_1'), \ldots, (B_m, F_m') \notin \mathrm{FSIm}(E', F_{E'})$. Thus, T is finitely consistent and so it is consistent. Also, it is complete by Remark 4(1). □

Claim 2. For a model S of T, S is a prime model of T over $\mathcal{S}(H, F_H)$ if and only if $\mathcal{G}(S)$ is a universal SEP-cover of (H, F_H).

Proof. Fix a model S of T. Let $\mathcal{G}(S) := (G, F)$ and $\pi := \mathcal{G}(\iota) : (G, F) \to (H, F_H)$ be the morphism dual to the inclusion $\iota : \mathcal{S}(H, F_H) \to S$.

($\Rightarrow$) Suppose S is a prime model of T over $\mathcal{S}(H, F_H)$. Let $\pi' : (G', F') \to (H, F_H)$ be a universal SEP-cover, which exists by ($\dagger_3$). Since π is a SEP-cover, there is a morphism from (G, F) to (G', F') and so we have that $\Gamma \subseteq \mathrm{FSIm}(G', F') \subseteq \mathrm{FSIm}(G, F) = \Gamma$. Thus, $\mathrm{FSIm}(G', F') = \Gamma$ and $\mathcal{S}(G', F')$ is a model of T.

Let $q : (M, F_M) \to (H, F_H)$ be a SEP-cover. Since π' is a universal SEP-cover, there is $p : (M, F_M) \to (G', F')$ such that $q = \pi' \circ p$. Put $S'' := \Im(\mathcal{S}(p)) \subseteq S(M)$, which is a model of T. Since S is a prime model of T, there is an embedding $\Phi : S \to S''$ such that $\Phi(x) = \mathcal{S}(q)(x)$ for each $x \in \mathcal{S}(H, F_H)$. Consider an embedding $\iota'' \circ \Phi : S \to S'' \to \mathcal{S}(M, F_M)$ where $\iota'' : S'' \to \mathcal{S}(M, F_M)$ is the inclusion. Then, the dual map $\varphi := \mathcal{G}(\iota'' \circ \Phi) : (M, F_M) \to (G, F)$ gives a morphism such that $q = \pi \circ \varphi$ and so $\pi : (G, F) \to (H, F_H)$ is a universal SEP-cover.

($\Leftarrow$) Suppose $\pi := \mathcal{G}(\iota) : (G, F) \to (H, F_H)$ is a universal SEP-cover of (H, F_H). Let S^* be a model of T. We will show that there is an elementary embedding over $\mathcal{S}(H, F_H)$ from S to S^*. Since T contains $Diag\,(\mathcal{S}(H, F_H))$, there is an embedding $\iota^* : \mathcal{S}(H, F_H) \to S^*$ and so the dual map $\mathcal{G}(\iota^*) : \mathcal{G}(S^*) \to (H, F_H)$ is a SEP-cover of (H, F_H). Since π is a universal SEP-cover of (H, F_H), there is a morphism $\varphi^* : \mathcal{G}(S^*) \to (G, F)$ such that $\mathcal{G}(\iota^*) = \pi \circ \varphi^*$. By taking the contravariant functor $\mathcal{S}$, we have an embedding $\mathcal{S}(\varphi^*) : S\,(= \mathcal{S}(G, F)) \to S^*$ such that $\iota^*(x) = (\mathcal{S}(\varphi^*) \circ \iota)\,(x)$ for all x in $\mathcal{S}(H, F_H)$, which is the desired elementary embedding over $\mathcal{S}(H, F_H)$ by Remark 6. □

Therefore, we conclude that the dual sorted profinite group of any prime model of T over $\mathcal{S}(H, F_H)$ is a universal SEP-cover of (H, F_H). Conversely, the sorted complete system of a universal SEP-cover of (H, F_H) is a prime model of T over $\mathcal{S}(H, F_H)$. Note that there is a unique prime model of Y over an arbitrary set of parameters because T is ω-stable by $(\dagger_1)$ and Remark 4(2). So, all universal SEP-covers of (H, F_H) are isomorphic. □

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