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RESEARCH ARTICLE

Secondary Teachers' Perspectives on Mathematical Modeling and Modeling Mathematics: Discovery, Appreciation, and Conflict

Ahmad M. Alhammouri[1], Joseph DiNapoli[2]

[1]Associate Professor, Department of Teacher Education, Jacksonville State University
[2]Associate Professor, Department of Mathematics, Montclair State University

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Abstract

Recent international reform movements call for attention on modeling in mathematics classrooms. However, definitions and enactment principles are unclear in policy documents. In this case study, we investigated United States high-school mathematics teachers’ experiences in a professional development program focused on modeling and its enactment in schools. Our findings share teachers’ experiences around their discovery of different conceptualizations, appreciations, and conflicts as they envisioned incorporating modeling into classrooms. These experiences show how professional development can be designed to engage teachers with forms of modeling, and that those experiences can inspire them to consider modeling as an imperative feature of a mathematics program.

Keywords: mathematical modeling, modeling mathematics, professional development, modeling with algebra, modeling with geometry

• Corresponding Author: Ahmad M. Alhammouri, email: aalhammouri@jsu.edu

I. INTRODUCTION

The international mathematics education community has long understood the importance of including modeling as a focus of learning in school systems (Zawojewski & Phillips, 2016). Since the Common Core State Standards for Mathematics (CCSSM) were released in the United States in 2010, there has been increased attention worldwide on incorporating authentic experiences with mathematical modeling in K-12 classrooms. In fact, one of the major standards for practice in the CCSSM is MP.4: Model with mathematics. To achieve this standard for practice, students should be able to "apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (NGA Center & CCSSO, 2010, p. 7). Additionally, beyond this standard for practice, 40% (61 out of 156) of the high school mathematics content standards address mathematical modeling in some meaningful way (Foley et al., 2014). Because of this focus on mathematical modeling education, there has been a concerted effort to support high school mathematics teachers' own learning about mathematical modeling and its affordances in their classrooms (Gould, 2013). However, despite this increased recent attention on privileging modeling in K-12 mathematics education, its exact definition and principles for enactment are not always clear in the CCSSM and in the mathematics education literature (Cirillo et al., 2016; Felton-Koestler, 2017).

Furthermore, modeling has been conceptualized by international mathematics education researchers and policy makers in two distinct ways: mathematical modeling and modeling mathematics. Mathematical modeling is the process of using mathematics to solve real world problems that may not be inherently mathematical. The construct of mathematical modeling has been described as "using mathematics or statistics to describe (i.e., model) a real-world situation and deduce additional information about the situation by mathematical or statistical computation and analysis" (Common Core Standards Writing Team, 2013, p. 5). Alternatively, modeling mathematics is using concrete or virtual representations of mathematics to communicate mathematical concepts or ideas. The construct of modeling mathematics has been described as "any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed" (van de Walle, 2007, p. 31). Therefore, a distinguishing characteristic of these two terms is their process origin; that is, the process of mathematical modeling begins in the real world while the process of modeling mathematics begins in the mathematical world (Cirillo et al., 2016). Elevating both of these practices to a more central role in students' K-12 experiences can pay great dividends toward mathematics learning.

Although mathematical modeling and modeling mathematics have been distinctly defined by international researchers and policy makers, confusion persists about how modeling is incorporated into K-12 classrooms (Felton-Koestler, 2017). For teachers, it is often unclear the ways in which these two constructs differ, and the affordances of each in the classroom (Cirillo et al., 2016). Therefore, teachers need to be prepared to enact both mathematical modeling and modeling mathematics in their classrooms such that students can benefit from these processes. Engaging mathematics teachers in professional development (PD) programs that advance their capacity to enact both mathematical modeling and modeling mathematics effectively offers promise (Groshong, 2016; Gould, 2016). During such PD programs, teachers need to learn first-hand the difference between mathematical modeling and modeling mathematics, be prepared to select and design tasks that support mathematical modeling and modeling mathematics, and be prepared to monitor students' engagement in these tasks in such a way that engages them in deep real-world and mathematical thinking.

The purpose of this paper is to describe teachers' experiences in a United States' PD program that focused on learning about mathematical modeling and modeling mathematics for the first time and how those processes could be applied into their own classrooms. From the high school mathematics teachers' perspective, we will describe (a) teachers' experiences discovering first-hand what it means to be a mathematical modeler, (b) the aspects of mathematical modeling and modeling mathematics teachers most appreciated in the context of mathematics education, and (c) the conflicts that emerged for teachers as they meaningfully incorporated mathematical modeling and modeling mathematics in their own classrooms.

As such, in the context of participation in a PD program focused on mathematical modeling education, the three research questions that guided this study were:

1. What did high school mathematics teachers discover about the ways mathematical modeling and modeling mathematics can be conceptualized?
2. What did high school mathematics teachers appreciate about incorporating mathematical modeling and modeling mathematics into their classrooms?
3. What did high school mathematics teachers find conflicting about incorporating mathematical modeling and modeling mathematics into their classrooms?

II. RELATED LITERATURE

In this section, we discuss how mathematical modeling and modeling mathematics are alike and different, and present frameworks that depict both processes. In addition, we review aspects of the mathematics education research literature concerning these two terms. Then, we discuss how PD programs on modeling have influenced teachers' beliefs and instruction for teaching mathematics.

The major common aspect between mathematical modeling and modeling mathematics is that they both connect mathematics and the real world. Figure 1 depicts both of these processes. However, the major difference between them is where each process starts, where each process ends, and the goal of each process.

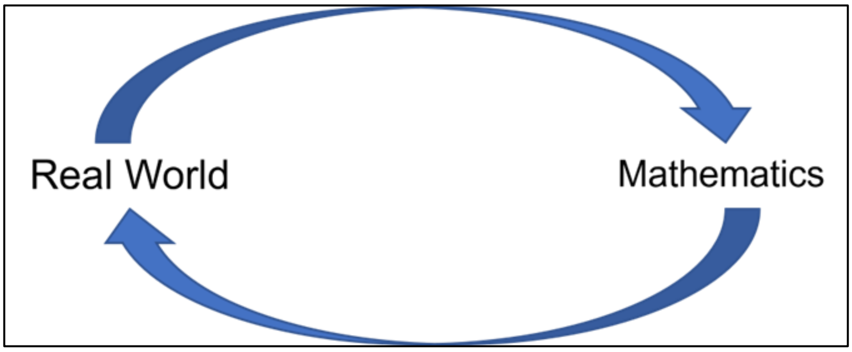


Figure 1. Representing mathematical modeling and modeling mathematics

In mathematical modeling, students start from the real world with a problem that needs to be solved using mathematics (Bliss et al., 2014). After students use mathematics to solve the problem, they go back to the real world they started with to make a decision. In contrast, modeling mathematics starts from the mathematical world to develop an understanding of mathematical concepts (van de Walle, 2007). Then, students use real-world objects and materials to develop a such understanding which leads them back to the mathematical world from where they started. Next, we review the history of each term and explain in more detail what each term means.

Background of Mathematical Modeling and Modeling Mathematics

Unlike modeling mathematics, mathematical modeling has historically gained a lot of attention, primarily in the United States. According to Pollak's (2003) historical account of mathematical modeling education, the focus on mathematical modeling started in 1949 when mathematician Richard Stevens Burington emphasized the importance of applied mathematics and connecting mathematics to the real world. While Burington defined and explained mathematical modeling, his work did not gain enough attention in the mathematics education field at that time.

In the 1950s, the focus on connecting mathematics with science and the real world started to occur in school mathematics. This was met with criticisms by various mathematicians who believed that mathematics curricula should focus on pure mathematics. In the 1960s, mathematical modeling was explicitly advocated to be addressed in school mathematics. Pollak (1966), a member of the School Mathematics Study Group's 9th grade writing team, emphasized the importance of mathematical modeling for school students. Pollak indicated that students must experience mathematics by enacting it to solve real-world problems. Pollak argued that if teachers want to reflect the true picture of mathematics, they need to present mathematical tasks in the form of "here is a situation—think about it," instead of "here is a problem—solve it" (Pollak, 1966, p. 117). Pollak added, "many teachers of mathematics have never been involved in the process of building mathematical models of situations in the outside world" (p. 122). Later that year, the School Mathematics Study Group held a conference to discuss new mathematics curricula for the high school level. During that conference, a report of the modeling committee was presented that emphasized the need to address mathematical modeling in school mathematics starting from 7th grade.

In 1970, mathematical modeling was addressed in a school mathematics textbook for the first time. It appeared as a topic of study in the textbook *Secondary School Mathematics*, published by Leland Stanford Junior University. In Chapter 4: Problem Formulation, mathematical modeling problems were characterized by the need to be strongly connected to the real world via real-world scenario, and being able to be solved using mathematics. Furthermore, the author team of *Secondary School Mathematics* made clear that a mathematical model is never a perfect representation of the real-world situation. Usually, many simplifying assumptions have been made before the mathematical model is finally constructed.

During the 1980s, the focus on mathematical modeling in mathematics education had increased. Mathematical modeling was addressed by several mathematics and mathematics education organizations, including the National Council of Teachers of Mathematics (NCTM), the Consortium for Mathematics and Its Applications (COMAP), and the Woodrow Wilson National Fellowship Foundation (WW). In 1980, NCTM published its 42nd yearbook on annual perspectives in mathematics education, which included a chapter focused on different types of mathematical problems (Butts, 1980/2013). In this chapter, Butts alluded to mathematical modeling in his discussion of "problem situations." In 1983, COMAP published several mathematical modeling projects as part of their *High School Mathematics and its Applications Project*. The course was one of the first books that addressed mathematical modeling for the school level. The book included various projects from pre-algebra and pre-calculus areas. In 1985, COMAP established an international competition called the *Mathematical Contest in Modeling*[1] (MCM), in which high school and undergraduate students collaboratively competed on open-ended and real-world mathematical modeling problems. In 1987, WW held a workshop for mathematics teachers that addressed mathematical modeling. This was one of the first workshops of its kind (Pollak, 2003) and, as a result of the workshop, some American school systems started to offer mathematical modeling courses in their high schools.

Also in the 1980s, the National Assessment of Educational Progress showed that mathematics students in the United States were achieving at a low level. In response to this, Lesh emphasized the importance of including mathematical modeling in school mathematics experiences, positing that these experiences may enhance students' achievement in mathematics (1981). As such, Lesh defined four features of the mathematical modeling process:

1. Simplifying the original situation by ignoring irrelevant characteristics in a real situation in order to focus on other characteristics;
2. establishing a mapping between the original situation and the model,
3. investigating the properties of the model in order to generate predictions about the original situation; and
4. translating (or mapping) the predictions back into the original situation and

[1] The Mathematical Contest in Modeling is still active today.

checking whether the results fit (p. 246).

During the 1990s, the focus on mathematical modeling continued by developing curricula that addressed mathematical modeling. Such curricula were developed by science and educational organizations such as COMAP and the National Science Foundation (NSF). For instance, NSF funded a project called Core-Plus Mathematics. Through this project, a series of mathematics curricula were developed for Grades 9-12. One of the major goals of this series was to “explicitly develop student understanding and skill in use of mathematical modeling, including the processes of data collection, representation, interpretation, prediction, and simulation” (Fey & Hirsch, 2007, p. 130).

During the 2000s, two international comparison studies were conducted that suggested concern around mathematical modeling education, especially in the United States. In 2000, the Program for International Student Assessment (PISA) test showed that American students' achievement was below the world average. The test focused on applying mathematics to solve real-world problems. In 2003, the Trends in International Mathematics and Science Study (TIMSS) showed that American students’ achievement in mathematics was below (but close to) the world average. The test addressed three cognitive demand components (e.g., knowing, applying, and reasoning). According to TIMISS, the United States scored the lowest in applying mathematics, which was below the world average. These findings motivated several American studies to emphasize the importance of mathematical modeling for school students (e.g., Doerr & English, 2003; Niss, 2003; Lesh & Zawojewski, 2007; Doerr & Pratt, 2008; NCTM, 2000) and the need for mathematical modeling to enhance students’ achievement in mathematics. For example, as a result of their study, Doerr and English (2003) espoused that "modeling tasks provide a rich platform for students’ independent development of powerful mathematical ideas" (p. 130).

All the previous studies and initiatives provided suggestions on the importance of mathematical modeling and its meaning. However, arguably none of them has a major impact on teachers’ practices in classrooms. In contrast, in 2010 the Common Core State Standards for Mathematics (CCSSM) were released, emphasizing the importance of mathematical modeling in the American education system. CCSSM includes a framework that depicts the mathematical modeling process, as shown in Figure 2.

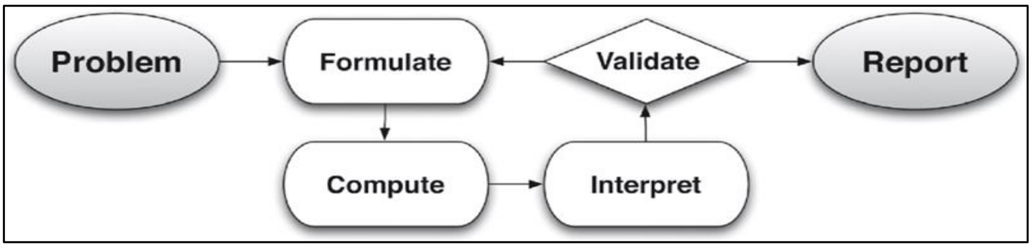


Figure 2. CCSSM modeling cycle (NGA Center & CCSSO, 2010, p.72).

However, the CCSSM presented two forms of modeling without clearly distinguishing between them, which led to confusion. These forms are mathematical modeling and modeling mathematics (Felton-Koestler, 2017). Next, we define and elaborate on these terms.

What is Mathematical Modeling? After the release of CCSSM in 2010, several frameworks for the mathematical modeling process were presented. All of these frameworks suggest that mathematical modeling is defined as using mathematics to solve real-world, open-ended, and messy problems. The goal of the mathematical modeling process is to make decisions, which aims to solve problems, develop situations, or predict scenarios (Alhammouri et al., 2018; Bliss et al., 2014; COMAP & Society for Industrial and Applied Mathematics, 2016; Kasier & Stender, 2013). For instance, the framework presented by Bliss and colleagues (2014) shows mathematical modeling as a process that starts with an open-real-world problem (see Figure 3).

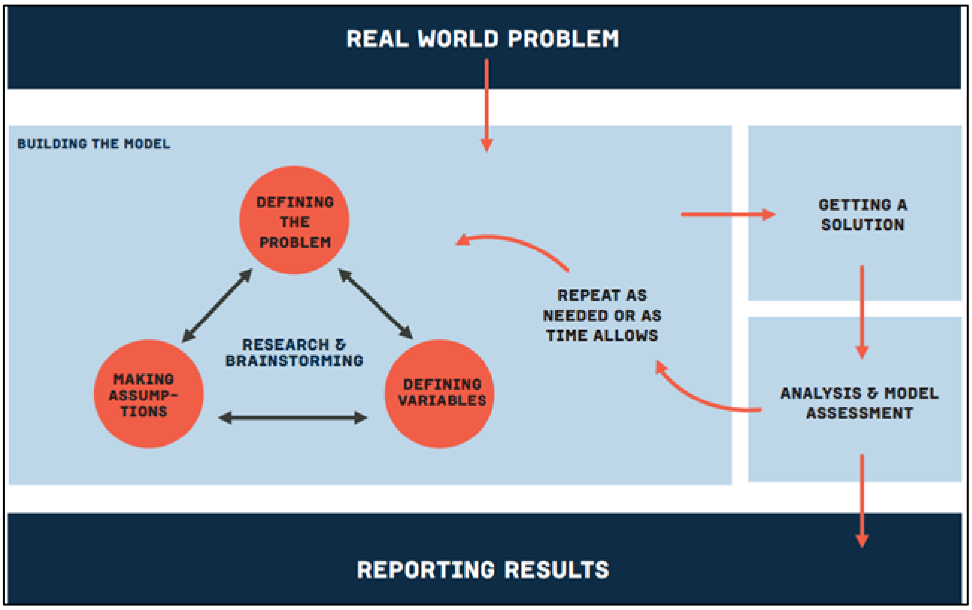


Figure 3. Mathematical modeling cycle (Bliss et al., 2014, p. 5).

Then, according to Bliss and colleagues (2014), the modeler needs to understand the problem, define a problem statement, make assumptions, and define variables. In fact, Alhammouri and colleagues (2018) found that the steps of defining the problem, making assumptions, and defining variables are interactive and can occur in any order. Next, the modeler needs to define a mathematical model representing the variables and the assumptions defined in the previous step. This mathematical model can be an algebraic, geometric, numerical, or graphical model. Then, the modeler performs mathematical procedures to obtain mathematical results. Next, the framework shows that analysis and model assessment (i.e., validation) can occur after getting a solution. However, related research suggests that validation can occur at any stage of the modeling process (Alhammouri et al., 2017; Alhammouri & Foley, 2019). Finally, the mathematical modeler reports and presents the findings in real-world contexts.

What is Modeling Mathematics? Modeling mathematics uses the real world to represent mathematical concepts. van de Walle (2007) stated, "a model for a mathematical concept refers to any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed" (p. 31). Cirillo and colleagues (2016) cited five different representations of mathematical concepts, including manipulative modeling, pictures, symbols, oral language, and real-world situations. When students engage in modeling mathematics, they use such representations to develop their understanding of a mathematical concept. For example, students may use an orange fruit, as shown in Figure 4, to develop an understanding of concepts such as antipodal points, longitude, and latitude.

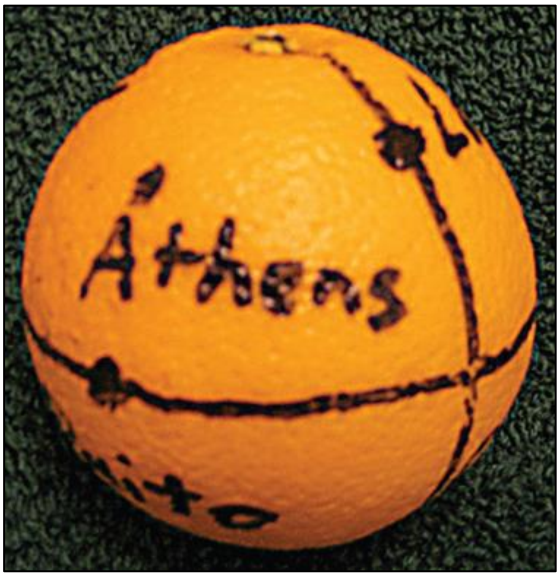


Figure 4. Modeling the globe using an orange (Foley et al., 2016, p. 19)

Modeling mathematics deepens students' understanding of these concepts' nature and characteristics. The modeling mathematics process starts and ends in the mathematics world, in contrast to mathematical modeling, which starts and ends in the real world.

Previously, we provided a background for mathematical modeling and modeling mathematics and an explanation to distinguish between them. Mathematics teachers must be well prepared and supported to engage their students in the mathematical modeling process and in effectively modeling mathematics. One way to prepare and support these teachers is to offer PD experiences. In the next section, we examine the literature on PD programs for mathematics teachers to advance their teaching capacity for modeling.

Professional Development Programs on Modeling

While more research is needed on PD programs explicitly focused on mathematical modeling or modeling mathematics, we reviewed five relevant studies that informed our work. One common factor of these PD studies was a focus on student-centered pedagogies (Kaput, 2018) that can support student engagement with modeling. Such student-centered pedagogy refers to teacher moves that prioritize student agency and exploration with new mathematical ideas, while teachers act as a facilitator and support system to support students' productive struggle. Student-centered pedagogy is in contrast to a more traditional teacher-centered pedagogy, in which the teacher is the focal point of the classroom, often lecturing or leading the entirety of a mathematics lesson.

Maaß and Gurlitt (2011) conducted a study investigating the effectiveness of a PD program called Learning and Education In and Through Modelling and Applications (LEMA). The LEMA PD program was designed for mathematics teachers to enhance their beliefs, knowledge, and teaching efficacy of mathematical modeling. The LEMA program included five modules:

1. Background information about modeling
2. Task selection
3. Lesson design
4. Assessment
5. Reflection on implemented lessons

The results showed that the LEMA PD program positively influenced the participants' knowledge and teaching of mathematical modeling. Specific to pedagogy, participants learned that a student-centered classroom was most advantageous for student learning about modeling. However, participants' beliefs about mathematical modeling not belonging in the classroom remained unchanged. In turn, teachers' instruction about mathematical modeling did not change significantly. The researchers posited that teaching efficacy likely did not change because teacher beliefs remained unchanged.

Tan and Ang (2015) conducted a study to evaluate a school-based program designed to enhance secondary school teachers' capacity to teach mathematical modeling. The PD program included four phases:

1. Background understanding to introduce the PD program and its goals
2. Developing knowledge about mathematical modeling and how it can be taught
3. Applying their knowledge in classrooms by planning, designing, and teaching mathematical modeling lessons
4. Reflecting on their applications by analyzing their lessons and how to improve

Tan and Ang (2015) found that the PD program enhanced the teachers' capacity to select, design, and enact mathematical tasks that engage their students in effective modeling processes within a student-centered classroom. Teachers were also better able to support students' mathematical reasoning while working on activities developed during the PD program.

Jung and Brady (2016) conducted a study in which a researcher implemented a series of mathematical modeling activities where the teacher observed the researcher's practices and shared her concerns with the researcher. These concerns included students' capability to interpret data tables, students' unfamiliarity with the open-ended nature of modeling activities, and the challenges she anticipated with students' working collaboratively in groups. The researcher then opened a discussion with the teacher to respond to those concerns and help the teacher develop more student-centered pedagogical moves to prepare for them. This study supports the effectiveness and utility of this form of "in-situ professional development, indicating that significant changes in teachers' thinking about their students' mathematical model development can occur in relatively short periods of time" (p. 2). However, we believe that it would be more beneficial for the teacher if she took part in enacting the series of modeling activities using the co-teaching model so that she can develop stronger mathematical modeling teaching practices.

Gaston and Lawrence (2015) conducted a research evaluation to define strategies to help mathematics teachers to develop knowledge of and teach mathematical modeling. In addition, they aimed to define strategies of assessing students' engagement in mathematical modeling. They suggested that the field of mathematics education must acknowledge the diversity of teacher backgrounds, beliefs and preparedness in mathematics, especially in mathematical modeling, and select ways to build upon it to improve teachers' expertise in mathematical modeling. Any interventions for teacher learning should closely consider teachers' prior experiences (or lack thereof) with modeling; a successful intervention will successfully utilize teachers' "academic background and professional expertise as unique resources to learn how to best facilitate mathematical modeling in the classroom" (p. 9). Additionally, the research on mathematical modeling PD "does not reveal one perfect method or set of techniques that can prepare each individual teacher to effectively teach mathematical modeling as a transferable process" (p. 9). This means that PD must focus on helping practicing teachers build conceptions of different types of modeling, like mathematical modeling and modeling mathematics. Gaston and Lawrence suggest such conceptual work should involve actually developing, doing, and assessing mathematical modeling activities. Furthermore, teachers need support in learning about student-centered pedagogical practices that can best support modeling in the classroom, as well as multidisciplinary content since many modeling activities span multiple dimensions of science, technology, engineering, and mathematics.

In an action research study, Taite and colleagues (2023) designed, implemented, and investigated the impact of a PD program on mathematical modeling alongside secondary teachers. Across this year-long PD program, the research team used qualitative methods to help reveal challenges associated with teachers implementing mathematical modeling in their classrooms. The researchers found that teachers were frustrated most with supporting their students to productively struggle with the uncertain mathematics involved in open-ended mathematical modeling activities. Often, teachers in this study shared that there never seemed to be enough time for sufficient productive struggle as students engaged in modeling tasks, especially to understand how algebra concepts connected to real-world scenarios. Furthermore, the researchers reported that student engagement and facilitating student discourse during modeling activities proved to be a challenge. In all, it was many facets of establishing and facilitating a student-centered classroom that most challenged teachers in this study.

This literature review section provided a historical timeline of modeling development for the school level in the United States, and discussed how mathematical modeling and modeling mathematics are alike and different. In mathematical modeling, we start with a real-world and open-ended scenario before using mathematics to engage with the scenario. In modeling mathematics, we begin with the mathematical world before using real-world representations to express the mathematical world. Also in this section, we discussed several research studies that addressed PD programs to enhance teachers' capacity to teach mathematical modeling effectively. The criteria of such PD programs should include effective modeling and consider distinguishing the terms of mathematical modeling and modeling mathematics, make apparent the affordances of including modeling in student-centered classrooms, and allow for teachers' voices to be heard about areas in which they need support to implement modeling in their practice.

Recall that this study, in the context of participation in a PD program focused on mathematical modeling education, aims to address these criteria via our research questions:

1. What did high school mathematics teachers discover about the ways mathematical modeling and modeling mathematics can be conceptualized?
2. What did high school mathematics teachers appreciate about incorporating mathematical modeling and modeling mathematics into their classrooms?
3. What did high school mathematics teachers find conflicting about incorporating mathematical modeling and modeling mathematics into their classrooms?

III. METHODS

Context

This study employed a case study approach (Creswell, 2012) to help us answer our research questions, specifically a multiple case study methodology (Stake, 2006). The leaders of this project designed, enacted, and evaluated a PD program for high school mathematics teachers in a Midwestern region of the United States. This PD program was part of a larger quantitative reasoning outreach program, led by the local University, that helped support mathematics teachers to strengthen their content knowledge, pedagogical expertise, and facility with technology in the areas of algebra, functions, geometry, modeling, and spatial reasoning. The PD program had two institutes: Modeling with Algebra and Modeling with Geometry. The modeling institutes were designed around Algebra and Geometry to best align with the goals of the quantitative reasoning outreach program. Each institute was one-week long and met for approximately seven hours per day. There were 28 teacher-participants in total: 10 teachers attended the Modeling with Algebra Institute only, 8 teachers attended the Modeling with Geometry Institute only, and 10 teachers attended both institutes. In sum, there were 20 mathematical activities with which teachers engaged across both institutes.

From a multiple case study perspective, each institute represented a bounded system within which we explored the experiences of our teacher-participants. Nested within these systems, each teacher-participant's experience with each mathematical activity represented a single case (Patton, 2002). Thus, we considered 28 nested cases such as these in our research. In all, the multiple case study was an appropriate methodology because it allowed us to conduct an in-depth exploration of the bounded system based on extensive data collection, specifically by analyzing multiple nested cases and investigating the similarities and differences between them and how they interacted with each other (Creswell, 2013; Stake, 2006).

Data Sources

Data were collected in the forms of participant interviews, observations, and artifacts of participant's engagement with activities. Each PD instructor (one of the project leaders) also collected field notes and made daily reflection journal entries. Furthermore, participants were interviewed after they attended the institutes and returned to their classrooms.

Data Analysis

To help us answer research question one, we analyzed data using a deductive coding process informed by Bliss et al. (2014) framework for mathematical modeling. These frameworks helped us ascertain the ways in which teachers were conceptualizing mathematical modeling as a result of their engagement with the activities across both institutes. To help us answer research questions two and three, we analyzed interview, observation, and artifact data using inductive and deductive codes to help establish categorical aggregation, and thus, to help create themes or patterns (Creswell, 2013) about participants' appreciations of modeling and about participants' conflicts as they envisioned incorporating modeling in their own classrooms. We present such themes in the Findings via participant accounts of their experiences. We also chose representative participant accounts (e.g., Teacher A, Teacher B, Teacher C) that illustrated the themes that emerged from our coding. We chose Teachers A, B, and C as representative teachers because their experiences in the PD program most closely mirrored the majority of the themes and patterns we found across all the nested cases in this study. We report on these representative participant accounts in the Findings, as well.

To help build the overall trustworthiness of our research, we used Shenton's (2004) criteria for qualitative research trustworthiness as a guide. Shenton urges that qualitative inquiry should consider credibility [internal validity], transferability [external validity], dependability [reliability], and confirmability [objectivity]. Thus, we incorporated strategies from Shenton to help align our work to these criteria. For instance, in this study, we used triangulation of data sources (e.g., interviews, observations, artifacts), incorporated member checking (i.e., shared findings with participants to consider their perspective), provided rich and thick descriptions of the data collected, spent ample time in the environment being studied, and identified and clarified potential researcher biases within the research team and with participants during member checking. Incorporating these strategies helped us share a realistic picture of what happened in this study, and helped us demonstrate that our findings emerged from our data sources and not from our own predispositions.

Mathematical Activities

Teacher-participants[2] attending the Modeling with Algebra Institute engaged with the NEXT-NOW activity. The NEXT-NOW activity aimed to introduce the participants to a NEXT-NOW formula in the context of finance. This activity started by introducing the following real-world situation: "Suppose that starting on the day you were born, your parents put 25¢ in a piggy bank every week on the same day of the week on which you were born" (Foley et al., 2016a, p. 16). From here, participants were invited to make their own assumptions, define their problem statements, and ultimately solve their problem statements. Additionally, participants used the digital platform Padlet as a place to collaborate, show their work, and share their solutions.

Teacher-participants attending the Modeling with Geometry Institute engaged with the polar functions activity. The polar functions activity leveraged Desmos, an advanced online graphing calculator, to introduce the participants to several types of polar functions and their algebraic and graphical representations, including $r = a\, cos\, cos\, n\theta$ and $r = a\, sin\, sin\, n\theta$. The participants were invited to examine these functions and their representations, and try to find patterns between them. Next, we detail participants' engagement with both of these activities in the Findings section.

Ⅳ. FINDINGS

In this section, we present our findings in three ways. First, we describe participants' engagement with the NEXT-NOW activity in the Modeling with Algebra Institute. Second, we describe participants' engagement with the polar functions activity in the Modeling with Geometry Institute. Third, we present participant accounts of their experiences with the NEXT-NOW activity and/or the polar functions activity and from their post-institute interviews to explicitly answer our research questions; in other words, we present evidence about participants' discovery of different conceptualizations, appreciation, and conflict around incorporating modeling in their classrooms.

Participants' Engagement with the NEXT-NOW Activity (in the Modeling with Algebra Institute)

In the Modeling with Algebra Institute, participants worked in small groups on a NEXT-NOW formula in the context of finance. The activity started by introducing the following real-world situation to the participants: "Suppose that starting on the day you were born, your parents put 25¢ in a piggy bank every week on the same day of the week on which you were born" (Foley et al., 2016a, p. 16). The instructor of the PD program asked the participants to represent this real-world situation using a NEXT-NOW formula. The participants had to share their mathematical representations on Padlet (see Figure 5). Several groups used algebraic expressions to represent the situation and one group used

[2] We use the term "teacher-participant" and "participant" interchangeably in this paper. All participants in this study were teachers.

textual expressions.

Next, the PD instructor presented the participants with the following real-world situation, which was built on the previous one: “Johnny puts a quarter each Monday in a piggy bank for his grandson since he was born.” The PD instructor asked the groups to define and share their assumptions on Padlet. Figure 6 shows the assumptions that the participants discussed, including that no monetary interest was building, no withdrawals were being made, and specifying the day on which the grandson goes to college. These assumptions led the participants to define two problem statements: (1) How much Johnny’s grandson will save after 2 months, and (2) How much Johnny’s grandson will have by the time he goes to college.

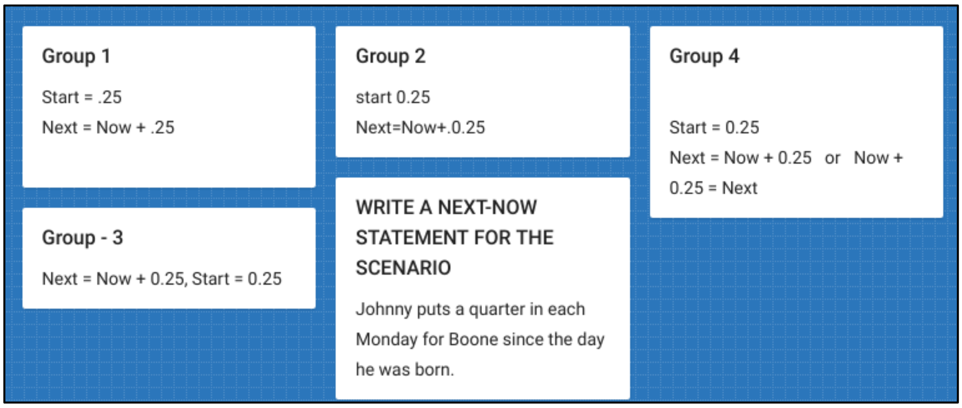


Figure 5. The participants’ NEXT-NOW formulas via Padlet

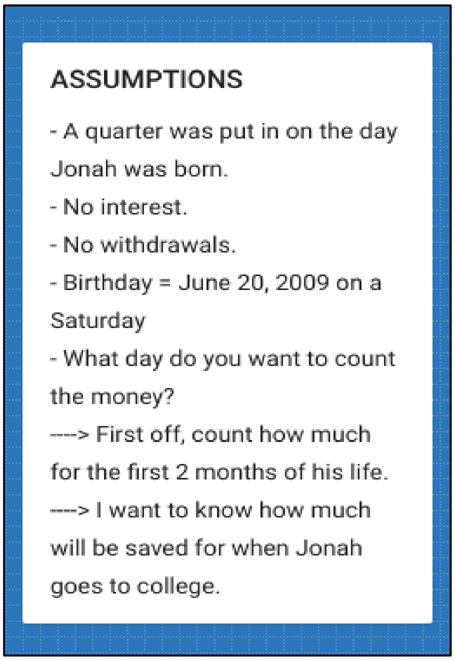


Figure 6. The participants’ assumptions via Padlet.

Then, all groups had to solve the defined problem statements. Figure 7 shows the approach that Group 2 used to solve the first problem statement, in which they wanted to know how much money would be saved after two months. First, Group 2 determined that there were 9 Saturdays between June 20 and August 20. Then, they used their NEXT-NOW formula to find that there would be \$2.25 in Johnny's grandson's piggy bank over that time span.

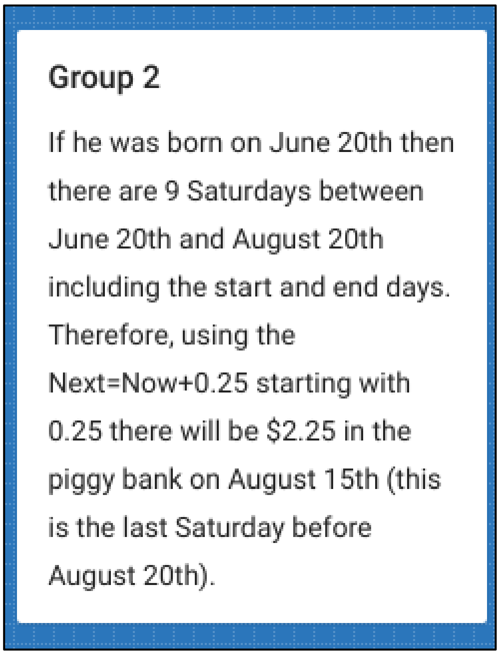
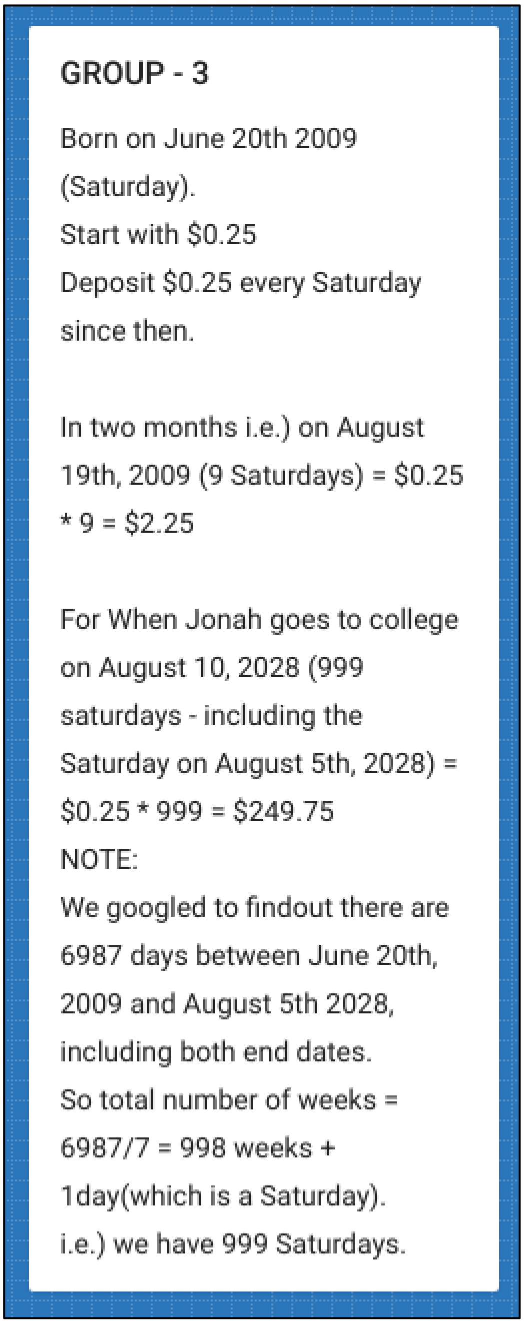


Figure 7. Group 2's solution for how much will be in Johnny's grandson's piggy bank after 2 months, via Padlet

Figure 8 shows the approach that Group 3 used to solve the second problem statement, in which they assumed that Johnny's grandson will go to college on August 10, 2028. They used online sources to define how many days from June 20th, 2009 to August 10, 2028. Then, they divided the number of days by 7 to find how many Saturdays (i.e., weeks) in this time period. They found that there were 6987 days over that span, which included 998 Saturdays. The group found that from the day that the grandson was born until he goes to college, there are 999 Saturdays. The group did not find a final answer assuming it is a matter of simple mathematics procedures (i.e., 999 weeks × 25¢ = \$249.75).

Group 4 considered almost the same approach that Group 3 used (see Figure 9). Group 4 used an online calendar calculator to find how many days between June 20, 2009 and August 15, 2028. They found that there are 999 days in that span, so they divided the number of days by 7 to find out how many weeks. They found that there are 999.4 weeks. Then they rounded this number to 999 weeks to find out that the grandson will have \$250 by the time he goes to college.


GROUP - 3

Born on June 20th 2009 (Saturday).
Start with $0.25
Deposit $0.25 every Saturday since then.

In two months i.e.) on August 19th, 2009 (9 Saturdays) = $0.25 * 9 = $2.25

For When Jonah goes to college on August 10, 2028 (999 saturdays - including the Saturday on August 5th, 2028) = $0.25 * 999 = $249.75
NOTE:
We googled to findout there are 6987 days between June 20th, 2009 and August 5th 2028, including both end dates.
So total number of weeks = 6987/7 = 998 weeks + 1day(which is a Saturday).
i.e.) we have 999 Saturdays.

Figure 8. Group 3's solution for how much Johnny's grandson will have in his piggy bank when he goes to college, via Padlet

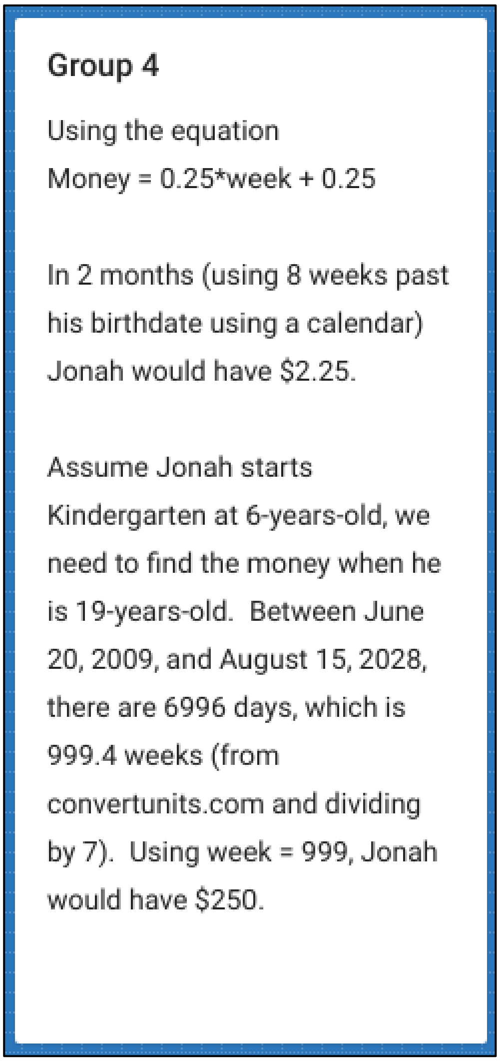


Figure 9. Group 4's solution to how much Johnny's grandson will have in his piggy bank when he goes to college, via Padlet

Considering the Modeling Perspective. In the finance NEXT-NOW activity, the participants engaged in mathematical modeling. The problem was presented to the participants as an open-ended, real-world situation about which for them to think freely (Butts, 1980/2013; Pollak, 1966). Participants were encouraged to ask their own questions about the situation and to follow their own problem-solving pathways. Additionally, the focus of the finance NEXT-NOW activity in the Modeling with Algebra Institute was more about engaging participants in the mathematical modeling process than addressing the mathematical content.

Building the model (defining the problem, making assumptions, & defining variables). The participants were prompted to generate their own questions about the situation and share these questions using Padlet. By doing so, the PD instructor monitored the participant groups' engagement to make sure that they were on the right trajectory of the modeling engagement (Smith & Stein, 2018). Posing original questions helped the participants define problem statements, make assumptions, and then define variables. This process is cyclic (Alhammouri et al., 2018; Bliss et al., 2014), so groups were free to revisit earlier questions and repose questions at any time.

By defining the problem statement, the participants defined exactly the output of their models (Bliss et al., 2014). Several groups decided to find out how much the grandson will have in his piggy bank when he goes to college. Whereas, one group decided to find out how much the grandson will have after two months. By making assumptions, the participants simplified and sharpened their focus (Bliss et al., 2014). The participants also defined the major factors that influenced their models. The major assumptions focused on that there is no interest, no withdrawals, and specifying the day on which the grandson goes to college. Then, the participants defined variables for their assumptions and problem statements. At this stage, the participants defined the factors as quantifiable variables and defined the inputs and outputs of their models (Bliss et al., 2014). Mainly, the participants defined how many Saturdays had occurred after two months or by when the grandson goes to college.

Getting a solution. After the participants built their models, they used mathematics to obtain mathematical results (Bliss et al., 2014). The group that decided to find how much the grandson will save after two months used the NEXT-NOW formula, which was introduced to them by the PD instructor at the beginning of the activity. Whereas, the other two groups used an online calendar calculator to define how many weeks are there between the time the grandson was born until he goes to college. All the groups multiplied the number of weeks by 25¢.

Analysis and model assessment (validation). The participants analyzed and assessed their work throughout the entire modeling process. The validation of the model occurred in several forms and during several stages of the mathematical modeling process (Alhammouri et al., 2017; Alhammouri et al., 2018; Alhammouri & Foley, 2019). First, the participants collaborated in groups, which supported all participants to engage in mathematical discussions that prompted them to reflect on each other's work and ideas. Second, the PD instructor monitored the participants' engagement and provided each group with feedback. Third, each group shared their work with the rest of the groups using Padlet; this gave each participant an opportunity to reflect on their ideas and to provide feedback to each other as individuals and as a large group. Lastly, participants used online calendar calculators to validate their mathematical solutions.

Reporting results. At the end of the modeling process, groups used Padlet to report their results and findings with all participants in the Modeling with Algebra Institute. Each group explained how they developed and used their models. Groups used Bliss et al.'s (2014) conception of the mathematical modeling cycle to structure their explanations.

Participants' Engagement with the Polar Functions Activity (in the Modeling with Geometry Institute)

In the Modeling with Geometry Institute, participants worked in small groups on an introduction to polar functions activity. This activity was situated in Desmos and was created by the PD instructor. Each participant collaborated with their peers within their groups, however everyone used their own laptop to explore the Desmos activity.

The participants were introduced to several types of polar functions. The Desmos activity included algebraic and graphical expressions of these functions simultaneously. The participants had to examine these functions and their representations and try to find patterns between them. Included with these functions were $r = a \cos\cos n\theta$ and $r = a \sin\sin n\theta$.

In one aspect of the activity, the participants engaged with Desmos as it appears in Figure 10. This polar graphs exploration activity displays graphs for the polar function $r = n\theta$; in particular, $r = 5\theta$ and $r = 5\theta$. Moreover, the participants could activate graphs for $r = -5\theta$ and $r = 5\theta$ by selecting them from the upper left side corner of the screen. The graphs were represented in different colors to distinguish between them. Groups had 5 minutes to make connections between the algebraic expression of each function and its graph in order to find a pattern between them. To better understand the nature of this activity, a version of this activity has been created on Desmos for the reader's perusal: https://www.desmos.com/calculator/knq6ryuwva

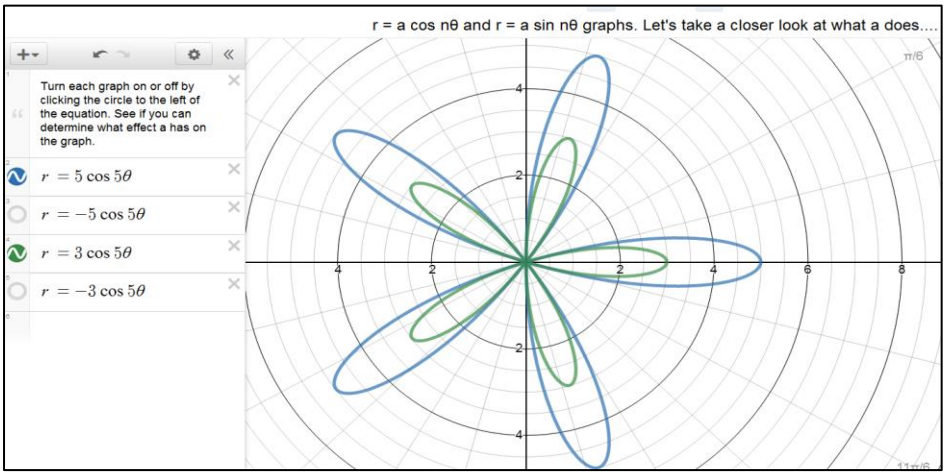


Figure 10. A group's graphs for $r = n\theta$ from the polar graph exploration activity, via Desmos

Then, the participants engaged with the Desmos activity as it appears in Figure 11. The slide displays graphs for the polar function $r = a \sin\sin n\theta$; in particular, $r = 2\theta$ and $r = 4\theta$. Moreover, the participants could activate graphs for $r = 3\theta$ and $r = 5\theta$ by selecting them from the upper left side corner of the screen. They had 5 minutes to make connections between the algebraic expression of each function and its graph in order to find a pattern between them. Similarly to above, to better understand the nature of this activity, a version of this activity has been created on Desmos for the reader's perusal: https://www.desmos.com/calculator/rpizb9fndv

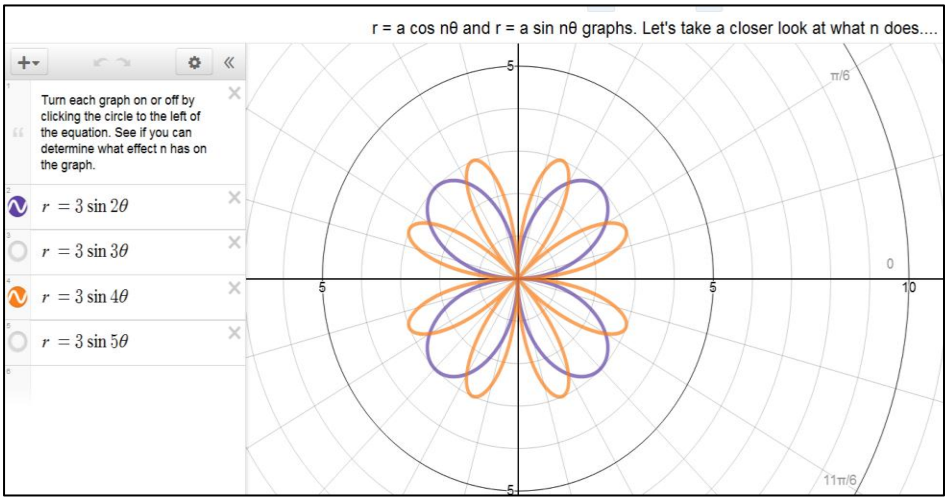


Figure 11. A group's graphs for $r = n\theta$ from the polar graph exploration activity, via Desmos

Next, the PD instructor asked the participants to consider the following prompt: "given $r = a\cos n\theta$ or $r = a\sin n\theta$, describe how each portion (a, cos/sin, n) affects the graph." The participants engaged in group discussions before they shared their answers with all participants in the Modeling with Geometry Institute. For instance, a participant mentioned that when the sign of a for the function $r = n\theta$ is changed, the graph will reflect on the y-axis. Whereas, in the case of the function $r = n\theta$, the graph will reflect on the x-axis.

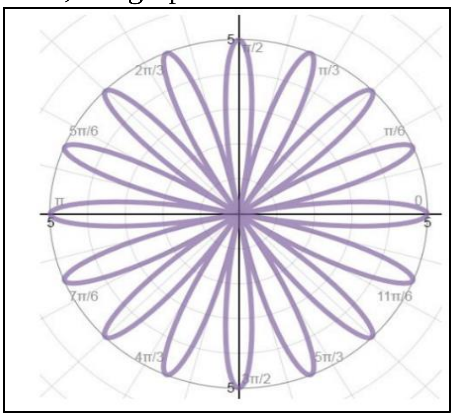


Figure 12. A group's graphs for $r = 8\theta$ from the polar graph exploration activity, via Desmos

As a final step, the participants moved to the next slide of the Desmos activity, which is displayed in Figure 12. The PD instructor asked the participants to define the polar function that represents the graph in the figure. The participants could move back to the graphs shown in Figures 10 and 11 in order to examine several patterns. The participants defined $r = 8\theta$ as an algebraic expression for the graphical expression shown in Figure 12. The PD instructor asked a participant to justify her answer to the rest of the groups.

Considering the Modeling Perspective. In the polar functions activity, the participants engaged in modeling mathematics. Participants in the Modeling with Geometry Institute were supported in their modeling of polar functions through multiple representations and the use of digital graphing tools (Bleiler et al., 2015). Groups were introduced to a Desmos activity that included algebraic and graphical expressions for polar functions. They had to observe, examine, and make connections between different representations of these functions in order to find patterns. Then, the participants had to share their thinking and ideas with the rest of the class. Finally, the participants were given graphical expressions for polar functions, and they had to define their algebraic expressions. In contrast to the NEXT-NOW activity in the Modeling with Algebra Institute in which participants engaged in mathematical modeling to use mathematics to solve a real-world problem that was not inherently mathematical, the polar functions activity engaged participants with technological tools (i.e., Desmos) to create mathematical representations and communicate mathematical ideas.

Participant Accounts of the NEXT-NOW and Polar Functions Activities

Through engagement in the Modeling with Algebra Institute and Modeling with Geometry Institute, participants were given opportunities to discover the different conceptions of modeling, namely mathematical modeling and modeling mathematics, respectively. Furthermore, since the institutes engaged practicing teachers, participants were able to share in their post-institute interviews about their appreciations of including modeling in their mathematics classrooms. Also, participants were able to share in their post-institute interviews about any conflicts that emerged as they incorporated modeling activities in their classrooms. We share three select participant accounts below.

Participant Account 1. A participant (Teacher A) who attended both institutes discovered a difference between the NEXT-NOW activity in the Modeling with Algebra Institute and the polar functions activity in Modeling with Geometry Institute. Teacher A shared:

> I reflected on the comparison between the [polar functions activity in the Modeling with Geometry Institute] versus the [NEXT-NOW activity in the Modeling with Algebra Institute] and how different they have been thus far. During the [NEXT-NOW activity] I felt like we were doing more data collection, and using the real-life situation to build the mathematical models. This week feels like the exact opposite. The [polar functions activity] feels like we begin with the model and use it to represent a real-life object or situation.

Teacher A went on to clarify that mathematical modeling problems like the NEXT-NOW activity involved "a situation, let us follow the modeling process and find a mathematical model like an equation that will help us predict something." Teacher A also clarified that modeling mathematics problems like the polar functions activities were different, namely it's [not an equation, [it's a model to] use to help you conceptualize the mathematical idea. Teacher A was reflecting on their experiences in the institutes and discovering that two different kinds of modeling exist: mathematical modeling and modeling mathematics. Their experience with the NEXT-NOW activity showed application of mathematics to make sense of a real-world problem, which is indicative of mathematical modeling. Their experience with the polar functions activity showed using an existing representation to communicate a mathematical idea, indicative of modeling mathematics.

Teacher A also shared some appreciations associated with modeling. They appreciated the cognitive demand opportunities of mathematical modeling and modeling mathematics, and the opportunities for productive struggle that they provided. They also appreciated how mathematical modeling and modeling mathematics activities afforded a more student-centered classroom. They explained how discovering mathematical modeling and modeling mathematics afforded this paradigm shift:

> It changed how I center my classroom. I have always been a teacher-centered classroom where I sit up front and I lecture them and I tell them this is how we're going to do this, and this is how you solve this problem, and these are the steps to follow, do it. And I find myself more saying, here is a problem, how are you going to solve it? Where I didn't do a lot of that before and modeling in this professional development program kind of changed my view on that. And more, I'm not going to answer every question they ask, but maybe throw the question back to them. Like, how are we going to do this? I don't know, how are you going to do this? And that's not something that I was very comfortable with before. And now I feel myself letting it be more of a student-driven classroom.

Teacher A went on to share how other aspects of their classroom had changed, including more group work, more open-ended activities, and more freedom for students to work at their own paces. Teacher A appreciated that activities like these would help students realize the benefits of productive struggle for learning something new.

However, Teacher A's appreciation of modeling was not without some associated conflicts. Students in their class were visibly frustrated, at times, during modeling activities. Teacher A shared:

> [My students] wanted to be told what to do and when to do it and be very spoon fed, like they have been their whole lives. That was not the way this lesson was prepared. It was very, 'you guys find an answer.' Whichever answer you find, make sure you can justify it. They didn't like that, because they wanted one specific correct answer.

Teacher A shared that most of their students "have never had to think on their own…they were always told what to think and when to think it" in school. Although Teacher A believed that modeling activities were useful in the classroom, this student frustration conflict certainly was apparent. However, Teacher A went on to share about more recent experiences incorporating modeling in their classroom, and that student frustration was starting to diminish, likely due to the students' growing experience with mathematical modeling and modeling mathematics.

Participant Account 2. Another participant (Teacher B) who attended both institutes discovered a difference similar to Teacher A's discovery between the activities in the Modeling with Algebra and Modeling with Geometry Institutes. Specifically, Teacher B discovered modeling mathematics may involve graphs and representations of new mathematical ideas, and that the mathematical modeling process connects mathematics with the real world. Furthermore, Teacher B discovered mathematical modeling was very open-ended, and needed to be open-ended to incorporate aspects of modeling like making assumptions and defining variables.

Teacher B appreciated many aspects of modeling. They primarily appreciated the utility of mathematical modeling in the NEXT-NOW activity, namely "how you can actually use [mathematical models], which was interesting to me because I had not seen that before." Teacher B also appreciated that modeling activities supported students to think on their own, which, like Teacher A, supported the development of a more student-centered classroom. Teacher B shared about the impact of their experiences at the institutes, "The teaching that I do is not so much lecturing the kids and getting them to just regurgitate something more or less, now it has helped me make them start to think on their own." In this way, Teacher B appreciated the opportunities for students to productively struggle with modeling activities, and that such struggle could help students see mathematics as a useful discipline.

Teacher B was conflicted with aspects of incorporating modeling in their classroom, too. Teacher B primarily taught in a special education setting, and lamented that he rarely had enough time to fully engage his students in mathematical modeling during class. When asked about whether they were often able to connect mathematics to the real world in class, Teacher B responded:

> Unfortunately, I would have to say no because we have such a rigorous thing they have to know how to do. Usually, we are far behind with our kids. Our [special education] kids are farther behind. Then we have to push so hard to get them so that they can take the test.

Teacher B clarified that they could not spend as much time on modeling as they preferred because of a school-wide pressure to prepare all students for national tests.

Participant Account 3. A participant, Teacher C, who attended only the Modeling with Algebra Institute also discovered something new about the conception of mathematical modeling, even though they did not have the opportunity to compare and contrast mathematical modeling with modeling mathematics (i.e., because they did not also attend the Modeling with Geometry Institute). Upon entering the Modeling with Algebra Institute, Teacher C shared that they conceived of mathematical modeling as a word problem where the problem statement was defined, all variables were defined, and students were prompted to solve the problem by substituting the given variables into a provided equation. Through engagement with experiences like the NEXT-NOW activity, Teacher C developed a conception of mathematical modeling that was closer to Bliss et al.'s (2014) conception. In a post-interview, Teacher C defined the mathematical modeling process as involving "presenting a problem without a lot of information, that [students] have to come up with the questions, define variables, and make assumptions." Teacher C added that "validation can be all the way around the modeling process…sometimes the students have thrown out questions that are not valid, so you have to make sure that they understand what they need to know and what they do not." In these ways, Teacher C had discovered, through their engagement in the Modeling with Algebra Institute, that mathematical modeling contained essential components like making assumptions, defining variables, and validation.

Teacher C developed many appreciations of mathematical modeling. Teacher C appreciated that mathematical modeling activities raised the cognitive demand for their students, in comparison to procedural word problems, which provided an expectation of productive struggle for students. Furthermore, like Teachers A and B, Teacher C appreciated that mathematical modeling activities helped cultivate a more student-centered classroom. Teacher C explained:

> I could finally envision what type of questions I might ask my own students and, frankly, I am pretty excited about it. I am starting to formulate ideas on why some problem-solving tasks I have given in the past have failed and how I can re-word problems; I have given in the past to elicit more student engagement. To rephrase prompts so that I am looking for questions and critical thinking rather than just evaluation and answers is what I believe will enhance student participation. Students do not have to be afraid of participating, especially in the beginning, because there will be no wrong questions to ask.

Teacher C's experiences in the Modeling with Algebra Institute seemed to influence their instructional choices, especially around who was doing the majority of the mathematical thinking: the teacher or the students. Teacher C strived for students to be the mathematical thinkers during modeling activities, and clarified "we are letting [students] figure it out on their own rather than just feeding them information that they have to use." Teacher C appreciated that incorporating mathematical modeling in the classroom helped students learn how to problem-solve and think critically and independently.

Teacher C was also conflicted with aspects of mathematical modeling in their classroom, especially with the time it takes to engage with an activity. Although Teacher C was excited about the student-centered paradigm shift, they still were reluctant to provide students with full autonomy to work through the mathematical modeling process. Teacher C admittedly scaffolded mathematical modeling tasks for their students to save time. For example, after the Modeling with Algebra Institute, Teacher C enacted a mathematical modeling activity in which students were investigating the ways in which certain objects could roll downhill. Teacher C scaffolded this activity by providing the students tools from the outset (e.g., grid paper, small boards, and the objects themselves), which made the activity less open-ended by introducing a direct way to investigate slope. This decision was made to save time, so students would be able to finish the activity during the class period. The conflict between providing an authentic mathematical modeling experience and saving classroom time was certainly present for Teacher C.

Summary of Findings

Across the data collected and analyzed from all participants and via the illustrative accounts of Teachers A, B, and C, we are able to answer our research questions regarding participants' discovery, appreciation, and conflict around modeling. First, participants were able to discover the different conceptions of mathematical modeling and modeling mathematics. Teacher-participants in the study recognized that mathematical modeling consists of using mathematics to solve open-ended and real-world problems, and that modeling mathematics consists of using different representations to communicate and explore a mathematical idea. For instance, we saw evidence of this from Teachers A and B as they juxtaposed their experiences with the NEXT-NOW activity from the Modeling with Algebra Institute and the polar functions activity from the Modeling with Geometry Institute. Furthermore, we saw evidence of this from Teacher C as their definition of mathematical modeling evolved as a result of their experiences with the NEXT-NOW activity from the Modeling with Algebra Institute.

Second, teacher-participants revealed appreciation for the student-centered nature of mathematical modeling and modeling mathematics activities. Teacher-participants appreciated that students engaging in this kind of activity had opportunities to construct new mathematical knowledge and deepen their understanding of such knowledge through productive struggle. Teacher-participants appreciated that through such engagement, students could perceive usefulness in mathematics as well as in mathematical practices like perseverance in problem-solving. For instance, as a result of their experiences in the PD program, Teacher A shared about their classroom focus completely shifted from teacher-centered to student-centered. Similarly, Teacher B shared about their new appreciation of the utility of modeling and how incorporating modeling in the classrooms could help students learn to think on their own. Moreover, Teacher C shared that a modeling-centric classroom could help raise and preserve the cognitive demand of a lesson.

Third, teacher-participants shared conflicts associated with how much time it takes to provide modeling experiences in their classrooms. The incongruity between needing the time to productively struggle with modeling to learn deeply, but not having the time to do so with their students in their own classrooms was an emergent theme in this study. Teacher-participants also reported student frustration levels and pressure to perform on national tests as other factors that made it difficult to spend the necessary time during class to engage with modeling activities. For instance, such themes were evident in Teacher A's account of how the novelty of modeling could lead to student frustration, or in Teacher B's account of how it was difficult to schedule time for modeling activities during class when their school administration was so concerned with national test preparations. Also, Teacher C's struggle with over-scaffolding modeling activities to save time illustrated a common conflict.

V. DISCUSSION

This case study focused on teachers' experiences in a PD program that was designed to support teacher learning about mathematical modeling and modeling mathematics. Regarding the first research question, our coding process (Creswell, 2013) helped us find evidence of teacher-participants discovering distinct features of mathematical modeling and modeling mathematics. Themes of appreciation and conflict also emerged, namely that teacher-participants appreciated the student-centered nature of modeling in the classroom, but were conflicted about the time investment of such lessons.

The fact that teacher-participants taking part in the Modeling with Algebra and Modeling with Geometry Institutes required clarification between mathematical modeling and modeling mathematics reiterates the point by Cirillo et al. (2016), that the exact definitions of these two terms are not clear in the CCSSM and in the mathematics education literature. All teacher-participants in this study were practicing mathematics teachers and were teaching with curriculum aligned to the CCSSM, yet it required ample engagement with mathematical modeling and modeling mathematics activities for teachers to discover the distinctions between these two conceptions of modeling. This suggests that such teacher knowledge will not develop from teaching alone, and teachers require additional support to discover the differences between mathematical modeling and modeling mathematics. Although wonderful modeling resources exist for teachers (e.g., Felton-Koestler, 2017), these findings echo Gatson & Lawrence's (2015) recommendations that PD concerning the use of modeling resources is necessary for teachers to build their expertise in conceptions of modeling.

Regarding the second and third research questions, considering the appreciations and conflicts associated with teacher-participants' modeling experiences, the dissonance between the student-centered affordances that modeling provides, but not having enough classroom time to provide them is apparent. These findings resonate with Taite et al.'s (2023) action research study, in which teachers claimed there was never enough time for their students to meaningfully struggle with the mathematical ideas associated with mathematical modeling activities. This suggests that more research is needed to determine how to effectively include modeling activities in mathematics lessons while considering the limited time available in many classrooms. One such option is to consider the inclusion of "stepping-stone problems" (Felton-Koestler, 2017, p. 270) into mathematics curricula, which help introduce mathematical modeling ideas to students in smaller pieces compared to full, open-ended modeling activities. Although some studies have shown that repeated engagement with mathematical tasks akin to stepping-stone problems can help develop students' perseverance in problem solving (see DiNapoli & Miller, 2022), more direct research is necessary to study the effects of using stepping-stone problems as a bridge to more authentic modeling activities.

Lastly, the appreciative findings associated with modeling in this study may extend the work by Maaß and Gurlitt (2011). We found evidence that teacher-participants in the Modeling with Algebra and/or Modeling with Geometry Institute may have changed their beliefs about what good mathematics classrooms look like. The representative teacher accounts shared in this paper help show that learning about and engaging in mathematical modeling and modeling mathematics supported a paradigm shift from teacher-centered to student-centered classrooms; classrooms in which students are given autonomy to think, explore, and productively struggle with mathematics and its connections to the real world. In their LEMA PD, Maaß and Gurlitt found that beliefs about teacher-centered teaching philosophies persisted, despite teacher-participants learning about mathematical modeling pedagogies. These persistent beliefs helped explain why teacher practice largely did not change for LEMA participants. Although it is encouraging that teachers participating in the Modeling with Algebra and Modeling with Geometry Institutes were able to change their beliefs, we are unable to claim exactly why. For this project, future research will consider this question about the effects of the PD program design on the evolution of teacher beliefs.

Limitations

There were several limitations associated with this study that help explain how we reported our findings. First, the PD instructor in the Modeling with Algebra Institute was different than the PD instructor in the Modeling with Geometry Institute. The two instructors were both members of the leadership team, and were both experts in the field of modeling, but this inconsistency made it difficult for us to claim specific relationships between how the PD program was designed and the teacher-participant outcomes. Second, although we had 28 total teacher-participants, only 10 people attended both institutes, which made it difficult to make clear comparisons between teachers' conceptions of mathematical modeling and modeling mathematics. There were many reasons for this and all of them were associated with teachers leading busy lives. We are grateful for the participation we received from the teachers involved. Third, we are unable to make generalizable claims about this work. Instead, this work focused on describing the rich nature of these institutes and the illuminating experiences of the teachers who participated. We hope these findings will inspire other mathematics educators to study modeling in their contexts.

VI. CONCLUSION

Recent international reform movements have called for increased attention on incorporating authentic experiences with modeling in K-12 mathematics classrooms. This also motivated a focus on supporting mathematics teachers' own learning about modeling and its affordances in their classrooms. However, definitions and enactment principles are often unclear in policy documents.

Modeling has been conceptualized in two primary ways: mathematical modeling (i.e., using mathematics to solve real-world problems that may not be inherently mathematical) and modeling mathematics (i.e., using representations to communicate mathematical ideas). Elevating both of these practices to a central role in students' experiences can pay dividends toward mathematics learning. Although such modeling has been distinctly defined by researchers, it is often unclear for teachers about the differences between these two constructs and their affordances in classrooms. PD programs that focus on these issues can offer one method of support.

In this study, we employed a nested case study methodology to examine 28 United States high-school mathematics teachers' experiences in a PD program that focused on learning about mathematical modeling and modeling mathematics and how those processes could be enacted in their own classrooms. The PD program had two week-long institutes: the Modeling Algebra Institute and the Modeling Geometry Institute. We used an inductive and deductive coding process to study teachers' (a) conceptualizations of modeling, (b) appreciations of modeling, and (c) conflicts as they envisioned incorporating modeling in their own classrooms.

Our findings showed that teachers perceived a difference between mathematical modeling and modeling mathematics during the PD program. In the context of a finance activity, they showcased their understanding of mathematical modeling in the Modeling Algebra Institute by engaging in aspects of the modeling cycle, including defining the problem, making assumptions, defining variables, obtaining a solution, reflecting on & continuously validating their solution, and reporting their results. In the context of polar functions, they showcased their understanding of modeling mathematics in the Modeling Geometry Institute by exploring and positing connections between graphical representations and equations in Desmos. Across both institutes, we also found that teachers appreciated both forms of modeling as a way to construct new mathematical knowledge and deepen their understanding of such knowledge through productive struggle. Contrarily, teachers voiced concern about enacting similar activities in their own classrooms, primarily because of time restrictions. This emerged as a point of conflict for many teachers because of the incongruity between needing the time to productively struggle with modeling to learn deeply, but not having the time to do so with their students in their own classrooms.

Collectively, these findings help show how a PD program can be designed to engage mathematics teachers with forms of modeling, and that those experiences can inspire mathematics teachers to consider modeling as an imperative feature of a mathematics program. However, these findings also revealed conflicts, specifically that teachers doubt that there is enough time in their curriculum to authentically engage their students in modeling activities. Ongoing research is needed to develop, evaluate, and refine such PD, as well as determine supports to help teachers use it in their classrooms with fidelity.

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