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Miyachi's Theorem for the k-Hankel Transform on $\mathbb{R}^d$

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Abstract. The classical Hardy Theorem on $\mathbb{R}$ states that a function f and its Fourier transform cannot be simultaneously very small; this fact was generalized by Miyachi in terms of $L^1 + L^\infty$ and $\log^+$-functions. In this paper, we consider the k-Hankel transform, which is a deformation of the Hankel transform by a parameter $k > 0$ arising from Dunkl's theory. We study Miyachi's theorem for the k-Hankel transform on $\mathbb{R}^d$.

1. Introduction

Let $\mathbb{R}^d$ be a real d-dimensional Euclidean space with scalar product $\langle x, y\rangle$ and norm $\|x\| = \sqrt{\langle x, x\rangle}$. Let S^{d-1} be the unit Euclidean sphere in $\mathbb{R}^d$, Δ be the Laplace operator, $d\mu(x) = (2\pi)^{-d/2}dx$ be the normalized Lebesgue measure, $L^p(\mathbb{R}^d), 1 \leq p < +\infty$ be the Lebesgue space with norm $\|f\|_p := (\int_{\mathbb{R}^d} |f|^p d\mu)^{1/p}$, and $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space.

The Euclidian Fourier transform is defined by

$$\mathcal{F}f(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-i\langle x,y\rangle}dx.$$

We introduce the real parameters α, β such that $\alpha, \beta > 0$ and let f be a measurable function on $\mathbb{R}$ satisfying $|f(x)| \leq \lambda e^{-\alpha x^2}$ and $|\mathcal{F}(y)| \leq \lambda e^{-\beta \xi^2}$. The function f reduces to the null function if $\alpha\beta \geq \frac{1}{4}$. A generalization of Hardy's theorem is estabilished by Miyachi in [18] where the following is shown.

If f is a measurable function on $\mathbb{R}$ such that

$$e^{\alpha x^2} f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$$

and

$$\int_{\mathbb{R}} \log^+ \frac{|\mathcal{F}(\xi)e^{\frac{\xi^2}{4\alpha}}|}{\lambda} d\xi < \infty,$$

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where α, λ are two positive constants, then f is a constant multiple of $e^{-\alpha x^2}$.

A large family of theorems have been investigated in recent years, the most classical one is Titchmarsh's theorem [9, 12, 17], which says that a function and its classical Fourier transform on the real line cannot both be clearly localized. To be more precise, it is impossible for a non-zero function and its classical Fourier transform (CFT) to both be small. The notion of smallness have been given many defintions. See, for example, Hardy's work in [13], Cowling et al. in [7] and Miyachi in [18].

In harmonic analysis theory, an important role is played by the following infinitisimal generator operator

$$T_{k,a} := \|x\|^{2-a}\Delta_k - \|x\|^a, \quad a > 0, \tag{1.1}$$

where Δ_k is the Dunkl Laplacian given by relation (2.1).
In the last decade, Ben Saïd et al. have generalized in [4] the classical situation by introducing a generalized integral transform $\mathcal{F}_{k,a}$, which is defined by

$$\mathcal{F}_{k,a} := e^{i\frac{\pi}{2}\left(\frac{2\langle k\rangle+d-a-2}{a}\right)} \exp\left(\frac{\pi i}{2a} T_{k,a}\right),$$

where k is a parameter comes from the Dunkl differential-difference operators, and a arises from the interpolation of two minimal unitary representations of two different reductive groups, see [4, 3]. More recently, a convolution structure has been studied for this transform by the author jointly Negzaoui and Sifi in [5].

The transform $\mathcal{F}_{k,a}$ specialises to various well-known integral transforms:

- ▶ the classical Fourier transform, [14] $(a = 2, k = 0)$.
- ▶ the classical Hankel transform, [15] $(a = 1, k = 0)$.
- ▶ the Dunkl transform, [11] $(a = 2, k > 0)$.
- ▶ the k-Hankel transform, [1] $(a = 1, k > 0)$.

In this paper, we pin down the last case (k-Hankel transform $\mathcal{F}_k$), we study Miyachi's theorem on $\mathbb{R}^d$. Analogous results have been studied by Chouchene et al. in [6] for the Dunkl transform, Loualid in [16] for the generalized Dunkl transform, by Daher in [8] for Jacobi-Dunkl transform, and Daher et al. in [10] for which a generalization of Miyachi's theorem on $\mathbb{R}^d$ is established for the generalized Fourier transforms, the Chébli-Trimèche and the Dunkl transforms.

We briefly summarize the contents of this paper. In §2, we collect some background materials for the harmonic analysis associated with the k-Hankel transform on $\mathbb{R}^d$. In §3, we provide keys lemmas used to prove our main result of Miyachi's theorem for the k-Hankel transform.

2. Background for the k-Hankel transform on $\mathbb{R}^d$

Let $\mathcal{R} \subset \mathbb{R}^d \setminus 0$ be a root system, $\mathcal{R}_+$ be a positive subsystem of $\mathcal{R}$, $G(\mathcal{R}) \subset O(d)$

be a reflection group formed by reflections $\sigma_a : a \in \mathcal{R}$, where σ_a is a reflection with respect to hyperplane $\langle a, x\rangle = 0$, and $k : \mathcal{R} \mapsto \mathcal{R}_+$ be a multiplicity function invariant under groups G. This is a G-invariant positive homogeneous of degree $2\gamma_k - 1$, where

$$\gamma_k = \sum_{\alpha\in\mathcal{R}_+} k_\alpha.$$

Let's consider the weight and the Dunkl measure given respectively on $\mathbb{R}^d$ by

$$\upsilon_k(x) = \|x\|^{-1} \prod_{\alpha\in\mathcal{R}_+} |\langle x, \alpha\rangle|^{2k(\alpha)}, \quad d\mu_k(x) = \upsilon_k(x)dx.$$

Denote by $\lambda_k = 2\gamma_k + d - 1$ the homogeneous dimension of the system.

The Dunkl operators $T_j, 1 \le j \le d$ on $\mathbb{R}^d$ are the first-order differential-difference operators, introduced by Dunkl in [11] are given by

$$T_j f(x) = \partial_j f(x) + \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\frac{f(x) - f(\sigma_\alpha x)}{\langle x, \alpha\rangle}\langle \alpha, e_j\rangle, \quad 1 \le j \le d,$$

where ∂_j denotes the usual partial derivatives and $e_1, ..., e_d$ the standard basis on $\mathbb{R}^d$. A fundamental property of these differential-difference operators is their commutativity:

$$T_k T_l = T_l T_k, \text{ for } 1 \le k, l \le d.$$

The Dunkl Laplacian $\Delta_k = \sum_{j=1}^{d} T_j^2$, is given explicitly for a regular function f, by

$$\Delta_k f = \Delta f + \sum_{\alpha\in\mathcal{R}} k(\alpha)\left(\frac{\langle\nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2}\right), x \in \mathbb{R}^d, \tag{2.1}$$

where ∇ and Δ are the classical gradient and Laplacian operators.

2.1. The k-Hankel transform

We define the kernel

$$B_k(x, y) = \Gamma\left(\frac{\lambda_k}{2}\right) V_k\left(\widetilde{J}_{\frac{\lambda_k}{2}-1}(z)\right)\left(\frac{y}{\|y\|}\right),$$

with $z = \sqrt{2\|x\|\|y\|(1 + \langle\frac{x}{\|x\|}, .\rangle)}$. Here, V_k denotes the Dunkl intertwining operator defined by

$$V_k f(x) = \int_{\mathbb{R}^d} f(y) d\sigma_x(y), \quad x \in \mathbb{R}^d, \tag{2.2}$$

where σ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0,\|x\|)$ of center 0 and radius $\|x\|$. The expression in (2.2) is Lebesgue integrable on $\mathbb{R}^d$, and $\widetilde{J}_\nu(z) = (\frac{z}{2})^\nu J_\nu(z)$, J_ν being the Bessel function of first kind and index ν.

Let us define the space:

$\mathcal{D}(\mathbb{R}^d)$ is the space of test functions (that is infinitely differentiable functions $f : \mathbb{R}^d \mapsto \mathbb{C}$ with compact support contained in $\mathbb{R}^d$).

Let ${}^tV_k$ denotes the dual operator of V_k on which is a topological automorphism of $\mathcal{D}(\mathbb{R}^d)$. It is defined by: There exists a positive probability measure ν_y on $\mathbb{R}^d$ with support in the closed ball $B(0,\|x\|)$ of center 0 and radius $\|x\|$ such that

$$ {}^tV_k f(y) = \int_{\mathbb{R}^d} f(x) d\nu_y(x), \quad x \in \mathbb{R}^d. \tag{2.3} $$

Relation (2.3) is also given in terms of the k-Hankel transform and the classical Fourier transform $\mathcal{F}$ by the following relation

$$ {}^tV_k(f) = \mathcal{F} \circ \mathcal{F}_k(f). \tag{2.4} $$

The operators V_k and ${}^tV_k$ possess the following property : For all $f \in \mathcal{D}(\mathbb{R}^d)$ and $g \in \mathcal{E}(\mathbb{R}^d)$ we have

$$ \int_{\mathbb{R}^d} {}^tV_k(f)(y) g(y) dy = \int_{\mathbb{R}^d} f(x) V_k(g)(x) d\mu_k(x). \tag{2.5} $$

If we take $g = 1$ in (2.5), we obtain

$$ \int_{\mathbb{R}^d} {}^tV_k(f)(y) dy = \int_{\mathbb{R}^d} f(x) d\mu_k(x), \tag{2.6} $$

Moreover, for all $x, y \in \mathbb{R}^d$, the kernel $B_k(x, .)$ possesses the following properties:

For all $x, y \in \mathbb{R}^d$, we have

$$ B_k(0, y) = 1, \qquad |B_k(x, y)| \leq 1. \tag{2.7} $$

$$ |\partial_z^\nu B_k(x, z)| \leq \|x\|^{|\nu|} e^{\|x\|\|\Re e z\|}, \tag{2.8} $$

where

$$ \partial_z^\nu = \frac{\partial^\nu}{\partial z_1^{\nu_1} ... \partial z_d^{\nu_d}} \quad \text{and} \quad |\nu| = \nu_1 + \nu_2 + ... + \nu_d. $$

The kernel B_k plays an important role in the development of the k-Hankel transform, for more details, we refer the reader to [1, 2]. Relation (2.7) asserts that the k-Hankel transform is well defined for all $f \in L^1(\mathbb{R}^d, \mu_k)$

$$ \mathcal{F}_k f(y) = c_k \int_{\mathbb{R}^d} f(x) B_k(x, y) d\mu_k(x), \quad y \in \mathbb{R}^d, \tag{2.9} $$

where c_k is the Macdonald-Mehta-Selberg integral given by

$$c_k^{-1} = \int_{\mathbb{R}^d} e^{-\|x\|} d\mu_k(x).$$

We collect some properties of the k-Hankel transform (for more details see [2]).

Proposition 2.1.1.

(i) (Inversion formula) The k-Hankel transform $\mathcal{F}_k$ is a topological isomorphism of $\mathcal{S}(\mathbb{R}^d)$ and its inverse is given by

$$\mathcal{F}_k^{-1} = \mathcal{F}_k.$$

(ii) (*Plancherel Theorem*) The k-Hankel transform extends to an isometry of $L^2(\mathbb{R}^d, \mu_k)$. In particular, we have

$$\|\mathcal{F}_k f\|_{L^2(\mathbb{R}^d,\mu_k)} = \|f\|_{L^2(\mathbb{R}^d,\mu_k)}.$$

The definition of the k-Hankel transform permets us to define the generalized translation operator on $L^2(\mathbb{R}^d, \mu_k)$.
The generalized translation operator $f \mapsto \tau_y^k f$, $y \in \mathbb{R}^d$ is defined on $L^2(\mathbb{R}^d, \mu_k)$ by

$$\mathcal{F}_k(\tau_y^k f)(\xi) = B_k(y, \xi)\mathcal{F}_k(f)(\xi), \quad \xi \in \mathbb{R}^d.$$

It plays the role of the arbitrary translation $\tau_y^k f(.) = f(. - y)$ in $\mathbb{R}^d$, since the Euclidean Fourier transform satisfies $\widehat{\tau_y^k f}(x) = e^{-i\langle x,y\rangle}\widehat{f}(x)$.

In the analysis of this translation a particular role is played by the space

$$A_k(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d, \mu_k)/\ \mathcal{F}_k f \in L^1(\mathbb{R}^d, \mu_k)\}.$$

Note that $A_k(\mathbb{R}^d) \subset L^1 \cap L^\infty(\mathbb{R}^d, \mu_k)$ and hence is a subspace of $L^2(\mathbb{R}^d, \mu_\kappa)$.
The operator τ_y^k satisfies the following properties:

Proposition 2.1.2. Assume that $f \in A_k(\mathbb{R}^d)$ and $g \in L^1 \cap L^\infty(\mathbb{R}^d, \mu_k)$. Then

(i) For every $x, y \in \mathbb{R}^d$, we have $\tau_y^k f(x) = \tau_x^k f(y)$.

(ii) For every $y \in \mathbb{R}^d$, the operator τ_y^k satisfies

$$\int_{\mathbb{R}^d} \tau_y^k f(x) g(x) d\mu_k(x) = \int_{\mathbb{R}^d} f(x)\tau_y^k g(x) d\mu_k(x).$$

A formula of $\tau_y^k f$ is known, at the moment, only in two cases.

Case 1. $G = \mathbb{Z}_2$ (see [1]).

Case 2. where a formula of $\tau_y^k f$ is known when f is a radial function in $A_k(\mathbb{R}^d)(f(x) = f_o(\|x\|))$, G being any reflection group(see [2])

$$\tau_y^k f(x) = \frac{\Gamma(\frac{\lambda_k}{2})}{\Gamma(\frac{\lambda_k}{2} - \frac{1}{2})} V_k \left[\int_{-1}^{1} f_0 \left(\preceq x, y, u; . \succeq\right) \left(1 - u^2\right)^{\frac{\lambda_k}{2} - \frac{3}{2}} du \right] (\frac{y}{\|y\|}),$$

where $\preceq x, y, u; . \succeq = \|x\| + \|y\| - \sqrt{2\|x\|\|y\|(1 + \langle \frac{x}{\|x\|}, .\rangle)u}$.

According to the positivity of the intertwining operator (2.2) it follows that $\tau_y^k f(x) \geq 0$ for all $y \in \mathbb{R}^d, f(x) = f_0(\|x\|) \geq 0$.

Some properties of $\tau_y^k f$ (f being radial) follow from this formula. This is collected in the following proposition.

Proposition 2.1.3. (See [2])

(i) For every $f \in L^1_{rad}(\mathbb{R}^d, \mu_k)$ the subspace of radial functions in $L^1(\mathbb{R}^d, \mu_k)$, we have:
$$\int_{\mathbb{R}^d} \tau_y^k f(x) d\mu_k(x) = \int_{\mathbb{R}^d} f(x) d\mu_k(x).$$

(ii) For $1 \leq p \leq 2$, $\tau_y^k : L^p_{rad}(\mathbb{R}^d, \mu_k) \mapsto L^p_{rad}(\mathbb{R}^d, \mu_k)$, is a bounded operator.

(iii) The generalized translation operator is well defined on $L^2(\mathbb{R}^d, \mu_k)$ by the relation
$$\mathcal{F}_k(\tau_x^k f)(y) = B_k(x, y)\mathcal{F}_k f(y).$$

2.2. The k-Hankel convolution product

The generalized translation operator can be used to define the k-Hankel convolution product.

Definition 2.2.1. For $f, g \in L^2(\mathbb{R}^d, \mu_k)$, we define the k-Hankel convolution product $*_k$, by
$$f *_k g(x) = c_k \int_{\mathbb{R}^d} f(y) \tau_x^k g(y) d\mu_k(y), \quad x \in \mathbb{R}^d.$$

Note that the generalized convolution $*_k$ is well defined since $\tau_x^k g \in L^2(\mathbb{R}^d, \mu_k)$ and it may be rewrite

$$f *_k g(x) = c_k \int_{\mathbb{R}^d} \mathcal{F}_k f(\lambda) \mathcal{F}_k g(\lambda) B_k(x, \lambda) d\mu_k(\lambda), \quad x \in \mathbb{R}^d.$$

Let $f \in L^2_{rad}(\mathbb{R}, \mu_k)$ and $g \in L^2(\mathbb{R}, \mu_k)$. Then

$$\int_{\mathbb{R}^d} |f *_k g(x)|^2 d\mu_k(x) = \int_{\mathbb{R}^d} |\mathcal{F}_k f(\lambda)|^2 |\mathcal{F}_k g(\lambda)|^2 d\mu_k(\lambda), \quad x \in \mathbb{R}^d.$$

This convolution has considered by [2]. It satisfies

$$f *_k g = g *_k f; \quad \mathcal{F}_k(f *_k g) = \mathcal{F}_k f \cdot \mathcal{F}_k g.$$

3. Miyachi's theorem for the k-Hankel transform

Our principal interest in this section is to prove a Miyachi's theorem associated with the k-Hankel transform.

Let us denote by $\mathcal{P}(\mathbb{R}^d)$ is the set of polynomials on $\mathbb{R}^d$.

For all $x \in \mathbb{R}^d$, $s > 0$ the k-Hankel heat kernel q_t^k is given by

$$q_t^k(x) = c_k t^{-\lambda_k} e^{-\frac{\|x\|}{t}}, \quad \text{for} \ \ t > 0$$

the function q_t^k is a solution of the heat equation $H_k u(x,t) = 0$ and $H_k = T_{k,1}$, where $T_{k,1}$ is the infinitisimal generator operator defined by (1.1). For more details we refer the reader to [2, 4].

Now, we state our principal theorem of this section.

Theorem 3.1. Let f be a measurable function on $\mathbb{R}^d$ such that

$$e^{\alpha\|x\|} f \in L^p(\mathbb{R}^d, \mu_k) + L^q(\mathbb{R}^d, \mu_k) \tag{3.1}$$

and

$$\int_{\mathbb{R}^d} \log^+ \frac{|\mathcal{F}_k(f)(\xi) e^{\beta\|\xi\|}|}{\lambda} d\xi < \infty \tag{3.2}$$

for some constants $\alpha, \beta, \lambda > 0$ and $1 \le p, q \le +\infty$.

Case 1. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e.

Case 2. If $\alpha\beta = \frac{1}{4}$, then $f = K q_\beta^k(.)$ with $|K| \le \lambda$.

Case 3. If $\alpha\beta < \frac{1}{4}$, then for all $\delta \in]\beta, \frac{1}{\alpha}[$, if f takes the form $f(x) = P(x) q_\delta^k(x)$, $P \in \mathcal{P}(\mathbb{R}^d)$, the relations (3.1), (3.2) hold. To achieve the proof of Theorem 3.1 we need the following auxiliary lemmas.

3.1. Auxiliary lemmas

Lemma 3.1.1. Let $g \in \mathbb{C}^d$ be an entire function, for some positive constants C_1 and C_2 such that

$$|g(z)| \le C_1 e^{C_2 \|\Re e z\|} \quad \wedge \quad \int_{\mathbb{R}^d} \log^+ |g(y)| dy < \infty \Rightarrow g \ \text{ is a constant.} \tag{3.3}$$

Proof. Using Fubini's theorem together with relation (3.3), there is s subset E of $\mathbb{R}^{d-1}$ with $\lambda(E^c) = 0$ (here λ denote the Lebsegue measure). Such that for all sequence $(x_i)_{2 \le i \le d} \in E$,

$$\int_{\mathbb{R}^d} \log^+ |g(x, (x_i)_{1\leq i\leq d})| dx < +\infty.$$

Additionally, the function $z_1 \mapsto g(z_1, (x_i)_{2\leq i\leq d})$ is an entire function and $O(e^{C_2(\Re z_1)^2})$ on $\mathbb{C}$. Then by Miyachi's Lemma [18, lemma 4], the function g is bounded in $\mathbb{C}$. Moreover, by using Liouville theorem, we see that for all $z_1 \in \mathbb{C}$ and all sequence $(x_i)_{2\leq i\leq d} \in E$

$$g((x_i)_{1\leq i\leq d}) = g(0, (x_i)_{2\leq i\leq d}).$$

For all $(z_i)_{1\leq i\leq d}$, the last equality has a sense because g is a continuous function. Then by induction we infer the result, which furnishes the proof of Lemma 3.1.1. □

Lemma 3.1.2. Let $r \in [1, +\infty[,\quad a > 0$. Then for $h \in L^r(\mathbb{R}^d, \mu_k)$, there is a constant $K > 0$ such that

$$\left(\int_{\mathbb{R}^d} e^{\alpha r\|x\|} |{}^tV_k(e^{-\alpha\|y\|}h)|^r dx\right)^{1/r} \leq K \left(\int_{\mathbb{R}^d} |h(x)|^r d\mu_k(x)\right)^{1/r}. \tag{3.4}$$

Proof. By means assertion (3.4), one can assert that $e^{-\alpha\|y\|}h \in L^1(\mathbb{R}^d, \mu_k)$. Then by relations (2.2), (2.5) and (2.6) ${}^tV_k(e^{-a\|y\|}h)$ is defined a.e on $\mathbb{R}^d$. Here two cases to be discussed:

Case 1. If $r \in [1, \infty[$, then

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\alpha r\|x\|} |{}^tV_k(e^{-\alpha\|y\|}h)|^r dx \\
&\leq \int_{\mathbb{R}^d} e^{\alpha r\|x\|} \left(\int_{\mathbb{R}^d} e^{-\alpha\|y\|} |h(y)| d\nu_x(y)\right)^r dx \\
&\leq \int_{\mathbb{R}^d} e^{\alpha r\|x\|} \left(\int_{\mathbb{R}^d} |h(y)|^r d\nu_x(y)\right) \left(\int_{\mathbb{R}^d} e^{-\alpha r'\|y\|} |h(y)| d\nu_x(y)\right)^{r/r'} dx,
\end{aligned}$$

where r' is the conjugate exponent of r.

Hence

$$\int_{\mathbb{R}^d} e^{-t\|y\|} d\nu_x(y) = K e^{-t\|x\|} \quad \text{for} \quad t > 0. \tag{3.5}$$

According to relation (2.6), we see that

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{\alpha r\|x\|} |{}^tV_k(e^{-\alpha\|y\|}h)|^r dx &\leq K \int_{\mathbb{R}^d} {}^tV_k(|h|^r)(x) dx \\
&= K \int_{\mathbb{R}^d} |h(x)|^r d\mu_k(x),
\end{aligned}$$

which gives the result for the case $r \in [1, \infty[$.

Case 2. If $r = +\infty$, then by relation (3.5), we have

$$\begin{aligned} e^{\alpha\|x\|}|{}^tV_k(e^{-\alpha\|y\|}h)(x)| &\le e^{\alpha\|x\|}{}^tV_k(e^{-\alpha\|y\|})(x)\|h\|_{k,\infty} \\ &= K\|h\|_{k,\infty} < \infty, \end{aligned}$$

which furnishes the case $r = +\infty$, and this infers the result. □

Lemma 3.1.3. Let $p, q \in [1, +\infty[$ and f a measurable function on $\mathbb{R}^d$, let $\alpha > 0$ such that

$$e^{\alpha\|x\|}f \in L^p(\mathbb{R}^d, \mu_k) + L^q(\mathbb{R}^d, \mu_k). \tag{3.6}$$

Then for all complex number $z \in \mathbb{C}^d$, $\mathcal{F}_k(f)(z)$, moreover it's entire, exists $K > 0$ such that for all $z \in \mathbb{C}^d$,

$$|\mathcal{F}_k(f)(z)| \le Ke^{\frac{\|v\|}{\alpha}}. \tag{3.7}$$

Proof. Using relation (2.8) together with Hölder's inequality , we infer the relation (3.6).

For the relation (3.7), observe that relations (3.6) and (2.6) assert that $f \in L^1(\mathbb{R}^d, \mu_k)$, and ${}^tV_k(f) \in L^1(\mathbb{R}^d, \mu_k)$, consequently, by (2.4), for all $z = u + iv \in \mathbb{C}^d$, $u, v \in \mathbb{R}^d$, we have

$$\mathcal{F}_k(f)(z) = \int_{\mathbb{R}^d} {}^tV_k(f)(x)e^{-i\langle x,z\rangle}dx.$$

Using Lemma 3.1.2, we can write

$$\begin{aligned} |\mathcal{F}_k(f)(z)| &\le \int_{\mathbb{R}^d} e^{\alpha\|x\|}|{}^tV_k(f)(x)|e^{-\alpha\|x\|+\|x\|\|v\|}dx \\ &\le \int_{\mathbb{R}^d} e^{\alpha\|x\|}|{}^tV_k(f)(x)|e^{-\alpha\|y\|}dx \end{aligned}$$

with $\|y\| = \|x\|(1 - \|v\|)$.
Relation (3.6) yields that there exists $f_1 \in L^p(\mathbb{R}^d, \mu_k)$ and $f_2 \in L^q(\mathbb{R}^d, \mu_k)$ for which

$$\int_{\mathbb{R}^d} e^{\alpha\|x\|}|{}^tV_k(f)(x)|e^{-\alpha\|y\|}dx \le K(\|f_1\|_{k,p} + \|f_2\|_{k,q}) < +\infty,$$

which furnishes the proof of Lemma 3.1.3. □

Thanks to the tools collected above, we can now prove our theorem.

3.2. Proof of Theorem 3.1

Case 1. $\alpha\beta > \frac{1}{4}$. Let h be a function on $\mathbb{R}^d$ defined by

$$g(z) = \left(\prod_{i=1}^{d} e^{\frac{z_i}{\alpha}}\right)\mathcal{F}_k(f)(z)$$

g is an entire function belongs to $\mathbb{C}^d$, then according to relation (3.7), we write

$$|g(z)| \leq K e^{\frac{\|u\|}{\alpha}}, \qquad \text{for all} \qquad u \in \mathbb{R}^d. \tag{3.8}$$

Moreover, observe that

$$\begin{aligned}\int_{\mathbb{R}^d} \log^+ |g(y)| dy &= \int_{\mathbb{R}^d} \log^+ |e^{\frac{\|y\|}{\alpha}} \mathcal{F}_k(f)(y)| dy \\ &= \int_{\mathbb{R}^d} \log^+ \left(\frac{e^{\beta\|y\|} |\mathcal{F}_k(f)(y)|}{\lambda} \lambda e^{\left(\frac{1}{\alpha} - \beta\right)\|y\|} \right) dy.\end{aligned}$$

For all positive constants $a, b > 0$ and using the fact that $\log^+ ab \leq \log^+ a + b$, we get

$$\int_{\mathbb{R}^d} \log^+ |g(y)| dy = \int_{\mathbb{R}^d} \log^+ \left(\frac{e^{\beta\|y\|} |\mathcal{F}_k(f)(y)|}{\lambda} \right) dy + \int_{\mathbb{R}^d} \lambda e^{\left(\frac{1}{\alpha} - \beta\right)\|y\|} dy.$$

Since $\alpha\beta > \frac{1}{4}$, relation (3.2) yields that

$$\int_{\mathbb{R}^d} \log^+ |g(y)| dy < +\infty. \tag{3.9}$$

Relations (3.8) and (3.9) assert that the function g satisfies (3.3), consequently g is a constant, we have then

$$\mathcal{F}_k(f)(y) = K e^{-\frac{\|y\|}{\alpha}}.$$

Since we have $\alpha\beta > \frac{1}{4}$, relation (3.2) makes sense as $K = 0$, furthermore, the injectivity of the k-Hankel transform gives that $f = 0$ a.e.

Case 2. $\alpha\beta = \frac{1}{4}$, as in the first case, we have that $\mathcal{F}_k(f)(y) = K e^{-\frac{\|y\|}{\alpha}}$. So, (3.2) holds as $|K| \leq \lambda$. Consequently, we get $f = K q^k_\beta(.)$ whenever $|K| \leq \lambda$.

Now, it remains the third case when $\alpha\beta < \frac{1}{4}$. If f is given like the form $f = K q^k_\beta(.)$, then its k-Hankel transform takes the form $\mathcal{F}_k(f)(y) = P(y) e^{-\delta\|y\|}$, then f and $\mathcal{F}_k(f)$ satisfy (3.1) and (3.2) for all $\delta \in]\beta, \alpha^{-1}[$,

which furnishes the proof of Theorem 3.1. □

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