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Existence and Uniqueness Results for a Coupled System of Nonlinear Fractional Langevin Equations

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Abstract. In this paper, we present a sufficient condition for the unique existence of solutions for a coupled system of nonlinear fractional Langevin equations with a new class of multipoint and nonlocal integral boundary conditions. We define a $\mathcal{Z}_\lambda^*$-contraction mapping and present the sufficient condition by identifying the problem with an equivalent fixed point problem in the context of b-metric spaces. Finally, some numerical examples are given to validate our main results.

1. Introduction

Langevin[8] developed a mathematical equation, naturally termed as Langevin equation, of Brownian motion in 1908; later, Kubo[7] conceived a generalized Langevin differential equation, with a fractional memory kernel, in order to depict the fractal processes. The existence and uniqueness of solutions for a Langevin initial value problem with two fractional orders, is investigated by Yu et al.[13] in 2014; the existence of solutions of nonlinear fractional initial and boundary value problems in this context are discussed in [1, 4, 5, 9, 10, 13, 15].

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In this paper, we consider the following coupled system of nonlinear fractional Langevin equations:

$$\begin{aligned} {}^cD^{\beta}\left({}^cD^{\alpha}+\gamma\right)x(t) &= f(t,x(t),y(t)) \\ {}^cD^{\beta}\left({}^cD^{\alpha}+\gamma\right)y(t) &= f(t,y(t),x(t)), \quad t\in[0,1] \end{aligned} \tag{1.1}$$

supplemented with the boundary conditions:

$$x(0)=y(0)=1; \tag{1.2}$$

$${}^cD^{\alpha}x(1)+\gamma x(1)={}^c D^{\alpha}y(1)+\gamma y(1)=0; \tag{1.3}$$

$$x(1)+y(1)=\sum_{i=1}^{n}a_i\int_0^{\delta_i}(x(s)+y(s))ds+b_i(x(\delta_i)+y(\delta_i)); \tag{1.4}$$

$$\int_0^1(x(t)-y(t))dt=0. \tag{1.5}$$

where, $\alpha\in(0,1]$, $\beta\in(1,2]$, $\gamma\in\mathbb{R}^*$, $a_i,b_i\in\mathbb{R}$ for $i=1,2,\cdots,n$, $0<\delta_1<\delta_2<\cdots<\delta_n<1$ and $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

This paper is organised as follows. In Section 2, some necessary antecedents are provided. In Section 3, we define a $\mathcal{Z}_{\lambda}^{*}$-contraction mapping and prove a coupled fixed point theorem. We apply the obtained coupled fixed point theorem to establish the sufficient conditions for the existence and uniqueness of solutions to the system (1.1) in Section 4. Finally, numerical examples are provided to show the applicability of our results.

2. Preliminaries

Definition 2.1.([2]) Let X be a nonempty set and $b\geq 1$ be a given real number. A function $d:X^2\to[0,\infty)$ is said to be b-metric if for all $x,y,z\in X$,

(B1) $d(x,y)=0\Leftrightarrow x=y$;

(B2) $d(x,y)=d(y,x)$;

(B3) $d(x,z)\leq b[d(x,y)+d(y,z)]$.

The pair (X,d) is called a b-metric space.

Definition 2.2.([3]) A point $(x,y)\in X^2$ is said to be a coupled fixed point of $F:X^2\to X$ if $F(x,y)=x$ and $F(y,x)=y$.

Definition 2.3.([6]) Let $\zeta:[0,\infty)^2\to\mathbb{R}$, then ζ is said to be a simulation function if it satisfies the following conditions:

($\zeta 1$) $\zeta(t,s) < s-t$, for all $t,s>0$;

($\zeta 2$) if $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$, then $\limsup_{n\to\infty} \zeta(t_n,s_n) < 0$.

We denote the collection of all simulation functions by $\mathcal{Z}$.

Definition 2.4.([13]) The Riemann-Liouville fractional integral of order q for a continuous function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds,\ q>0,$$

provided the integral exists, where Γ is the Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.

Definition 2.5.([13]) Let $f : [0,\infty) \to \mathbb{R}$ be an atleast n-times continuously differentiable function. Then the Caputo derivative of fractional order q of f is given by

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^n(s)}{(t-s)^{q+1-n}} ds,$$

$n-1 < q < n$, $n = [q]+1$, where $[q]$ denotes the integer part of the real number q.

Lemma 2.6.([13]) *For $q>0$, the general solution of the fractional differential equation $^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,\cdots,n-1$, $(n=[q]+1)$.

Lemma 2.7.([13]) *If $\beta > \alpha > 0$, $n=[\beta]+1$ and $x \in C^n[a,b]$, then*

i. *$^cD^\alpha I^\beta x(t) = I^{\beta-\alpha}x(t)$, holds almost everywhere on $[0,1]$ and it is valid at any point $t \in [0,1]$, whenever $x \in C[0,1]$; $^cD^\alpha I^\alpha x(t) = x(t)$, for all $t \in [0,1]$;*

ii. *$^cD^\alpha t^{\lambda-1} = \frac{\Gamma(\lambda)t^{\lambda-\alpha-1}}{\Gamma(\lambda-\alpha)}$, $\lambda > [\alpha]$ and $^cD^\alpha t^{\lambda-1} = 0$, $\lambda \le [\alpha]$,*

where $C^n[a,b]$ denotes set of all n times continuously differentiable function on $[a,b]$ and $C[0,1]$ denotes the set of all real valued continuous functions on $[0,1]$.

3. A New Coupled Fixed Point Theorem

In this section we define a particular class of simulation functions $\mathcal{Z}^*$ which we use to formulate a new contraction.

Definition 3.1. Let ζ be a simulation function. If we say ζ belongs to the class $\mathcal{Z}^*$ then $\zeta(t,s_1) \le \zeta(t,s_2)$ whenever $s_1 \le s_2$.

Here we note that the class $\mathcal{Z}^*$ is a proper subset of the class $\mathcal{Z}$, since the function $\zeta(t,s) = -t-s$ is in $\mathcal{Z}$, but not in $\mathcal{Z}^*$.

Definition 3.2. A 2-variable mapping $F : X^2 \to X$ is said to be a $\mathcal{Z}^*_\lambda$-contraction with respect to $\zeta \in \mathcal{Z}^*$ if there exists $\lambda \in [0, 1]$ such that

$$\zeta\left(bd\left(F(x, y), F(u, v)\right), (1-\lambda)\mathcal{M}_1(x, y, u, v) + \frac{\lambda}{4}\mathcal{M}_2(x, y, u, v)\right) \geq 0, \tag{3.1}$$

whenever

$$\frac{\lambda}{2}\mathcal{S}(x, y) \leq \mathcal{M}_1(x, y, u, v) + \frac{\lambda}{4}\mathcal{S}(u, v) \tag{3.2}$$

for all $x, y, u, v \in X$, where

$$\begin{aligned} \mathcal{S}(x, y) &= d(x, F(x, y)) + d(y, F(y, x)); \\ \mathcal{M}_1(x, y, u, v) &= \max\{d(x, u), d(y, v)\}; \\ \mathcal{M}_2(x, y, u, v) &= \max\{d(x, F(u, v)), d(y, F(v, u)), d(u, F(x, y)), d(v, F(y, x))\}. \end{aligned}$$

Theorem 3.3. *Let (X, d) be a complete b-metric space and $F : X^2 \to X$ be a $\mathcal{Z}^*_\lambda$-contraction with respect to $\zeta \in \mathcal{Z}^*$, then F has a unique coupled fixed point in X.*

Proof. Let $(x_0, y_0) \in X^2$, then we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n).$$

Suppose $\mathcal{M}_1(x_m, y_m, x_{m+1}, y_{m+1}) = 0$, for some $m \geq 0$, then $x_m = x_{m+1}$ and $y_m = y_{m+1}$, which implies (x_m, y_m) is a coupled fixed point of F as desired.

On the other hand, suppose $\mathcal{M}_1(x_n, y_n, x_{n+1}, y_{n+1}) \neq 0$ for all $n \geq 0$, then the following statements are hold:

1. The sequences $\{x_n\}$ and $\{y_n\}$ are asymptotically regular.
2. The sequences $\{x_n\}$ and $\{y_n\}$ must be Cauchy and has to converge to some x and y in X respectively, as X is complete.

From here the proof of the theorem proceeds in two natural steps: proving that (x, y) is the required coupled fixed point and then establishing its uniqueness. Indeed, suppose

$$\frac{\lambda}{2}\mathcal{S}(x_n, y_n) > \mathcal{M}_1(x_n, y_n, x, y) + \frac{\lambda}{4}\mathcal{S}(x, y) \tag{3.3}$$

for infinitely many n, then by letting limit $n \to \infty$, we have

$$\frac{\lambda}{4}(d(x, F(x, y)) + d(y, F(y, x))) \leq 0$$

which is possible only when $F(x,y) = x$ and $F(y,x) = y$; otherwise there must exists $N \in \mathbb{N}$ such that

$$\frac{\lambda}{2}\mathcal{S}(x_n, y_n) \leq \mathcal{M}_1(x_n, y_n, x, y) + \frac{\lambda}{4}\mathcal{S}(x, y)$$

for all $n \geq N$, but then by using contractive condition (3.1), we get

$$0 \leq \zeta\left(bd(x_{n+1}, F(x,y)), (1-\lambda)\mathcal{M}_1(x_n, y_n, x, y) + \frac{\lambda}{4}\mathcal{M}_2(x_n, y_n, x, y)\right),$$

for all $n \geq N$. Subsequently, by using ($\zeta1$), we get

$$bd(x_{n+1}, F(x,y)) \leq (1-\lambda)\mathcal{M}_1(x_n, y_n, x, y) + \frac{\lambda}{4}\mathcal{M}_2(x_n, y_n, x, y),$$

for all $n \geq N$. Thus by letting limit $n \to \infty$ on both sides of the above inequality, we get

$$bd(x, F(x,y)) \leq \frac{\lambda}{4}\max\{d(x, F(x,y)), d(y, F(y,x))\}. \tag{3.4}$$

Now by repeating the same arguments used to obtain Equation (3.4), just by interchanging x_n and y_n in Equation (3.3), we get

$$bd(y, F(y,x)) \leq \frac{\lambda}{4}\max\{d(y, F(y,x)), d(x, F(x,y))\}. \tag{3.5}$$

Consequently, by adding (3.4) and (3.5), we have

$$\begin{aligned} b(d(x, F(x,y)) + d(y, F(x,y))) &\leq \frac{\lambda}{2}\max\{d(x, F(x,y)), d(y, F(y,x))\} \\ &\leq \frac{\lambda}{2}(d(x, F(x,y)) + d(y, F(y,x))); \end{aligned}$$

but as $\lambda \in [0,1]$ and $b \geq 1$, it is easy to see that $d(x, F(x,y)) + d(y, F(x,y)) = 0$ which implies that (x,y) is a coupled fixed point of F as desired. Next, we claim that $x = y$. Suppose $x \neq y$,then

$$\begin{aligned} \frac{\lambda}{2}\mathcal{S}(x,y) &= \frac{\lambda}{2}(d(x, F(x,y)) + d(y, F(y,x))) \\ &= 0 \\ &\leq \mathcal{M}_1(x, y, y, x) + \frac{\lambda}{4}\mathcal{S}(y,x) \end{aligned}$$

by using contractive condition (3.1), we get

$$\begin{aligned} 0 &\leq \zeta\left(bd(x,y), (1-\lambda)d(x,y) + \frac{\lambda}{4}d(x,y)\right) \\ &\leq \zeta\left(bd(x,y), \left(1 - \frac{3\lambda}{4}\right)d(x,y)\right). \end{aligned}$$

By using ($\zeta 1$), we get

$$d(x,y) \le \left(\frac{4-3\lambda}{4b}\right) d(x,y).$$

It is not possible, since

$$\frac{4-3\lambda}{4b} \le 1.$$

Hence it results that $x = y$ as required.

To the end, suppose (u,u) is an another coupled fixed point of F, then we have

$$\begin{aligned} \frac{\lambda}{2}\mathcal{S}(x,x) &= 0 \\ &\le \mathcal{M}_1(x,x,u,u) + \frac{\lambda}{4}\mathcal{S}(u,u) \end{aligned}$$

and therefore by using contractive condition (3.1), we get

$$0 \le \zeta\left(bd(x,u), (1-\lambda)d(x,u) + \left(1 - \frac{3\lambda}{4}\right) d(x,y)\right);$$

consecutively, by using ($\zeta 1$), we get

$$d(x,u) \le \left(\frac{4-3\lambda}{4b}\right) d(x,u),$$

which implies $x = u$, as $\frac{4-3\lambda}{4b} \le 1$, which proves the uniqueness of (x,x). □

Example 3.4. Let $X = [0, 1.4]$ and $d : X^2 \to [0,\infty)$ be the mapping defined by $d(x,y) = |x-y|^2$, then clearly (X,d) is a complete b-metric space with coefficient 2. Let $F : X^2 \to X$ be the mapping defined by

$$F(x,y) = \begin{cases} \frac{\sin(x+y)}{2} & \text{if } (x,y) \in [1, 1.4] \\ \frac{x}{3} & \text{otherwise} \end{cases}$$

If we let $\lambda = 0.5$ and $\zeta(t,s) = 0.9s - t$, then the contractive condition (3.1) in Theorem 3.3 is satisfied and it is visible that, $(0,0)$ is a unique coupled fixed point of F.

Corollary 3.5. *Let (X,d) be a complete b-metric space and $F : X^2 \to X$ be a 2-variable mapping. If there exists $\zeta \in \mathcal{Z}^*$ such that*

$$0 \le \zeta\left(bd\left(F(x,y), F(u,v)\right), \max\{d(x,u), d(y,v)\}\right), \tag{3.6}$$

for all $x,y,u,v \in X$, then F has a unique coupled fixed point in X.

Proof. By letting $\lambda = 0$ in Theorem 3.3, we get the proof. □

Remarks 3.6. In corollary 3.5,

(i) If we take $b = 1$, then we get the result of Santhi et al.[11].

(ii) If we let $\zeta(t,s) = ks - t$, where $k \in (0,1)$ then we get the result posted by Bhaskar and Lakshmikantham[3].

(iii) If we let $\zeta(t,s) = \phi(s) - t$ where $\phi : [0,\infty) \to [0,\infty)$ is an increasing function with $0 = \phi(0) < \phi(t) < t$ and $\lim\limits_{r\to t^+} \phi(r) < t$ for each $t > 0$, then we get the results of Sintunavarat et al.[12] and Zlatanov et al.[14].

4. Existence Results

Let us define some notations for our convenience.

$$I_0^\theta = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\theta(s)ds; \qquad I_1^\theta = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\theta(s)ds;$$

$$I_2^\theta = \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\theta(s)ds; \qquad I_3^\theta = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\theta(s)ds;$$

$$I_4^\theta = \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\theta(s)ds; \qquad I_5^\theta = \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\theta(s)ds;$$

$$I_6^\theta = \int_0^1 \frac{(1-s)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\theta(s)ds; \qquad I_7^\theta = \int_0^{\delta_i} \frac{(\delta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\theta(s)ds;$$

$$I_8^\theta = \int_0^{\delta_i} \frac{(\delta_i-s)^{\alpha}}{\Gamma(\alpha+1)}\theta(s)ds; \qquad I_9^\theta = \int_0^{\delta_i} \frac{(\delta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\theta(s)ds;$$

$$I_{10}^\theta = \int_0^{\delta_i} \frac{(\delta_i-s)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\theta(s)ds.$$

In addition, let

$$\Delta_1 = \frac{1}{\Gamma(\alpha+1)} - \frac{\sum\limits_{i=1}^n a_i\delta_i^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{\sum\limits_{i=1}^n b_i\delta_i^{\alpha}}{\Gamma(\alpha+1)}; \quad \Delta_2 = \frac{1}{\Gamma(\alpha+2)} - \frac{\sum\limits_{i=1}^n a_i\delta_i^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{\sum\limits_{i=1}^n b_i\delta_i^{\alpha+1}}{\Gamma(\alpha+2)};$$

$$G = \frac{1}{\Gamma(\alpha+3)} - \frac{1}{\Gamma(\alpha+2)}; \qquad G^* = \frac{1}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)};$$

$$A = \frac{\sum\limits_{i=1}^n a_i}{\Delta_2 - \Delta_1}; \qquad B = \frac{\sum\limits_{i=1}^n b_i}{\Delta_2 - \Delta_1}.$$

Here note that, none of the constants G, Δ_1, Δ_2 are zero and $\Delta_1 \neq \Delta_2$.

Lemma 4.1. *Let $h, k : [0,1] \to \mathbb{R}$ be continuous functions. Then the solution of the system of fractional differential equations*

$$\begin{aligned} {}^cD^\beta \left({}^cD^\alpha + \gamma\right) x(t) &= h(t) \\ {}^cD^\beta \left({}^cD^\alpha + \gamma\right) y(t) &= k(t), \quad t \in [0,1] \end{aligned} \tag{4.1}$$

supplemented with the boundary conditions (1.2-1.5) is equivalent to the solution of the following system of integral equations

$$x(t) = \begin{cases} -\gamma I_0^x + I_1^h - I_2^h \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{2}\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\ \left(\frac{I_2^{h-k}}{G\Gamma(\alpha+2)} + \frac{\Delta_1 I_2^{h+k}}{\Delta_2-\Delta_1} + \frac{\gamma I_3^{x+y}}{\Delta_2-\Delta_1} + \frac{\gamma I_4^{x-y}}{G} - \frac{I_5^{h+k}}{\Delta_2-\Delta_1}\right. \\ \left. -\frac{I_6^{h-k}}{G} - \gamma B I_7^{x+y} - \gamma A I_8^{x+y} + B I_9^{h+k} + A I_{10}^{h+k}\right); \end{cases} \tag{4.2}$$

$$y(t) = \begin{cases} -\gamma I_0^y + I_1^k - I_2^k \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{2}\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\ \left(-\frac{I_2^{h-k}}{G\Gamma(\alpha+2)} + \frac{\Delta_1 I_2^{h+k}}{\Delta_2-\Delta_1} + \frac{\gamma I_3^{x+y}}{\Delta_2-\Delta_1} - \frac{\gamma I_4^{x-y}}{G} - \frac{I_5^{h+k}}{\Delta_2-\Delta_1}\right. \\ \left. +\frac{I_6^{h-k}}{G} - \gamma B I_7^{x+y} - \gamma A I_8^{x+y} + B I_9^{h+k} + A I_{10}^{h+k}\right). \end{cases} \tag{4.3}$$

Proof. Using Lemmas 2.6 and 2.7, we can reduce the fractional differential equation (4.1) to the following system of integral equations

$$\begin{aligned} x(t) &= I_1^h + c_0 \frac{t^\alpha}{\Gamma(\alpha+1)} + c_1 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \gamma I_0^x + c_2; \\ y(t) &= I_1^k + d_0 \frac{t^\alpha}{\Gamma(\alpha+1)} + d_1 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \gamma I_0^y + d_2, \end{aligned}$$

where $c_0, c_1, c_2, d_0, d_1, d_2 \in \mathbb{R}$ are arbitrary constants. Thus to substantiate our claim, it is enough to find the constants c_0, c_1, c_2, d_0, d_1 and d_2, which can be done by simple computations as follows: using the boundary condition (1.2), we get

$$c_2 = d_2 = 0.$$

Subsequently, it is easy to derive the following expressions:

$$c_0 + c_1 = -I_2^h \tag{4.4}$$

and

$$d_0 + d_1 = -I_2^k \tag{4.5}$$

by using the boundary condition (1.3).

Adding the above two equations, we get

$$c_0 + d_0 = -I_2^{h+k} - (c_1 + d_1). \tag{4.6}$$

Subtracting the above two equations, we get

$$c_0 - d_0 = -I_2^{h-k} - (c_1 - d_1). \tag{4.7}$$

Further, we have

$$\int_0^{\delta_i} (x(s) + y(s))ds = I_{10}^{h+k} + \frac{\delta_i^{\alpha+1}}{\Gamma(\alpha+2)}(c_0 + d_0) + \frac{\delta_i^{\alpha+2}}{\Gamma(\alpha+3)}(c_1 + d_1) - \gamma I_8^{x+y}$$

and

$$x(\delta_i) + y(\delta_i) = I_9^{h+k} + \frac{\delta_i^{\alpha}}{\Gamma(\alpha+1)}(c_0 + d_0) + \frac{\delta_i^{\alpha+1}}{\Gamma(\alpha+2)}(c_1 + d_1) - \gamma I_7^{x+y}$$

for all $1 \leq i \leq n$ and therefore from the boundary conditions (1.4), it is easy to derive

$$\begin{aligned}(c_0 + d_0)\Delta_1 + (c_1 + c_2)\Delta_2 \quad &= \quad -I_5^{h+k} + \gamma I_3^{x+y} + \sum_{i=1}^{n} a_i I_{10}^{h+k} \\ &\quad -\gamma \sum_{i=1}^{n} a_i I_8^{x+y} + \sum_{i=1}^{n} b_i I_9^{h+k} - \gamma \sum_{i=1}^{n} b_i I_7^{x+y}.\end{aligned} \tag{4.8}$$

On the other side, by computing the value of

$$\int_0^1 (x(t) - y(t))dt = I_6^{h-k} + \frac{c_0 - d_0}{\Gamma(\alpha+2)} + \frac{c_1 - d_1}{\Gamma(\alpha+3)} - \gamma I_4^{x-y};$$

and substituting it in boundary condition (1.5), we have

$$\frac{c_0 - d_0}{\Gamma(\alpha+2)} + \frac{c_1 - d_1}{\Gamma(\alpha+3)} = -I_6^{h-k} + \gamma I_4^{x-y}. \tag{4.9}$$

Consecutively, by substituting (4.6) in (4.8) and (4.7) in (4.9), we get

$$c_1 + d_1 = \frac{I_2^{h-k}}{G\Gamma(\alpha+2)} - \frac{I_6^{h-k}}{G} + \gamma \frac{I_4^{x-y}}{G} \tag{4.10}$$

and

$$\begin{aligned}c_1 - d_1 \quad &= \quad \frac{I_2^{h+k}\Delta_1}{\Delta_2 - \Delta_1} - \frac{I_5^{h+k}}{\Delta_2 - \Delta_1} + \gamma \frac{I_3^{x+y}}{\Delta_2 - \Delta_1} \\ &\quad +AI_{10}^{h+k} - \gamma A I_8^{x+y} + BI_9^{h+k} - \gamma B I_7^{x+y}\end{aligned} \tag{4.11}$$

respectively. Further, by adding (4.10) and (4.11), it results that

$$\begin{aligned} c_1 &= \frac{1}{2}\Bigg(\frac{I_2^{h-k}}{G\Gamma(\alpha+2)} + \frac{I_2^{h+k}\Delta_1}{\Delta_2-\Delta_1} + \gamma\frac{I_3^{x+y}}{\Delta_2-\Delta_1} + \gamma\frac{I_4^{x-y}}{G} \\ &\quad -\frac{I_5^{h+k}}{\Delta_2-\Delta_1} - \frac{I_6^{h-k}}{G} - \gamma B I_7^{x+y} - \gamma A I_8^{x+y} + B I_9^{h+k} + A I_{10}^{h+k}\Bigg). \end{aligned} \tag{4.12}$$

Subsequently, by subtracting (4.10) from (4.11), we get

$$\begin{aligned} d_1 &= \frac{1}{2}\Bigg(-\frac{I_2^{h-k}}{G\Gamma(\alpha+2)} + \frac{I_2^{h+k}\Delta_1}{\Delta_2-\Delta_1} + \gamma\frac{I_3^{x+y}}{\Delta_2-\Delta_1} - \gamma\frac{I_4^{x-y}}{G} \\ &\quad -\frac{I_5^{h+k}}{\Delta_2-\Delta_1} + \frac{I_6^{h-k}}{G} - \gamma B I_7^{x+y} - \gamma A I_8^{x+y} + B I_9^{h+k} + A I_{10}^{h+k}\Bigg). \end{aligned} \tag{4.13}$$

Finally, by using (4.12) and (4.13), we get

$$x(t) = -\gamma I_0^x + I_1^h - I_2^h\frac{t^\alpha}{\Gamma(\alpha+1)} + c_1\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^\alpha}{\Gamma(\alpha+1)}\right)$$

and

$$y(t) = -\gamma I_0^y + I_1^k - I_2^k\frac{t^\alpha}{\Gamma(\alpha+1)} + d_1\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^\alpha}{\Gamma(\alpha+1)}\right)$$

as desired. □

The following theorem gives a sufficient condition for the unique existence of a solution for a system of non-linear fractional differential equations.

Theorem 4.2. *Suppose there exists a constant $L > 0$ such that*

$$|f(t,x(t),y(t)) - f(t,u(t),v(t))| \le L\max\{|x(t)-u(t)|, |y(t)-v(t)|\},$$

for each $t \in [0,1]$ and $x(t), y(t), u(t), v(t) \in C[0,1]$ and if

$$2L^2R^2 < 1,$$

where

$$\begin{aligned} R &= \frac{\gamma}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &\quad + \frac{G^*}{2}\Bigg(\frac{\rho_1}{\Gamma(\alpha+2)} + \frac{\Delta_1}{(\Delta_2-\Delta_1)(\Gamma(\beta+1))} + \frac{2\rho_2}{\Gamma(\alpha+1)} \\ &\qquad + \frac{\rho_3}{\Gamma(\alpha+\beta+1)} + \frac{\rho_4}{\Gamma(\alpha+\beta+2)}\Bigg); \end{aligned}$$

with

$$\begin{aligned}
\rho_1 &= \frac{1}{G\Gamma(\beta+1)} + \frac{2\gamma}{G} + 2A\gamma\sum_{i=1}^{n}\delta_i^{\alpha+1};\\
\rho_2 &= \frac{\gamma}{\Delta_2-\Delta_1} + \gamma B\sum_{i=1}^{n}\delta_i^{\alpha};\\
\rho_3 &= \frac{1}{\Delta_2-\Delta_1} + B\sum_{i=1}^{n}\delta_i^{\alpha+\beta};\\
\rho_4 &= \frac{1}{G} + A\sum_{i=1}^{n}\delta_i^{\alpha+\beta+1}.
\end{aligned}$$

Then the system of fractional boundary value problem (1.1) has a unique solution on $[0,1]$.

Proof. Indeed, in order to establish the inference of the theorem, we exhibit the existence of solution for the coupled system of integral equations obtained from previous lemma through our theory.

Let $\mathcal{C}$ be the complete b-metric space of all continuous functions from $[0,1]$ to $\mathbb{R}$ with the metric

$$d(x(t),y(t)) = \sup_{t\in[0,1]} |x(t)-y(t)|^2.$$

Then it is clear to see that, the solutions of the derived coupled system of integral equations are elements of $\mathcal{C}$. Let

$$\begin{aligned}
f_{xy} &= f(t,x(t),y(t));\\
\psi_1 &= |f(t,x(t),y(t)) - f(t,u(t),v(t))|;\\
\psi_2 &= |f(t,x(t),y(t)) + f(t,y(t),x(t)) - f(t,u(t),v(t)) - f(t,v(t),u(t))|;\\
\psi_3 &= |x(t)-u(t)|;\\
\psi_4 &= |x(t)+y(t)-u(t)-v(t)|.
\end{aligned}$$

Let $F:\mathcal{C}^2\to\mathcal{C}$ be the function defined by

$$F(x(t),y(t)) = \begin{cases} -\gamma I_0^x + I_1^{f_{xy}} - I_2^{f_{xy}}\frac{t^\alpha}{\Gamma(\alpha+1)} \\ +\frac{1}{2}\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(\frac{I_2^{f_{xy}-f_{yx}}}{G\Gamma(\alpha+2)} + \frac{\Delta_1 I_2^{f_{xy}+f_{yx}}}{\Delta_2-\Delta_1}\right. \\ +\frac{\gamma I_3^{x+y}}{\Delta_2-\Delta_1} + \frac{\gamma I_4^{x-y}}{G} - \frac{I_5^{f_{xy}+f_{yx}}}{\Delta_2-\Delta_1} - \frac{I_6^{f_{xy}-f_{yx}}}{G} \\ \left. -\gamma B I_7^{x+y} - \gamma A I_8^{x+y} + B I_9^{f_{xy}+f_{yx}} + A I_{10}^{f_{xy}+f_{yx}}\right). \end{cases}$$

Then

$$
\begin{aligned}
&|F(x(t),y(t)) - F(u(t),v(t))| \\
&\leq \gamma I_0^{\psi_3} + I_1^{\psi_1} + I_2^{\psi_1}\frac{|t|^\alpha}{\Gamma(\alpha+1)} + \frac{1}{2}\left(\frac{|t|^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{|t|^\alpha}{\Gamma(\alpha+1)}\right) \\
&\quad \left(\frac{I_2^{\psi_2}}{G\Gamma(\alpha+2)} + \frac{\Delta_1 I_2^{\psi_2}}{\Delta_2-\Delta_1} + \frac{\gamma I_3^{\psi_4}}{\Delta_2-\Delta_1} + \frac{\gamma I_4^{\psi_4}}{G} + \frac{I_5^{\psi_2}}{\Delta_2-\Delta_1}\right. \\
&\quad \left. + \frac{I_6^{\psi_2}}{G} + \gamma B I_7^{\psi_4} + \gamma A I_8^{\psi_4} + B I_9^{\psi_2} + A I_{10}^{\psi_2}\right) \\
&\leq L\max\{|x(t)-u(t)|, |y(t)-v(t)|\} \\
&\quad \left(\frac{\gamma|t^\alpha|}{\Gamma(\alpha+1)} + \frac{|t^{\alpha+\beta}|}{\Gamma(\alpha+\beta+1)} + \frac{|t^\alpha|}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right. \\
&\quad + \frac{1}{2}\left(\frac{|t^{\alpha+1}|}{\Gamma(\alpha+2)} + \frac{|t^\alpha|}{\Gamma(\alpha+1)}\right)\left(\frac{1}{G\Gamma(\alpha+2)\Gamma(\beta+1)} + \frac{\Delta_1}{(\Delta_2-\Delta_1)(\Gamma(\beta+1))}\right. \\
&\quad + \frac{2\gamma}{(\Delta_2-\Delta_1)(\Gamma(\alpha+1))} + \frac{2\gamma}{G\Gamma(\alpha+2)} + \frac{1}{(\Delta_2-\Delta_1)\Gamma(\alpha+\beta+1)} \\
&\quad + \frac{1}{G\Gamma(\alpha+\beta+2)} + \sum_{i=1}^n \delta_i^\alpha \frac{2\gamma B}{\Gamma(\alpha+1)} + \sum_{i=1}^n \delta_i^{\alpha+1}\frac{2\gamma A}{\Gamma(\alpha+2)} \\
&\quad \left.\left. + \sum_{i=1}^n \delta_i^{\alpha+\beta}\frac{B}{\Gamma(\alpha+\beta+1)} + \sum_{i=1}^n \delta_i^{\alpha+\beta+1}\frac{A}{\Gamma(\alpha+\beta+2)}\right)\right) \\
&\leq LR\max\{|x(t)-u(t)|, |y(t)-v(t)|\},
\end{aligned}
$$

where

$$
\begin{aligned}
R &= \frac{\gamma}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad + \frac{G^*}{2}\left(\frac{\rho_1}{\Gamma(\alpha+2)} + \frac{\Delta_1}{(\Delta_2-\Delta_1)(\Gamma(\beta+1))} + \frac{2\rho_2}{\Gamma(\alpha+1)}\right. \\
&\qquad\qquad \left. + \frac{\rho_3}{\Gamma(\alpha+\beta+1)} + \frac{\rho_4}{\Gamma(\alpha+\beta+2)}\right).
\end{aligned}
$$

Therefore

$$bd(F(x(t),y(t)), F(u(t),v(t))) \leq 2L^2R^2\max(d(x(t),u(t)), d(u(t),v(t))),$$

which implies

$$0 \leq 2L^2R^2\max(d(x(t),u(t)), d(u(t),v(t))) - bd(F(x(t),y(t)), F(u(t),v(t))).$$

Since $2L^2R^2 < 1$, by letting $\zeta(t,s) = 2L^2R^2 s - t$ in Corollary 3.5, it is clear to see that the 2-variable mapping F satisfies the hypothesis of 3.5 and therefore F has a unique coupled fixed point in $\mathcal{C}$. Thus the coupled system of Langevin fractional differential equation has a unique solution. □

Example 4.3. Consider the coupled system of fractional Langevin boundary value problem given by

$$\begin{aligned} {}^cD^{1.75}({}^cD^{0.5}+5)x(t) &= t+t^3x+\sin y; \\ {}^cD^{1.75}({}^cD^{0.5}+5)y(t) &= t+t^3y+\sin x; \end{aligned}$$

$$x(0)=y(0)=0;$$
$${}^cD^{0.5}(x(1))+5x(1)={}^c D^{0.5}(y(1))+5y(1)=0;$$

$$\begin{aligned} x(1)+y(1) \;=\; & 8\int_0^{0.1}(x(s)+y(s))ds-1.75\int_0^{0.4}(x(s)+y(s))ds \\ & +x(0.1)+y(0.1)-7.25(x(0.4)+y(0.4)); \end{aligned}$$

and

$$\int_0^1 (x(t)+y(t))dt=0,$$

where $\alpha=0.5$, $\beta=1.75$, $\gamma=0.5$, $n=2$, $a_1=8$, $a_2=-1.75$, $b_1=1$, $b_2=-7.25$, $\delta_1=0.1$, $\delta_2=0.4$, $f(t,x(t),y(t))=t+t^3x+\sin y$. Then
$|f(t,x(t),y(t))-f(t,u(t),v(t))|$

$$\begin{aligned} &= |t+t^3x(t)+\sin y(t)-t-t^3u(t)-\sin v(t)| \\ &\leq |t^3||x(t)-u(t)|+|y(t)-v(t)| \\ &\leq 2\max\{|x(t)-u(t)|,|y(t)-v(t)|\}. \end{aligned}$$

Calculating Δ_1, Δ_2, G and R, we get $\Delta_1=6.0882$, $\Delta_2=2.1539$, $G=-0.4513$ and $R=-0.2278$. Consequently, by letting $L=2$ and computing $2L^2R^2$, we get

$$2L^2R^2=0.4153<1$$

as desired. Thus the system satisfies all the conditions of Theorem 4.2 and hence possess a unique solution.

References

[1] H. Baghani, J. Alzabut and J. J. Nieto, *A coupled System of Langevin Differential Equations of Fractional Order and Associated to Antiperiodic Boundary Conditions*, Math Meth Appl Sci., (2020), 1–11.

[2] I. A. Bakhtin, *The Contraction Mapping Principle in Quasi Metric Spaces*, Funct. Anal. Unianowsk Gos. Ped. Inst., **30**(1989), 26–37.

[3] T. G. Bhaskar and V. Lakshmikantham, *Fixed Point Theorems in Partially Ordered Metric Spaces and Applications*, Nonlinear Analysis, **65(7)**(2006), 1379–1393.

[4] R. Darzi, B. Agheli and J. J. Nieto, *Langevin Equation Involving Three Fractional Orders*, J. Stat. Phys., **178**(2020), 986–995.

[5] H. Fazli, H. G. Sun and J. J. Nieto, *Fractional Langevin Equation Involving Two Fractional Orders: Existence and Uniqueness Revisited*, Mathematics, **8(5)**(2020), 1–10.

[6] F. Khojasteh, S. Shukla and S. Radenovic, *A New Approach to the Study of Fixed Point Theory for Simulation Functions*, Filomat, **29(6)**(2015), 1189–1194.

[7] R. Kubo, *The fluctuation-dissipation theorem*, Rep. Prog. Phys., **29**(1966), 255–284.

[8] P. Langevin, *On the theory of Brownian motion*, Comptes Rendus, **146**(1908), 530–533.

[9] A. K. Mirmostafaee, *Coupled Fixed Points for Mappings on a b-metric space with a Graph*, Mathematicki Vesnik, **69(3)**(2017), 214–225.

[10] A. Salem, F. Alzahrani and M. Alnegga, *Coupled System of Nonlinear Fractional Langevian Equation with Multipoint and Nonlocal Integral Boundary Conditions*, Mathematical Problems in Engineering, (2020), 1–15.

[11] A. Santhi, S. Muralisankar and R. P. Agarwal, *A New Coupled Fixed Point Theorem via Simulation Function with Application*, Matematicki Vesnik, **73(3)**(2021), 209–222.

[12] W. Sintunavarat, P. Kumam and Y. J. Cho, *Coupled Fixed Point Theorems for Nonlinear Contractions without Mixed Monotone Property*, Fixed Point Theory Appl., **170**(2012), 1–12.

[13] T. Yu and M. Luo, *Existence and Uniqueness of Solutions of Initial Value Problems for Nonlinear Langevin Equation Involving Two Fractional Orders*, Commun. Nonlinear. Sci. Numer. Simulat., **19**(2014), 1661–1668.

[14] B. Zlatanov, *A Variational Principle and Coupled Fixed Points*, J. Fixed Point Theory Appl., **21(69)**(2019), 1–13.

[15] H. Zhou, J. Alzabut and L. Yang, *On fractional Langevin differential equations with anti-periodic boundary conditions*, Eur. Phys. J. Spec. Topics., **226**(2017), 3577–3590.