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On f-cosymplectic and (k,μ)-cosymplectic Manifolds Admitting Fischer -Marsden Conjecture

Sangeetha Mahadevappa* and Halammanavar Gangadharappa Nagaraja
Department of Mathematics, Bangalore University, Bengaluru 560056, India
e-mail : msangeetha336@gmail.com *and* hgnraj@yahoo.com

Abstract. The aim of this paper is to study the Fisher-Marsden conjucture in the frame work of f-cosymplectic and (k,μ)-cosymplectic manifolds. First we prove that a compact f-cosymplectic manifold satisfying the Fisher-Marsden equation $R_g^{'*}=0$ is either Einstein manifold or locally product of Kahler manifold and an interval or unit circle S^1. Further we obtain that in almost (k,μ)-cosymplectic manifold with $k<0$, the Fisher-Marsden equation has a trivial solution.

1. Introduction

Let $\Im$ denote the set of all smooth Riemannian metrics on a $2n+1$,$(n>1)$-dimensional closed, connected and orientable Riemannian manifold (M,g) of unit volume, whose derivatives are L_2-integrable. Then for any $g\in\Im$ we can define the scalar curvature operator from $\Im$ to the set of all C^∞ functions $\Re$ on M, which is quasi linear differential operator of second order. Here we can see that for any $g\in\Im$ its scalar curvature R_g is a non-linear function of the metric g. So that the linearization of R_g is $R_g^{'}(\overline{g})$, for any symmetric bilinear 2-tensor $\overline{g}$ such that

$$R_g^{'}(\overline{g})=-\triangle_g(tr_g\overline{g})+\delta\delta(\overline{g})-g(Ric_g,\overline{g}),$$

where tr is the trace, $\delta\delta$ is the double covariant divergence, $\triangle_g$ is the Laplacian, Ric_g is the Ricci curvature tensor of g and $\overline{g}$ is the $(0,2)$ symmetric tensor on the manifold M.

Now it is easy to derive the L_2-adjoint operator $Rg^{'*}$ of $Rg^{'}$, which linearizes the scalar curvature operator $Rg^{'}$ and it is given by

$$Rg^{'*}(\lambda)=-(\triangle_g\lambda)g+\nabla_g^2\lambda-\lambda Ric_g, \quad (1.1)$$

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where $\nabla_g^2\lambda$ is the Hessian of the smooth function λ on M. On contracting the above equation, we obtain

$$v = \triangle_g \lambda = -\frac{R_g \lambda}{2n}. \tag{1.2}$$

Here we designate $Rg^{'*}(\lambda) = 0$ as Fischer-Marsden equation and the ordered pair (g, λ) is the solution for the Fischer-Marsden equation.

In [4], Bourguignon and in [3], Fischer-Marsden proved that if (g, λ) is a non-trivial solution for the equation $Rg^{'*}(\lambda) = 0$ on the complete Riemannian manifold then the scalar curvature of the manifold is constant. Corvino [8] showed that on a compact Riemannian manifold, the warped product metric $g^* = g - \lambda^2 dt^2$ has a non-trivial solution for the Fischer Marsden equation $Rg^{'*}(\lambda) = 0$ if and only if the metric is Einstein.

In[17] Shen studied the Fischer Marsden conjecture using some ideas of general relativity and find Robinson-type identity for the over determined system of partial differential equation and they also showed that if a 3-dimensional closed manifold of positive curvature has a non-trivial solution for the over determined system then the manifold contains a totally geodesic 2-sphere.

We call back Fischer-Marsden conjecture in [10] as "A compact Riamannian manifold that satisfies the equation $Rg^{'*}(\lambda) = 0$ with non-trivial solution is necessarily an Einstein manifold".

Later Kobayashi gave the counter example for the Fischer-Marsden Conjecture in [11]. However J. Lafontaine [12] also showed the counter example for the conjecture when metric g is conformally flat. The counter examples showed in [11] and [12], almost all are homogeneous, but we can see the non-homogeneous counter examples for Fischer-Marsden equation which are proved by Cernea and Guan in [5].

Furthermore they showed that for closed homogeneous Riemannian manifold (M^n, g), if λ is the non-trivial solution of the Fischer-Marsden equation $Rg^{'*}(\lambda) = 0$ then M is isomorphic to $S^p \times N$, where S^p denotes the Euclidean sphere and N represents the Einstein manifold. In [15] Patra and Ghosh studied the Fischer-Marsden conjecture in the frame work of K-contact and (k, μ)-contact manifolds and proved that, if λ is non-trivial solution of the Fischer Marsden equation $Rg^{'*}(\lambda) = 0$ for complete K-contact manifold then it is Einstein and isometric to the unit sphere of dimension $(2n + 1)$. Further they considered the non-trivial solution for the Fischer-Marsden equation in the case of (k, μ)-contact metric manifold of non-Sasakian type of dimension $(2n + 1)$ and showed that for $n = 3$, M^3 is flat and for $n > 1$, M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$.

Later in [13] Li studied the Fischer-Marsden conjecture and gave a counter example to the conjecture on vacuum static spaces. Recently Mandal [17] showed that if, 3-dimensional non-Kenmotsu $(k, \mu)'$-almost Kenmotsu manifold satisfies the Fischer-Marsden equation then it is locally isometric to the product space $H^2(-4) \times R$, along with this they also showed that, if conformal Reeb foliation admits the

Fischer Marsden equation in complete almost Kenmotsu manifold then the manifold is Einstein.

Further Prakasha et. al., [16] and Chaubey et. al., [6] studied the conjecture for non-Kenmotsu $(k, \mu)^{'}$-almost Kenmotsu manifold of dimension $(2n+1)$ and Kenmotsu manifold respectively and proved some interesting results.

Motivated by the above research, we examine the properties of f-cosymplectic and (k, μ) cosymplectic manifolds satisfying Fischer Marsden equation.

2. Preliminaries

Let M be a differential manifold of dimension $(2n+1)$ admitting a triple (ϕ, ξ, η), where ϕ is the $(1,1)$ tensor such that $\phi^2 = -I + \eta \otimes \xi$, ξ is the unit vector field and η is the global 1-form satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on M such that $\eta(\xi) = 1$, $\phi\xi = 0$, $\eta \circ \phi = 0$ and $rank\phi = 2n$. Now M along with the triple (ϕ, ξ, η) is called almost contact manifold. A Riemannian metric g on M is said to be compatible if $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ and $g(X, \xi) = \eta(X)$, for all vector fields X and Y on M. An almost contact structure along with the Riemannian metric is called almost contact metric structure and (M, ϕ, ξ, η, g) is called almost contact metric manifold. An almost contact metric structure is normal if Nijehuis torsion tensor

$$N_\phi(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] - 2d\eta[X, Y]\xi,$$

vanishes everywhere on M.

By defining a fundamental 2-form $\omega(X, Y) = g(\phi X, Y)$ for all $X, Y \in TM$, if 1-form η and ω are closed on an almost contact manifold (M, ϕ, ξ, η) then the structure (ϕ, ξ, η) becomes almost cosymplectic. If almost cosymplectic structure is normal then it is called cosymplectic. Similarly an almost contact structure with closed η and $d\omega = 2\alpha\, \eta \wedge \omega$ for some non-zero constant α is called almost α-Kenmotsu manifold. If α is any real number then the almost contact structure becomes α-almost cosymplectic manifold [8].

Aktan et. al., [1] generalizes the almost α-cosymplectic manifold to almost f-cosymplectic manifold by assuming real number α to be any smooth function f on M in which $d\eta = 0$ and $d\omega = 2f\eta \wedge \omega$ for some smooth function f satisfying $df \wedge \eta = 0$. Similarly, if M is normal then almost f-cosymplectic structure becomes f-cosymplectic. Clearly, a f-cosymplectic manifold becomes either α-Kenmotsu manifold or cosymplectic manifold concerning the condition α is a non-zero real number or $f = 0$ respectively. Further $D = ker\eta$, distribution on f-cosymplectic manifold exists and integrable since $d\eta = 0$. For an almost contact manifold we can define a self adjoint operator $h = \frac{1}{2}L_\xi\phi$ and $h^{'} = h \circ \phi$ satisfying $tr(h) = tr(h\phi) = 0$, $h\xi = 0$ and $\phi h = -h\phi$ for all vector fields X and Y on M.

Let M be an f-cosymplectic manifold. The Levi-civita connection ∇ on M is given by

$$2g((\nabla_X\phi)Y, Z) = 2fg(g(\phi X, Y)\xi - \eta(X)\phi X, Z) + g(N_\phi(Y, Z), \phi X),$$

which implies

$$\nabla_X \xi = -f\phi^2 X - \phi h X \ \ and \ \ \nabla_\xi \phi = 0, \tag{2.1}$$

for any vector fields X and Y on M.

Moreover, if $(M^{2n+1}, \phi, \xi, \eta, g)$ is an almost f-cosymplectic manifold then it satisfies the following equations [14],

$$R(X,\xi)\xi - \phi R(\phi X, \xi) = 2[\tilde{f}\phi^2 X - h^2 X], \tag{2.2}$$

$$Ric(\xi,\xi) = -2n\tilde{f} - tr(h^2), \tag{2.3}$$

$$R(X,\xi)\xi = \tilde{f}\phi^2 X - 2f\phi h X - h^2 X + \phi(\nabla_\xi h)X, \tag{2.4}$$

$$(\nabla_\xi h)X = -\phi R(X,\xi)\xi - f^2\phi X - 2fhX - \phi h^2 X, \tag{2.5}$$

for every vector fields X and Y on M, where

$$\tilde{f} = f^2 + (\xi f). \tag{2.6}$$

In the following sections (M, ϕ, ξ, η, g) denotes an almost f-cosymplectic manifold of dimension $(2n+1)$ which is normal so implies that $h = 0$ [9].

3. f-cosymplectic Manifold Satisfying Fischer-Marsden Equation

In this section, we study the f-cosymplectic satisfying (1.1) with $Rg^{'*}(\lambda) = 0$. Therefore M is a f-cosymplectic manifold of dimension $(2n+1)$ and by Proposition 9 and Proposition 10 of [14], we have the following:

$$\nabla_X \xi = -f\phi^2 X, \tag{3.1}$$

$$Q\xi = -2n\tilde{f}\xi, \tag{3.2}$$

$$R(X,Y)\xi = \tilde{f}[\eta(X)Y - \eta(Y)X], \tag{3.3}$$

where ∇ is a Levi Civita connection and Q is the Ricci Operator on M.
We use the following results later.

Proposition 3.1.([15]) *In a f-cosymplectic manifold, if $(\xi\tilde{f}) = 0$, then $\tilde{f}$ is constant.*

Proposition 3.2.([15]) *A compact f-cosymplectic manifold M^{2n+1} with $(\xi\tilde{f}) = 0$ is α-cosymplectic. In particular if $\tilde{f} = 0$, M is cosymplectic.*

Proposition 3.3.([4]) *Let M be a cosymplectic manifold. It is a local product of a Kahler manifold and an interval or unit sphere S^1.*

Proposition 3.4.([14]) *Let M be an almost f-cosymplectic manifold of dimension $(2n+1)$ and $\tilde{M}$ be an integral manifold of a distribution D of M defined by $D = ker\eta$. Then*
1. when $f = 0$, $\tilde{M}$ is totally geodesic if and only if the operator h vanishes.
2. when $f \neq 0$, $\tilde{M}$ is totally umbilic if and only if the operator h vanishes.

Lemma 3.5.([1]) *Let M be a $(2n+1)$-dimensional contact metric manifold having the non-trivial solution (g,λ) for the Fischer-Marsden equation $Rg^{'*}(\lambda)=0$. Then the Riemannian curvature tensor can be expressed as*

$$R(X,Y)D\lambda = (X\lambda)QY-(Y\lambda)QX-\lambda\{(\nabla_X Q)Y-(\nabla_Y Q)X\} + (Xv)Y-(Yv)X, \tag{3.4}$$

where $v=\frac{-r\lambda}{2n}$. □

We now consider an f-cosymplectic manifold M satisfying Fischer-Marsden equation:

Theorem 3.6. *Let (M,ϕ,ξ,η,g) be a compact f-cosymplectic manifold. Suppose that M admits a non-trivial solution (g,λ) for the Fischer -Marsden equation $Rg^{'*}(\lambda)=0$ with $(\xi\tilde{f})=0$ then either M is Einstein or M is locally a product of a Kahler manifold and an interval or unit circle S^1.*

Proof. Replacing ξ with X in (3.4), employing (3.2) and (3.3), we get

$$\tilde{f}\{(\xi\lambda)g(X,Y)-(Y\lambda)\eta(X)\} = (\xi\lambda-\lambda f)Ric(X,Y) + \left\{2n\tilde{f}(Y\lambda)-\lambda(Y\tilde{f})-(Yv)\right\}\eta(X)-\left\{\lambda(\xi\tilde{f})+2n\lambda f\tilde{f}-(\xi v)\right\}g(X,Y). \tag{3.5}$$

Now antisymmetrizing the above equation, we have

$$\left\{2n\tilde{f}(Y\lambda)+\lambda(Y\tilde{f})-(Yv)+\tilde{f}(Y\lambda)\right\}\eta(X) = \left\{2n\tilde{f}(X\lambda)+\lambda(X\tilde{f})-(Xv)+\tilde{f}(X\lambda)\right\}\eta(Y). \tag{3.6}$$

Replacing X by ϕX and taking $Y=\xi$ in the above equation, we get

$$(2n+1)\tilde{f}\phi D\lambda+\lambda\phi D\tilde{f}-\phi Dv=0. \tag{3.7}$$

Notice that from the Proposition 3.1., if $(\xi\tilde{f})=0$ then f is a constant. Hence the above equation becomes

$$(2n+1)\tilde{f}\phi D\lambda=\phi Dv. \tag{3.8}$$

From [7], we can see that if (g,λ) is non-trivial solution of the equation (1.1), then the scalar curvature R_g is constant. By virtue of (1.2), we thus obtain

$$2nDv=-R_g D\lambda. \tag{3.9}$$

From (3.8) and (3.9), we arrive at

$$(2n(2n+1)\tilde{f}+R_g)\phi D\lambda=0. \tag{3.10}$$

From this we conclude that either $R_g=-2n(2n+1)\tilde{f}$ or $\phi D\lambda=0$.
Suppose $R_g=-2n(2n+1)\tilde{f}$. Then (3.9) gives

$$X(v)=(2n+1)\tilde{f}(X\lambda),\ \ \xi v=(2n+1)\tilde{f}(\xi\lambda). \tag{3.11}$$

Therefore by replacing X by ξ in (3.4) and then by using (3.2) and $Q\xi = -2n\tilde{f}\xi$, we find

$$(3.12) R(\xi, Y)D\lambda = (\xi\lambda - \lambda f)QY + \Big\{2n(\lambda)\tilde{f} - \lambda(Y\tilde{f}) - (2n+1)\tilde{f}(Y\lambda)\Big\}\xi - \Big\{\lambda(\xi\tilde{f}) + 2n\lambda f\tilde{f} - (2n+1)\tilde{f}(\xi\lambda)\Big\}Y,$$

for all vector fields Y on M. Making use of equation (3.3), we get

$$R(\xi, X)D\lambda = \tilde{f}\left\{(\xi\lambda)X - (X\lambda)\xi\right\}. \tag{3.13}$$

Unifying equation (3.12) and (3.13), we reach at

$$((\xi\lambda) - \lambda f)QY + \Big\{2n\tilde{f}(Y\lambda) - \lambda(Y\tilde{f}) - 2n\tilde{f}(Y\lambda)\Big\}\xi - \Big\{\lambda(\xi\tilde{f}) + 2n\lambda f\tilde{f} - (2n+2)\tilde{f}(\xi\lambda)\Big\}Y = 0. \tag{3.14}$$

Further, the hypothesis $(\xi\tilde{f}) = 0$ implies that $\tilde{f}$ is constant and the above equation reduces to

$$((\xi\lambda) - \lambda f)QY - \Big\{2n\lambda f\tilde{f} - (2n+2)\tilde{f}(\xi\lambda)\Big\}Y = 0. \tag{3.15}$$

Letting $Y = \xi$ in (3.15) and f be not identically zero, then we see that $(\xi\lambda) = 0$. From this, we have $(X\lambda) = 0$, for all vector fields X on M, which implies λ is a non-zero constant. Substituting this in equation (3.15), we arrive at $QY = -2n\tilde{f}Y$. This shows that M is Einstein manifold.

Next we assume that $R_g \neq -2n(2n+1)\tilde{f}$. Then from (3.10) it follows that $\phi D\lambda = 0$. Applying ϕ and by virtue of $\phi^2 = -I + \eta \otimes \xi$, the last equation gives $D\lambda = (\xi\lambda)\xi$.

Now taking covariant derivative of this along arbitrary vector field X on M yields

$$\nabla_X D\lambda = X(\xi\lambda)\xi - (\xi\lambda)f\phi^2 X. \tag{3.16}$$

On contracting equation (1.1), we obtain

$$\nabla_X D\lambda = \lambda(QX) + \triangle_g \lambda X. \tag{3.17}$$

From (3.16) and (3.17), we get

$$\lambda(QX) + \triangle_g \lambda = X(\xi\lambda)\xi - (\xi\lambda)f\phi^2 X.$$

Taking $X = \xi$ in the preceding equation, using (3.2), we get

$$\xi(\xi\lambda) = -2n\tilde{f}\lambda + \triangle_g \lambda. \tag{3.18}$$

Contracting (3.16) along X implies

$$\triangle_g \lambda = \xi(\xi\lambda) + 2n(\xi\lambda)f. \tag{3.19}$$

Making use of (3.19) in (3.18), we obtain

$$(\xi\lambda)f - \tilde{f}\lambda = 0. \tag{3.20}$$

Differentiating (3.20) in the direction of ξ and using the hypothesis $\xi\tilde{f} = 0$. We get

$$\xi(\xi\lambda)f = (\xi\lambda)f^2, \tag{3.21}$$

since $\tilde{f} = (\xi f) + f^2$.
Suppose that $f \not\equiv 0$ on some neighborhood θ around $p \in M$.
Then (3.21) implies that $\xi(\xi\lambda)f = (\xi\lambda)f$ on θ. Inserting this into (3.19), we get $\triangle_g\lambda = (2n+1)(\xi\lambda)f$. Making use of this in (3.20), we obtain

$$\triangle_g \lambda = (2n+1)\tilde{f}\lambda. \tag{3.22}$$

Furthermore tracing the Fischer-Marsden equation in (1.1), we obtain

$$2n \triangle_g \lambda = -\lambda R_g. \tag{3.23}$$

Comparing (3.22) and (3.23), we obtain $R_g = -2n(2n+1)\tilde{f}$, which is contradictory to our assumption. Hence $f = 0$, and M is cosymplectic. □

Corollary 3.7.([9],Theorem 4) *Let (M, ϕ, ξ, η, g) be a compact f-cosymplectic manifold satisfying the Fischer-Marsden equation $Rg^{'*}(\lambda) = 0$ having non-trivial solution with $(\xi\tilde{f}) = 0$ and $\tilde{M}$ be an integral manifold of D. Then $\tilde{M}$ is either totally umbilical or totally geodesic.*

In Theorem 3.6., if f is a constant say α then f-cosymplectic manifold reduces to particular case i.e., α-cosymplectic manifold. From the Theorem 3.6., we arrive at the following:

Corollary 3.8. *Let M be a compact $(2n+1)$-dimensional α-cosymplectic manifold, where α is a real number. Suppose that M satisfies a Fischer-Marsden equation $Rg^{'*}(\lambda) = 0$ having non-trivial solution. Then M is either of constant scalar curvature $-2n(2n+1)\alpha^2$ or locally the product of Kahler manifold and an interval or unit circle S^1.*

Remark 3.9. If $f = 1$ and $n = 3$ in Theorem 3.6., then the manifold reduces to three-dimensional Kenmotsu manifold, and the result of the Theorem 3.6. coincides with the result of [6]. That is, suppose (g, λ) is a non-trivial solution of the Fischer Marsden equation $Rg^{'*}(\lambda) = 0$ of M, then M is locally isometric to the hyperbolic space of the form $H^3(-1)$.

We now prove the following theorem:

Theorem 3.10. *If (M, ϕ, ξ, η, g) is a compact orientable f-cosymplectic manifold* *satisfying Fischer-Marsden equation (1.1) with $\lambda = \tilde{f}$, then M is an Einstein manifold. Further the solution λ is either equal to $\frac{-R_g}{2n(2n+2)}$ or zero.*

Proof. Taking $\lambda = \tilde{f}$ in (3.7), we have $\left\{(2n+2)\lambda + \frac{R_g}{2n}\right\}\phi D\lambda = 0$.
Suppose that $(2n+2)\lambda + \frac{R_g}{2n} = 0$. Then it follows that λ is a constant, because of constancy of R_g in Fischer-Marsden conjecture [4]. Thus M is Einstein.

On the other hand, if we assume $(2n+2)\lambda + \frac{R_g}{2n} \neq 0$ in a neighborhood θ around $p \in M$, then we have $\phi D\lambda = 0$. Now by the action of ϕ in the preceding equation, we get $D\lambda = (\xi\lambda)\xi$. It follows immediately from (3.20) that

$$(\xi\lambda)f - \lambda^2 = 0. \tag{3.24}$$

Taking covariant differentiation of (3.24) along ξ, we obtain

$$2\lambda(\xi\lambda) = (\xi\lambda)(\xi f) + \xi(\xi\lambda f). \tag{3.25}$$

By our hypothesis $\lambda = \tilde{f}$, a direct calculation gives

$$\left\{\lambda + f^2\right\}(\xi\lambda) = \xi(\xi\lambda)f. \tag{3.26}$$

Suppose that $f = 0$. It follows immediately from (3.26) that $\lambda = 0$.
Next we assume that $f \neq 0$, substituting (3.26) in (3.19) we get

$$\triangle_g\lambda = \frac{\lambda}{f}(\xi\lambda) + (2n+1)\lambda^2. \tag{3.27}$$

Tracing the relation (1.1), with the help of (3.27), we obtain

$$-\frac{2n}{f}(\xi\lambda) - 2n(2n+1)\lambda = R_g. \tag{3.28}$$

Note that if a compact Riemannian manifold (M, g) satisfies a Fischer Marsden equation then the scalar curvature R_g is constant. Taking the covariant derivative of (3.28) and using (3.25), we obtain that $(2n+3)(\xi\lambda) = 0$, which implies $(\xi\lambda) = 0$.

Finally from (3.24), we have $\lambda = 0$. □

Lemma 3.11. *Let $(2n+1)$-dimensional f-cosymplectic manifold M admit a Reeb flow invariant Ricci operator Q. Then*

$$(\xi R_g) = -2\left\{fR_g + 2n(\xi\tilde{f}) + 2n(2n+1)f\tilde{f}\right\} \tag{3.29}$$

holds.

Proof. Suppose that the Ricci operator Q of an f-cosymplectic manifold is a Reeb flow invariant. Then we have

$$(L_\xi Q)X = 0, \tag{3.30}$$

for all vector fields X on M.

Using the definition of Lie derivative in (3.30), we obtain

$$(\nabla_\xi Q)\, X - \nabla_{QX}\xi + Q\,(\nabla_X \xi) = 0. \tag{3.31}$$

Using (3.1), (3.2) in (3.31) and by straight forward calculation, we get

$$(\nabla_\xi Q)\, X = 0, \tag{3.32}$$

for all vector fields X on M.
This shows that the Ricci operator Q is locally symmetric along Reeb vector field ξ.
From (3.2) one can easily see that,

$$(\nabla_\xi Q)\, X - (\nabla_X Q)\,\xi = fQX + 2n(X\tilde{f})\xi + 2nf\tilde{f}X,$$

for all vector fields X on M. On contracting the above equation, we get (3.29). □

Theorem 3.12. *Let M be a compact (2n+1)-dimensional f-cosymplectic manifold with the Reeb flow invariant Ricci operator. If (g,λ) is a non-trivial solution of the Fischer Marsden equation $Rg^{'*} = 0$, then the following hold*
i. M is a manifold of constant scalar curvature
ii. M reduces to α-cosymplectic manifold
iii. either $(\xi\lambda) = \lambda f$ or M is Einstein manifold.

Proof. Taking inner product of equation (3.4) with ξ and making use of (3.1) and (3.2), we have

$$\begin{aligned} g(R(\xi,Y)D\lambda,X) &= (\xi\lambda - \lambda f)Ric(X,Y) \\ &- \left\{\lambda(\xi\tilde{f}) + 2n\lambda f\tilde{f} - (\xi v)\right\} g(X,Y) \\ &+ \left\{2n\tilde{f}(Y\lambda) + \lambda(Y\tilde{f}) - Yv\right\}\eta(X). \end{aligned} \tag{3.33}$$

Taking scalar product of (3.3) with $D\lambda$ and comparing with (3.33), we obtain

$$\begin{aligned} \tilde{f}\,\{(\xi\lambda)g(X,Y) - \eta(X)(Y\lambda)\} &= (\xi\lambda - \lambda f)Ric(X,Y) \\ &- \left\{\lambda(\xi\tilde{f}) + 2nf\tilde{f}\lambda - (\xi v)\right\} g(X,Y) \\ &+ \left\{2n\tilde{f}(Y\lambda) + \lambda(Y\tilde{f}) - (Yv)\right\}\eta(X). \end{aligned} \tag{3.34}$$

Let $\{e_i\}$ for $1 \leq i \leq 2n+1$ be an orthonormal basis at each point of the tangent space of the manifold M. Taking $X = Y = e_i$ in the above equation and then summing over i, we get

$$-2n(\xi v) = \left\{R_g(\xi\lambda) - \lambda f R_g - 2n\lambda(\xi\tilde{f}) - 2n(2n+1)f\tilde{f}\lambda\right\}. \tag{3.35}$$

Taking covariant derivative of $v = -\frac{R_g \lambda}{2n}$ along ξ, we obtain

$$(\xi v) = -\frac{1}{2n}\left\{R_g(\xi\lambda) + \lambda(\xi R_g)\right\}. \tag{3.36}$$

Unifying equation (3.35) and (3.36), and noting that (g, λ) is a non-zero solution of Fischer-Marsden equation, we obtain

$$(\xi R_g) = -\left\{f R_g + 2n(\xi \tilde{f}) + 2n(2n+1) f \tilde{f}\right\}. \tag{3.37}$$

From (3.29) and (3.37), we find that

$$f R_g = -\left\{2n(\xi \tilde{f}) + 2n(2n+1) f \tilde{f}\right\}. \tag{3.38}$$

From (3.37) and (3.38), we have $(\xi R_g) = 0$, implies that $(X R_g) = 0$ for all vector fields X on M. This shows that, whenever $f \neq 0$ then M is manifold of constant scalar curvature.
Using the above equation in (3.35), we get

$$-2n(\xi v) = R_g(\xi\lambda). \tag{3.39}$$

In (3.33), replacing X by ξ and making use of (3.2), we conclude that $(\xi \tilde{f}) = 0$. By the Proposition 3.1., f is a constant.
According to the Proposition 3.2., M is α-cosymplectic manifold.
Thus the relation (3.38), show that $R_g = -2n(2n+1)\tilde{f}$. Inserting this into (3.39), we find

$$\xi v = (2n+1)\tilde{f}(\xi\lambda), \quad X v = (2n+1)\tilde{f}(X\lambda), \tag{3.40}$$

for all vector fields X on M. Substituting (3.40) in (3.34), one can easily obtain

$$(\xi\lambda - \lambda f)\left\{Ric(X, Y) + 2n\tilde{f} g(X, Y)\right\} = 0,$$

which implies that,

$$(\xi\lambda - \lambda f) = 0 \ \ or \ \ Ric(X, Y) + 2n\tilde{f} g(X, Y) = 0. \tag{3.41}$$

The proof completes. □

Remark 3.13. For $f = 1$ in Theorem 3.12., M is a (2n+1)-dimensional Kenmotsu manifold. In this case suppose that Ricci operatior Q is Reeb flow invariant and (g, λ) is a non-trivial solution of the Fischer Mersden equation $R_g^{'*} = 0$, then either $\xi\lambda = \lambda$ or M is an Einstien manifold.

4. Almost Cosymplectic (k,μ) Metrics Satisfying Fischer Marsden Equation

A $(2n+1)$-dimensional almost cosymplectic manifold (M,ϕ,ξ,η,g) is said to be almost cosymplectic (k,μ)-manifold, if the charactrestic vector field belongs to the (k,μ)-nullity distribution. That is

$$N_p(k,\mu)=\{Z\in T_pM/R(X,Y)Z=k(g(Y,Z)X-g(X,Z)Y)+\mu(g(Y,Z)hX-g(X,Z)hY\},$$

for certain constants k and μ, where X and Y are arbitrary vector fields on M. By the definiton of (k,μ)-almost cosymplectic manifold, we have

$$R(X,Y)\xi=k\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\}. \tag{3.1}$$

In addition by [7], if M is a $(2n+1)$-dimesional almost cosymplectic manifold satisfying (k,μ)-nullity condition, we have

$$\begin{aligned}
\nabla_X\xi &= -\phi hX, && (3.2)\\
Q &= 2nk\eta\otimes\xi+\mu h, && (3.3)\\
h^2 &= k\phi^2, && (3.4)\\
Q\xi &= 2nk\xi, && (3.5)\\
R_g &= 2nk. && (3.6)
\end{aligned}$$

Lemma 4.1. *Let M be an almost (k,μ)-cosymplectic manifold of dimension $2n+1(n\geq 1)$, with $k<0$. Then we have*

$$(\nabla_X Q)\xi=2nk(h^{'}X)-Qh^{'}X, \tag{3.7}$$

for all vector fields X on M.

Proof. Taking covariant derivative of (3.4), along the arbitrary vector field X on M and with the help of (3.2), we get the desired equation (3.7). □

Lemma 4.2. *Let g be a metric of an almost (k,μ)-cosymplectic manifold, $(k<0)$ satisfying the Fischer-Marsden equation (1.1) with non-zero solution λ. Then the Riemannian curvature tensor can be expressed as*

$$\begin{aligned}
R(X,Y)D\lambda &= (X\lambda)QY-(Y\lambda)QX+\lambda\{(\nabla_X Q)Y-(\nabla_Y Q)X\}\\
&\quad +k\{(Y\lambda)X-(X\lambda)Y\}.
\end{aligned} \tag{3.8}$$

Theorem 4.3. *If the metric g of an almost (k,μ)-cosymplectic manifold with $k<0$ satisfies the Fischer-Marsden equation, then either $4n^2k+\mu^2=0$ or it has a trivial solution only.*

*Proof.*By definition, if the metric g of an almost (k,μ)-cosymplectic manifold satisfies the Fischer-Marsden equation (1.1) with $k<0$, then we get

$$\begin{aligned}
\nabla_X D\lambda &= \lambda\{QX-kX\}, && (3.9)\\
\textit{with}\ \triangle_g\lambda &= \upsilon=-k\lambda, && (3.10)
\end{aligned}$$

where we have used (3.6).
Now by Lemma 4.2., taking inner product of (3.8) with ξ and using (3.7), we have

$$(3.11)\quad \begin{aligned} g(R(X,Y)D\lambda,\xi) &= 2nk\{(X\lambda)\eta(Y)-(Y\lambda)\eta(X)+\lambda g(X,h^{'}Y) \\ &\quad -\lambda g(h^{'}X,Y)\}+\lambda\{g(h^{'}X,QY)-g(h^{'}Y,QX)\} \\ &\quad -k\{(X\lambda)\eta(Y)-Y(\lambda)\eta(X)\}, \end{aligned}$$

for all vector fields X and Y on M. Plugging $Y=\xi$, in the previous equation, we obtain

$$(3.12)\quad g(R(X,\xi)D\lambda,\xi)=(2n-1)k\{(X\lambda)-(\xi\lambda)\eta(X)\},$$

for all vector fields X on M. Since $\xi\in(k,\mu)$-nullity distribution, from (3.2), we see that

$$(3.13)\quad g(R(X,\xi)\xi,D\lambda)=k\{(X\lambda)-(\xi\lambda)\eta(X)\}+\mu g(hX,D\lambda)\},$$

for any vector field X on M. Moreover, we have

$$(3.14)\quad g(R(X,\xi)D\lambda,\xi)+g(R(X,\xi)\xi,D\lambda)=0.$$

Inserting (3.12) and (3.13) in (3.14) and by straightforward calculation, we obtain

$$(3.15)\quad \mu(hD\lambda)=2nk\{(\xi\lambda)\xi-D\lambda\}.$$

Application of h on both sides of the above relation yields,

$$(3.16)\quad -2n(hD\lambda)=\mu\{(\xi\lambda)\xi-D\lambda\}.$$

By comparing (3.15) and (3.16), we get

$$(3.17)\quad (4n^2k+\mu^2)\{(\xi\lambda)\xi-D\lambda\}=0,$$

which implies that either $(4n^2k+\mu^2)=0$ or $\{(\xi\lambda)\xi-D\lambda\}=0$.
Suppose that $(4n^2k+\mu^2)\neq 0$. Then

$$(3.18)\quad D\lambda=(\xi\lambda)\xi.$$

Differentiating this along arbitrary vector field X, we get

$$(3.19)\quad \nabla_X D\lambda=X(\xi\lambda)\xi+(\xi\lambda)h^{'}X.$$

Comparing (3.9) and (3.19), we get

$$(3.20)\quad \lambda(QX)-(k\lambda)X=X(\xi\lambda)\xi+(\xi\lambda)h^{'}X.$$

Now, by using (3.3) in (3.20), we obtain

$$(3.21)\quad 2nk\lambda\eta(X)\xi-X(\xi\lambda)\xi-k\lambda X+\mu\lambda(hX)-(\xi\lambda)h^{'}X=0,$$

for any vector field X on M.

Finally, contracting (3.21) over X and using $tr(h)=tr(\phi h)=0$, we get $\lambda=0$, which proves the theorem. □

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