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THE u-S-GLOBAL DIMENSIONS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity and S a multiplicative subset of R. First, we introduce and study the u-S-projective dimension and u-S-injective dimension of an R-module, and then explore the u-S-global dimension u-S-gl.dim(R) of a commutative ring R, i.e., the supremum of u-S-projective dimensions of all R-modules. Finally, we investigate u-S-global dimensions of factor rings and polynomial rings.

1. Introduction and preliminary concepts

Throughout this article, R is always a commutative ring with identity and S is always a multiplicative subset of R, that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S, s_2 \in S$. In 2002, Anderson and Dumitrescu [1] defined S-Noetherian rings R in which any ideal of R is S-finite. Recall from [1] that an R-module M is called S-finite (with respect to s) provided that $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule F of M. An R-module T is called u-S-torsion ("u" abbreviates "uniformly" throughout this article) if $sT = 0$ for some $s \in S$ (see [4]). So an R-module M is S-finite if and only if M/F is u-S-torsion for some finitely generated submodule F of M. The idea derived from u-S-torsion modules is deserved to be further investigated.

In [7], the authors of this paper introduced the class of u-S-projective modules P for which the functor $\mathrm{Hom}_R(P, -)$ preserves u-S-exact sequences. The class of u-S-projective modules can be seen as a "uniform" generalization of that of projective modules, since an R-module P is u-S-projective if and only if $\mathrm{Ext}^1_R(P, M)$ is u-S-torsion for any R-module M (see [7, Theorem 2.5]). The class of u-S-projective modules owns the following u-S-hereditary property: let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence, if B and C are u-S-projective so is A (see [7, Proposition 2.8]). So it is worth to study the u-S-analogue of projective dimensions of R-modules. Similarly, by the discussion of u-S-injective modules in [2], we can study the u-S-analogue of injective dimensions

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of R-modules. Together these, a u-S-analogue of global dimensions of commutative rings can also be introduced and studied.

In this article, we define the u-S-projective dimension u-S-$\mathrm{pd}_R(M)$ (resp., u-S-injective dimension u-S-$\mathrm{id}_R(M)$) of an R-module M to be the length of the shortest u-S-projective (resp., u-S-injective) u-S-resolution (resp., u-S-coresolution) of M. We characterize u-S-projective dimensions (resp., u-S-injective) of R-modules using the uniform torsion property of the "Ext" functors in Proposition 3.3 (resp., Proposition 3.4). Besides, we obtain some characterizations of projective dimensions and injective dimensions of R-modules in Corollary 3.19. The u-S-global dimension u-S-gl.dim(R) of a commutative ring R is defined to be the supremum of u-S-projective dimensions of all R-modules. We find that the u-S-global dimension of a commutative ring is also the supremum of u-S-injective dimensions of all R-modules. A new characterization of global dimensions is given in Corollary 3.19. u-S-semisimple rings are firstly introduced in [7] for which any free R-module is u-S-semisimple. By [4, Theorem 3.11], a ring R is u-S-semisimple if and only if all R-modules are u-S-projective (resp., u-S-injective). So u-S-semisimple rings are exactly commutative rings with u-S-global dimension equal to 0 (see Corollary 3.20). In the final section, we investigate the u-S-global dimensions of factor rings and then give a complete description of u-S-global dimensions of polynomial rings (see Theorem 4.6).

Since this paper involves uniformly S-torsion theory, we give a quick review. For more details, please refer to [4–7].

An R-module T is called a u-S-torsion module provided that there exists an element $s \in S$ such that $sT = 0$. An R-sequence $M \xrightarrow{f} N \xrightarrow{g} L$ is called *u-S-exact* (at N) provided that there is an element $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. We say a long R-sequence $\cdots \to A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \to \cdots$ is u-S-exact, if it is u-S-exact at every A_n, that is, for any n there is an element $s \in S$ such that $s\mathrm{Ker}(f_{n+1}) \subseteq \mathrm{Im}(f_n)$ and $s\mathrm{Im}(f_n) \subseteq \mathrm{Ker}(f_{n+1})$. A u-S-exact sequence $0 \to A \to B \to C \to 0$ is called a short u-S-exact sequence. Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-short exact sequence. Then ξ is said to be u-S-split provided that there are $s \in S$ and R-homomorphism $f' : B \to A$ such that $f'(f(a)) = sa$ for any $a \in A$, that is, $f' \circ f = s\mathrm{Id}_A$.

An R-homomorphism $f : M \to N$ is a *u-S-monomorphism* (resp., *u-S-epimorphism*, *u-S-isomorphism*) provided $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N \to 0$) is u-S-exact. It is easy to verify an R-homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(f)$, both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$) is a u-S-torsion module. Let R be a ring and S a multiplicative subset of R. Suppose M and N are R-modules. We say M is *u-S-isomorphic* to N if there exists a u-S-isomorphism $f : M \to N$. A family $\mathcal{C}$ of R-modules is said to be closed under u-S-isomorphisms if M is u-S-isomorphic to N and M is in $\mathcal{C}$, then N is also in $\mathcal{C}$. It follows from [7, Lemma 2.1] that if $f : M \to N$ is a u-S-isomorphism. Then there is a u-S-isomorphism $g : N \to M$ such that $f \circ g = s\mathrm{Id}_N$ and $g \circ f = s\mathrm{Id}_M$ for some $s \in S$.

An R-module F is called *u-S-flat* provided that for any u-S-exact sequence $0 \to A \to B \to C \to 0$, the induced sequence

$$0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$$

is u-S-exact. It follows from [4, Theorem 3.2] that an R-module F is u-S-flat if and only if $\mathrm{Tor}_1^R(F, M)$ is u-S-torsion for any R-module M. An R-module P is called *u-S-projective* provided that the induced sequence

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0$$

is u-S-exact for any u-S-exact sequence $0 \to A \to B \to C \to 0$. And recall from [2, Definition 4.1] that an R-module E is called *u-S-injective* provided that the induced sequence

$$0 \to \mathrm{Hom}_R(C, E) \to \mathrm{Hom}_R(B, E) \to \mathrm{Hom}_R(A, E) \to 0$$

is u-S-exact for any u-S-exact sequence $0 \to A \to B \to C \to 0$. Following from [4, Theorem 3.2], an R-module P is u-S-projective if and only if $\mathrm{Ext}_R^1(P, M)$ is u-S-torsion for any R-module M. Similarly, an R-module E is u-S-injective if and only if $\mathrm{Ext}_R^1(M, E)$ is u-S-torsion for any R-module M by [2, Theorem 4.3] and [7, Proposition 2.3].

2. Long u-S-exact sequences induced by Ext functors

The following result says that a short u-S-exact sequence induces a long u-S-exact sequence by the "Ext" functor as in the classical case.

Lemma 2.1. *Let R be a ring and S a multiplicative subset of R. Let L, M and N be R-modules. If $f : M \to N$ is a u-S-isomorphism, then $\mathrm{Ext}_R^n(L, f) : \mathrm{Ext}_R^n(L, M) \to \mathrm{Ext}_R^n(L, N)$ and $\mathrm{Ext}_R^n(f, L) : \mathrm{Ext}_R^n(N, L) \to \mathrm{Ext}_R^n(M, L)$ are all u-S-isomorphisms for any $n \geq 0$.*

Proof. We only show $\mathrm{Ext}_R^n(L, f) : \mathrm{Ext}_R^n(L, M) \to \mathrm{Ext}_R^n(L, N)$ is a u-S-isomorphism for any $n \geq 0$ since the other one is similar. Consider the exact sequences: $0 \to \mathrm{Ker}(f) \to M \xrightarrow{\pi_{\mathrm{Im}(f)}} \mathrm{Im}(f) \to 0$ and $0 \to \mathrm{Im}(f) \xrightarrow{i_{\mathrm{Im}(f)}} N \to \mathrm{Coker}(f) \to 0$ with $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ u-S-torsion. Then there are long exact sequences

$$\begin{aligned}\mathrm{Ext}_R^n(L, \mathrm{Ker}(f)) \to \mathrm{Ext}_R^n(L, M) &\xrightarrow{\mathrm{Ext}_R^n(L, \pi_{\mathrm{Im}(f)})} \mathrm{Ext}_R^n(L, \mathrm{Im}(f)) \\ &\to \mathrm{Ext}_R^{n+1}(L, \mathrm{Ker}(f))\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ext}_R^{n-1}(L, \mathrm{Coker}(f)) \to \mathrm{Ext}_R^n(L, \mathrm{Im}(f)) &\xrightarrow{\mathrm{Ext}_R^n(L, i_{\mathrm{Im}(f)})} \mathrm{Ext}_R^n(L, N) \\ &\to \mathrm{Ext}_R^n(L, \mathrm{Coker}(f)).\end{aligned}$$

Since $\mathrm{Ext}_R^n(L, \mathrm{Ker}(f))$, $\mathrm{Ext}_R^{n+1}(L, \mathrm{Ker}(f))$, $\mathrm{Ext}_R^{n-1}(L, \mathrm{Coker}(f))$ and $\mathrm{Ext}_R^n(L,$ $\mathrm{Coker}(f))$ are all u-S-torsion by [2, Lemma 4.2], we have the composition

$$\begin{aligned}\mathrm{Ext}_R^n(L, f) : \mathrm{Ext}_R^n(L, M) &\xrightarrow{\mathrm{Ext}_R^n(L, \pi_{\mathrm{Im}(f)})} \mathrm{Ext}_R^n(L, \mathrm{Im}(f)) \\ &\xrightarrow{\mathrm{Ext}_R^n(L, i_{\mathrm{Im}(f)})} \mathrm{Ext}_R^n(L, N)\end{aligned}$$

is a u-S-isomorphism. $\square$

Theorem 2.2. *Let R be a ring and S a multiplicative subset of R. Let M and N be R-modules. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a u-S-exact sequence of R-modules. Then for any $n \geq 1$ there are R-homomorphisms $\delta_n : \mathrm{Ext}_R^{n-1}(M, C) \to \mathrm{Ext}_R^n(M, A)$ and $\sigma_n : \mathrm{Ext}_R^{n-1}(A, N) \to \mathrm{Ext}_R^n(C, N)$ such that the induced sequences*

$$\begin{aligned}0 &\to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to \mathrm{Ext}_R^1(M, A) \to \cdots \\ &\to \mathrm{Ext}_R^{n-1}(M, B) \to \mathrm{Ext}_R^{n-1}(M, C) \xrightarrow{\delta_n} \mathrm{Ext}_R^n(M, A) \to \mathrm{Ext}_R^n(M, B) \to \cdots\end{aligned}$$

and

$$\begin{aligned}0 &\to \mathrm{Hom}_R(C, N) \to \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N) \to \mathrm{Ext}_R^1(C, N) \to \cdots \\ &\to \mathrm{Ext}_R^{n-1}(B, N) \to \mathrm{Ext}_R^{n-1}(A, N) \xrightarrow{\sigma_n} \mathrm{Ext}_R^n(C, N) \to \mathrm{Ext}_R^n(B, N) \to \cdots\end{aligned}$$

are u-S-exact.

Proof. The proof is similar to the classical case. But we give a proof for completeness. We only show the first sequence is u-S-exact since the other one is similar. Since the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is u-S-exact at B, there is an exact sequence $0 \to \mathrm{Ker}(g) \xrightarrow{i_{\mathrm{Ker}(g)}} B \xrightarrow{\pi_{\mathrm{Im}(g)}} \mathrm{Im}(g) \to 0$. So there is a long exact sequence of R-modules:

$$\begin{aligned}0 &\to \mathrm{Hom}_R(M, \mathrm{Ker}(g)) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, \mathrm{Im}(g)) \\ &\to \mathrm{Ext}_R^1(M, \mathrm{Ker}(g)) \to \cdots \to \mathrm{Ext}_R^{n-1}(M, B) \to \mathrm{Ext}_R^{n-1}(M, \mathrm{Im}(g)) \\ &\xrightarrow{\delta'_n} \mathrm{Ext}_R^n(M, \mathrm{Ker}(g)) \to \mathrm{Ext}_R^n(M, B) \to \cdots\end{aligned}$$

Note that there are u-S-isomorphisms $t_1 : A \to \mathrm{Ker}(g)$, $t'_1 : \mathrm{Ker}(g) \to A$, $t_2 : \mathrm{Im}(g) \to C$ and $t'_2 : C \to \mathrm{Im}(g)$ by [7, Lemma 2.1]. So, by Lemma 2.1, $\mathrm{Ext}_R^n(M, t'_1) : \mathrm{Ext}_R^n(M, \mathrm{Ker}(g)) \to \mathrm{Ext}_R^n(M, A)$ and $\mathrm{Ext}_R^n(M, t'_2) : \mathrm{Ext}_R^n(M, C)$ $\to \mathrm{Ext}_R^n(M, \mathrm{Im}(g))$ are u-S-isomorphisms for any $n \geq 0$. Setting

$$\delta_n = \mathrm{Ext}_R^n(M, t'_1) \circ \delta'_n \circ \mathrm{Ext}_R^n(M, t'_2),$$

we have a u-S-exact sequence:

$$\begin{aligned}0 &\to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to \mathrm{Ext}_R^1(M, A) \to \cdots \\ &\to \mathrm{Ext}_R^{n-1}(M, B) \to \mathrm{Ext}_R^{n-1}(M, C) \xrightarrow{\delta_n} \mathrm{Ext}_R^n(M, A) \to \mathrm{Ext}_R^n(M, B) \to \cdots\end{aligned}$$

$\square$

Following from Theorem 2.2, we have the following result.

Corollary 2.3. *Let R be a ring, S a multiplicative subset of R and M and N R-modules. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a u-S-exact sequence of R-modules.*

(1) *If B is u-S-projective, then $\mathrm{Ext}_R^n(C, N)$ is u-S-isomorphic to $\mathrm{Ext}_R^{n+1}(A, N)$ for any $n \geq 0$.*
(2) *If B is u-S-injective, then $\mathrm{Ext}_R^n(M, A)$ is u-S-isomorphic to $\mathrm{Ext}_R^{n+1}(M, C)$ for any $n \geq 0$.*

3. On u-S-projective (u-S-injective) dimensions of modules and u-S-global dimensions of rings

It is well known that the projective (resp., injective) dimension of an R-module M is the shortest length of projective (resp., injective) resolution of M. Recall from [5] that the u-S-flat dimension of M is the shortest length of u-S-flat u-S-resolutions of M. We first introduce the u-S-versions of projective dimensions and injective dimensions of R-modules.

Definition 3.1. Let R be a ring, S a multiplicative subset of R and M an R-module. We write $u\text{-}S\text{-}\mathrm{pd}_R(M) \leq n$ (u-S-pd abbreviates *u-S-projective dimension*) if there exists a u-S-exact sequence of R-modules

$$(\diamondsuit) \qquad 0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where each F_i is u-S-projective for $i = 0, \ldots, n$. The u-S-exact sequence ($\diamondsuit$) is said to be a u-S-projective u-S-resolution of length n of M. If such a finite u-S-projective u-S-resolution does not exist, then we say $u\text{-}S\text{-}\mathrm{pd}_R(M) = \infty$; otherwise, define $u\text{-}S\text{-}\mathrm{pd}_R(M) = n$ if n is the length of the shortest u-S-projective u-S-resolution of M.

Similarly, one can define the u-S-injective dimension $u\text{-}S\text{-}\mathrm{id}_R(M)$ and u-S-injective u-S-coresolution of an R-module M.

Trivially, $u\text{-}S\text{-}\mathrm{pd}_R(M) \leq \mathrm{pd}_R(M)$ and $u\text{-}S\text{-}\mathrm{id}_R(M) \leq \mathrm{id}_R(M)$. And if S is composed of units, then $u\text{-}S\text{-}\mathrm{pd}_R(M) = \mathrm{pd}_R(M)$. It is also obvious that an R-module M is u-S-projective if and only if $u\text{-}S\text{-}\mathrm{pd}_R(M) = 0$, and is u-S-injective if and only if $u\text{-}S\text{-}\mathrm{id}_R(M) = 0$.

Lemma 3.2. *Let R be a ring, S a multiplicative subset of R. If A is u-S-isomorphic to B, then $u\text{-}S\text{-}\mathrm{pd}_R(A) = u\text{-}S\text{-}\mathrm{pd}_R(B)$ and $u\text{-}S\text{-}\mathrm{id}_R(A) = u\text{-}S\text{-}\mathrm{id}_R(B)$.*

Proof. We only prove $u\text{-}S\text{-}\mathrm{pd}_R(A) = u\text{-}S\text{-}\mathrm{pd}_R(B)$ as the u-S-injective dimension is similar. Let $f : A \to B$ be a u-S-isomorphism. If $\cdots \to P_n \to \cdots \to P_1 \to P_0 \xrightarrow{g} A \to 0$ is a u-S-projective resolution of A, then $\cdots \to P_n \to \cdots \to P_1 \to P_0 \xrightarrow{f \circ g} B \to 0$ is a u-S-projective resolution of B. So $u\text{-}S\text{-}\mathrm{pd}_R(A) \geq u\text{-}S\text{-}\mathrm{pd}_R(B)$. Similarly we have $u\text{-}S\text{-}\mathrm{pd}_R(B) \geq u\text{-}S\text{-}\mathrm{pd}_R(A)$ by [7, Lemma 2.1]. $\square$

Proposition 3.3. *Let R be a ring and S a multiplicative subset of R. The following statements are equivalent for an R-module M:*

(1) u-S-$\mathrm{pd}_R(M) \leq n$;
(2) $\mathrm{Ext}_R^{n+k}(M,N)$ *is u-S-torsion for all R-modules N and all $k > 0$*;
(3) $\mathrm{Ext}_R^{n+1}(M,N)$ *is u-S-torsion for all R-modules N*;
(4) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is a u-S-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are u-S-projective R-modules, then F_n is u-S-projective*;
(5) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is a u-S-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are projective R-modules, then F_n is u-S-projective*;
(6) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are u-S-projective R-modules, then F_n is u-S-projective*;
(7) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are projective R-modules, then F_n is u-S-projective*;
(8) *there exists a u-S-exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_{n-1}$ are projective R-modules and F_n is u-S-projective*;
(9) *there exists an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_{n-1}$ are projective R-modules and F_n is u-S-projective*;
(10) *there exists an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_n$ are u-S-projective R-modules.*

Proof. (1) $\Rightarrow$ (2): We prove (2) by induction on n. For the case $n = 0$, we have M is u-S-projective, then (2) holds by [7, Theorem 2.5]. If $n > 0$, then there is a u-S-exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is u-S-projective for $i = 0, 1, \ldots, n$. Set $K_0 = \mathrm{Ker}(F_0 \to M)$ and $L_0 = \mathrm{Im}(F_1 \to F_0)$. Then both $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to L_0 \to 0$ are u-S-exact. Since u-S-$\mathrm{pd}_R(L_0) \leq n-1$ and L_0 is u-S-isomorphic to K_0, u-S-$\mathrm{pd}_R(K_0) \leq n-1$ by Lemma 3.2. By induction, $\mathrm{Ext}_R^{n-1+k}(K_0, N)$ is u-S-torsion for all R-modules N and all $k > 0$. It follows from Corollary 2.3 that $\mathrm{Ext}_R^{n+k}(M,N)$ is u-S-torsion.

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (7) and (4) $\Rightarrow$ (6) $\Rightarrow$ (7): Trivial.

(3) $\Rightarrow$ (4): Let $0 \to F_n \xrightarrow{d_n} F^{n-1} \xrightarrow{d^{n-1}} F^{n-2} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ be a u-S-exact sequence, where $F_0, F_1, \ldots, F^{n-1}$ are u-S-projective. Then F_n is u-S-projective if and only if $\mathrm{Ext}_R^1(F_n, N)$ is u-S-torsion for all R-modules N, if and only if $\mathrm{Ext}_R^2(\mathrm{Im}(d^{n-1}), N)$ is u-S-torsion for all R-modules N. Iterating these steps, we can show F_n is u-S-projective if and only if $\mathrm{Ext}_R^{n+1}(M,N)$ is u-S-torsion for all R-modules N.

(9) $\Rightarrow$ (10) $\Rightarrow$ (1) and (9) $\Rightarrow$ (8) $\Rightarrow$ (1): Trivial.

(7) $\Rightarrow$ (9) : Let $\cdots \to P_n \to P^{n-1} \xrightarrow{d^{n-1}} P^{n-2} \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M. Set $F_n = \mathrm{Ker}(d^{n-1})$. Then we have an exact

sequence $0 \to F_n \to P^{n-1} \xrightarrow{d^{n-1}} P^{n-2} \to \cdots \to P_0 \to M \to 0$. By (7), F_n is u-S-projective. So (9) holds. $\square$

Similarly, we have the following result.

Proposition 3.4. *Let R be a ring and S a multiplicative subset of R. The following statements are equivalent for an R-module M:*

(1) u-S-$\mathrm{id}_R(M) \leq n$;
(2) $\mathrm{Ext}_R^{n+k}(N, M)$ *is u-S-torsion for all R-modules N and all $k > 0$;*
(3) $\mathrm{Ext}_R^{n+1}(N, M)$ *is u-S-torsion for all R-modules N;*
(4) *if $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ is a u-S-exact sequence, where $E_0, E_1, \ldots, E_{n-1}$ are u-S-injective R-modules, then F_n is u-S-injective;*
(5) *if $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ is a u-S-exact sequence, where $E_0, E_1, \ldots, E_{n-1}$ are injective R-modules, then E_n is u-S-injective;*
(6) *if $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ is an exact sequence, where $E_0, E_1, \ldots, E_{n-1}$ are u-S-injective R-modules, then E_n is u-S-injective;*
(7) *if $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ is an exact sequence, where $E_0, E_1, \ldots, E_{n-1}$ are injective R-modules, then E_n is u-S-injective;*
(8) *there exists a u-S-exact sequence $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$, where $E_0, E_1, \ldots, E_{n-1}$ are injective R-modules and E_n is u-S-injective;*
(9) *there exists an exact sequence $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$, where $E_0, E_1, \ldots, E_{n-1}$ are injective R-modules and E_n is u-S-injective;*
(10) *there exists an exact sequence $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$, where $E_0, E_1, \ldots, E_n$ are u-S-injective R-modules.*

Lemma 3.5. *Let R be a ring, S a multiplicative subset of R, and M an R-module. Then u-S-$\mathrm{pd}_R(M) \geq u$-S-$\mathrm{fd}_R(M)$.*

Proof. It follows from [7, Proposition 2.13] that any u-S-projective module is u-S-flat, and so u-S-$\mathrm{pd}_R(M) \leq u$-S-$\mathrm{fd}_R(M)$. $\square$

Proposition 3.6. *Let R be a ring and S a multiplicative subset of R. If R is a u-S-Noetherian ring, then the following statements hold.*

(1) *If M is an S-finite R-module, then there is a u-S-exact sequence*

$$\cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

with each F_n S-finite u-S-projective.

(2) *If M is an S-finite R-module, then u-S-$pd_R(M) = u$-S-$fd_R(M)$.*

Proof. (1) Since M is an S-finite R-module, there is a short u-S-exact sequence $0 \to K_0 \to F_0 \to M \to 0$, with F_0 a finitely generated free R-module. Since R

is a u-S-Noetherian ring, F_0 is u-S-Noetherian by [6, Theorem 2.7], and hence K_0 is S-finite. So there is a u-S-exact sequence $0 \to K_1 \to F_1 \to K_0 \to 0$ with F_1 a finitely generated free R-module. Iterating these steps, we can obtain a u-S-exact sequence $\cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_n finitely generated free and hence S-finite u-S-projective.

(2) It follows by Lemma 3.5 that u-S-$\mathrm{pd}_R(M) \geq u$-S-$\mathrm{fd}_R(M)$. On the other hand, assume u-S-$\mathrm{fd}_R(M) = n < \infty$. Then there is a u-S-exact sequence $0 \to K_{n-1} \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i u-S-projective for each $i < n$ and K_n u-S-Noetherian by (1) and its proof. It follows from [5, Proposition 2.3] that K_n u-S-finitely presented and u-S-flat. So K_n is u-S-projective by [6, Proposition 2.8]. Hence u-S-$\mathrm{fd}_R(M) \leq n$. □

Lemma 3.7. *Let R be a ring and $S' \subseteq S$ multiplicative subsets of R. Suppose M is an R-module, then u-S-$\mathrm{pd}_R(M) \leq u$-S'-$\mathrm{pd}_R(M)$ and u-S-$\mathrm{id}_R(M) \leq u$-S'-$\mathrm{id}_R(M)$.*

Proof. Suppose $S' \subseteq S$ are multiplicative subsets of R. Let M and N be R-modules. If $\mathrm{Ext}_R^{n+1}(M,N)$ is u-S'-torsion, then $\mathrm{Ext}_R^{n+1}(M,N)$ is u-S-torsion. The result follows by Proposition 3.3. □

Let S be a multiplicative subset of R. The saturation S^* of S is defined as $S^* = \{s \in R \mid s_1 = s_2 s \text{ for some } s_1, s_2 \in S\}$. A multiplicative subset S of R is called saturated if $S = S^*$. Note that S^* is always a saturated multiplicative subset containing S.

Proposition 3.8. *Let R be a ring, S be a multiplicative subset of R and S^* be the saturation of S. Suppose M is an R-module. Then u-S-$\mathrm{pd}_R(M) = u$-S^*-$\mathrm{pd}_R(M)$ and u-S-$\mathrm{id}_R(M) = u$-S^*-$\mathrm{id}_R(M)$.*

Proof. We only prove u-S-$\mathrm{pd}_R(M) = u$-S^*-$\mathrm{pd}_R(M)$ since the other one is similar. Certainly, u-S-$\mathrm{pd}_R(M) \geq u$-S^*-$\mathrm{pd}_R(M)$. On the other hand, we may assume that u-S^*-$\mathrm{pd}_R(M) = n < \infty$. Then there is $s \in S^*$ such that $s\mathrm{Ext}_R^{n+1}(N,M) = 0$ for all R-modules N. Then there are $s_1, s_2 \in S$ such that $s_1 = s_2 s$. Hence $s_1\mathrm{Ext}_R^{n+1}(N,M) = s_2 s\mathrm{Ext}_R^{n+1}(N,M) = 0$ for all R-modules N. It follows that u-S-$\mathrm{pd}_R(M) = n$. □

Proposition 3.9. *Let R_i be a ring, S_i be a multiplicative subset of R_i and M_i be an R_i-module ($i = 1, \ldots, n$). Set $R = R_1 \times \cdots \times R_n$, $S = S_1 \times \cdots \times S_n$ a multiplicative subset of R_i and $M = M_1 \times \cdots \times M_n$ an R-module. Then u-S-$\mathrm{pd}_R(M) = \sup\limits_{1 \leq i \leq n} \{u\text{-}S_i\text{-}\mathrm{pd}_{R_i}(M_i)\}$ and u-S-$\mathrm{id}_R(M) = \sup\limits_{1 \leq i \leq n} \{u\text{-}S_i\text{-}\mathrm{id}_{R_i}(M_i)\}$.*

Proof. We only prove u-S-$\mathrm{pd}_R(M) = \sup\limits_{1 \leq i \leq n} \{u\text{-}S_i\text{-}\mathrm{pd}_{R_i}(M_i)\}$ since the other one is similar.

Suppose u-S-$\mathrm{pd}_R(M) \leq n$. Then for any R-modules N, there is $s = (s_1, \ldots, s_n) \in S$ such that $s\mathrm{Ext}_R^{n+1}(M, N) = 0$. So $s_i\mathrm{Ext}_{R_i}^{n+1}(M_i, K) = 0$ for any R_i-module K. Consequently, $\sup\limits_{1\leq i\leq n} \{u\text{-}S_i\text{-}\mathrm{pd}_{R_i}(M_i)\} \leq n$. On the other hand, suppose $\sup\limits_{1\leq i\leq n} \{u\text{-}S_i\text{-}\mathrm{pd}_{R_i}(M_i)\} \leq n$. Let N be an R-module. Then e_iN is an R_i-module where e_i is the element in R with 1 at i-th component and 0 at others. Then for any $i = 1, \ldots, n$, there is $s_i \in S_i$ such that $s_i\mathrm{Ext}_{R_i}^{n+1}(M_i, e_iN) = 0$. Set $s = (s_1, \ldots, s_n) \in S$. Then $s\mathrm{Ext}_R^{n+1}(M, N) = 0$. So u-S-$\mathrm{pd}_R(M) \leq n$. $\square$

Proposition 3.10. *Let R be a ring and S a multiplicative subset of R. Let $0 \to A \to B \to C \to 0$ be a u-S-exact sequence of R-modules. Then the following statements hold.*

(1) u-S-$\mathrm{pd}_R(C) \leq 1 + \max\{u\text{-}S\text{-}\mathrm{pd}_R(A), u\text{-}S\text{-}\mathrm{pd}_R(B)\}$.
(2) *If* u-S-$\mathrm{pd}_R(B) < u$-S-$\mathrm{pd}_R(C)$, *then* u-S-$\mathrm{pd}_R(A) = u$-S-$\mathrm{pd}_R(C) - 1 > u$-S-$\mathrm{pd}_R(B)$.
(3) u-S-$\mathrm{id}_R(A) \leq 1 + \max\{u\text{-}S\text{-}\mathrm{id}_R(B), u\text{-}S\text{-}\mathrm{id}_R(C)\}$.
(4) *If* u-S-$\mathrm{id}_R(B) < u$-S-$\mathrm{id}_R(A)$, *then* u-S-$\mathrm{id}_R(C) = u$-S-$\mathrm{id}_R(A) - 1 > u$-S-$\mathrm{id}_R(B)$.

Proof. The proof is similar with that of the classical case (see [3, Theorem 3.5.6] and [3, Theorem 3.5.13]). So we omit it. $\square$

Proposition 3.11. *Let $0 \to A \to B \to C \to 0$ be a u-S-split u-S-exact sequence of R-modules. Then the following statements hold.*

(1) u-S-$\mathrm{pd}_R(B) = \max\{u\text{-}S\text{-}\mathrm{pd}_R(A), u\text{-}S\text{-}\mathrm{pd}_R(C)\}$.
(2) u-S-$\mathrm{id}_R(B) = \max\{u\text{-}S\text{-}\mathrm{id}_R(A), u\text{-}S\text{-}\mathrm{id}_R(C)\}$.

Proof. We only show the first assertion since the other one is similar. Since the u-S-projective dimensions of R-modules are invariant under u-S-isomorphisms by Lemma 3.2, we may assume $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a u-S-split exact sequence. So there exist R-homomorphisms $f' : B \to A$ and $g' : C \to B$ such that $f' \circ f = s_1\mathrm{Id}_A$ and $g \circ g' = s_2\mathrm{Id}_C$ for some $s_1, s_2 \in S$. To prove (1), we just need to show that $0 \to \mathrm{Ext}_R^n(M, A) \xrightarrow{\mathrm{Ext}_R^n(M,f)} \mathrm{Ext}_R^n(M, B) \xrightarrow{\mathrm{Ext}_R^n(M,g)} \mathrm{Ext}_R^n(M, C) \to 0$ is a u-S-exact sequence for any R-module M. Since the composition map $\mathrm{Ext}_R^n(M, f') \circ \mathrm{Ext}_R^n(M, f) : \mathrm{Ext}_R^n(M, A) \to \mathrm{Ext}_R^n(M, A)$ is equal to $\mathrm{Ext}_R^n(M, s_1\mathrm{Id}_A)$ which is just the multiplication map by s_1, we have $\mathrm{Ext}_R^n(M, f)$ is a u-S-split u-S-monomorphism. Similarly, $\mathrm{Ext}_R^n(M, g)$ is a u-S-split u-S-epimorphism. $\square$

Let $\mathfrak{p}$ be a prime ideal of R and M an R-module. Set u-$\mathfrak{p}$-$\mathrm{pd}_R(M)$ (resp., u-$\mathfrak{p}$-$\mathrm{id}_R(M)$) to be u-$(R \setminus \mathfrak{p})$-$\mathrm{pd}_R(M)$ (resp., u-$(R \setminus \mathfrak{p})$-$\mathrm{id}_R(M)$) for simplification. The next result gives a new characterization of projective dimension and injective dimension of an R-module.

Proposition 3.12. *Let R be a ring and M an R-module. Then*

$$\mathrm{pd}_R(M)=\sup\{u\text{-}\mathfrak{p}\text{-}\mathrm{pd}_R(M)\mid \mathfrak{p}\in \mathrm{Spec}(R)\}=\sup\{u\text{-}\mathfrak{m}\text{-}\mathrm{pd}_R(M)\mid \mathfrak{m}\in \mathrm{Max}(R)\}$$

and

$$\mathrm{id}_R(M)=\sup\{u\text{-}\mathfrak{p}\text{-}\mathrm{id}_R(M)\mid \mathfrak{p}\in \mathrm{Spec}(R)\}=\sup\{u\text{-}\mathfrak{m}\text{-}\mathrm{id}_R(M)\mid \mathfrak{m}\in \mathrm{Max}(R)\}.$$

Proof. We only show the first equation since the other one is similar. Trivially, $\sup\{u\text{-}\mathfrak{m}\text{-}\mathrm{pd}_R(M)\mid \mathfrak{m}\in \mathrm{Max}(R)\}\leq \sup\{u\text{-}\mathfrak{p}\text{-}\mathrm{pd}_R(M)\mid \mathfrak{p}\in \mathrm{Spec}(R)\}\leq \mathrm{pd}_R(M)$. Suppose $\sup\{u\text{-}\mathfrak{m}\text{-}\mathrm{pd}_R(M)\mid \mathfrak{m}\in \mathrm{Max}(R)\}=n$. For any R-module N, there exists an element $s^{\mathfrak{m}}\in R-\mathfrak{m}$ such that $s^{\mathfrak{m}}\mathrm{Ext}_R^{n+1}(M,N)=0$ by Proposition 3.3. Since the ideal generated by all $s^{\mathfrak{m}}$ is R, we have $\mathrm{Ext}_R^{n+1}(M,N)=0$ for all R-modules N. So $\mathrm{pd}_R(M)\leq n$. Suppose $\sup\{u\text{-}\mathfrak{m}\text{-}\mathrm{pd}_R(M)\mid \mathfrak{m}\in \mathrm{Max}(R)\}=\infty$. Then for any $n\geq 0$, there exist a maximal ideal $\mathfrak{m}$ and an element $s^{\mathfrak{m}}\in R-\mathfrak{m}$ such that $s^{\mathfrak{m}}\mathrm{Ext}_R^{n+1}(M,N)\neq 0$ for some R-module N. So for any $n\geq 0$, we have $\mathrm{Ext}_R^{n+1}(M,N)\neq 0$ for some R-module N. Thus $\mathrm{pd}_R(M)=\infty$. So the equalities hold. $\square$

It is well known that the global dimension $\mathrm{gl.dim}(R)$ of a ring R is defined to be the supremum of projective dimensions of all R-modules. Recall from [5] that the u-S-weak global dimension u-S-w.gl.dim(R) of a ring R is the supremum of u-S-flat dimensions of all R-modules. Now, we introduce the u-S-global dimensions of rings R in terms of u-S-projective dimensions of R-modules.

Definition 3.13. The u-S-global dimension of a ring R is defined by

$$u\text{-}S\text{-}\mathrm{gl.dim}(R)=\sup\{u\text{-}S\text{-}\mathrm{pd}_R(M)\mid M \text{ is an } R\text{-module}\}.$$

Obviously, u-S-gl.dim$(R)\leq$gl.dim(R) for any multiplicative subset S of R. And if S is composed of units, then u-S-gl.dim$(R)=$gl.dim(R). The next result characterizes the u-S-global dimension of a ring R.

Proposition 3.14. *Let R be a ring and S a multiplicative subset of R. The following statements are equivalent for R:*

(1) u-S-gl.dim$(R)\leq n$;
(2) u-S-$\mathrm{pd}_R(M)\leq n$ *for all R-modules M*;
(3) $\mathrm{Ext}_R^{n+k}(M,N)$ *is u-S-torsion for all R-modules M,N and all $k>0$*;
(4) $\mathrm{Ext}_R^{n+1}(M,N)$ *is u-S-torsion for all R-modules M,N*;
(5) u-S-$\mathrm{id}_R(M)\leq n$ *for all R-modules M.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4): Trivial.

(2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (3): It follows from Proposition 3.3.

(4) $\Rightarrow$ (2): Let M be an R-module and $0\to F_n\to\cdots\to F_1\to F_0\to M\to 0$ an exact sequence, where $F_0,F_1,\dots,F^{n-1}$ are projective R-modules. To complete the proof, it suffices, by Proposition 3.3, to prove that F_n is u-S-projective. Let N be an R-module. Thus u-S-$\mathrm{pd}_R(N)\leq n$ by (4). It follows from Corollary 2.3 that $\mathrm{Ext}_R^1(N,F_n)\cong \mathrm{Ext}_R^{n+1}(N,M)$ is u-S-torsion. Thus F_n is u-S-projective.

(4) $\Rightarrow$ (5): Let M be an R-module and $0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ an exact sequence with $E_0, E_1, \ldots, E_{n-1}$ are injective R-modules. By dimension shifting, we have $\mathrm{Ext}_R^{n+1}(M,N) \cong \mathrm{Ext}_R^1(E_n,N)$. So $\mathrm{Ext}_R^1(E_n,N)$ is u-S-torsion for any R-module N. Thus E_n is u-S-injective by [2, Theorem 4.3]. Consequently, u-S-$\mathrm{id}_R(M) \leq n$ by Theorem 3.4. □

Consequently, we have

$$\begin{aligned} u\text{-}S\text{-gl.dim}(R) &= \sup\{u\text{-}S\text{-pd}_R(M) \mid M \text{ is an } R\text{-module}\} \\ &= \sup\{u\text{-}S\text{-id}_R(M) \mid M \text{ is an } R\text{-module}\}. \end{aligned}$$

Corollary 3.15. *Let R be a ring, S a multiplicative subset of R. Then u-S-gl.dim$(R) \geq u$-S-w.gl.dim(R).*

Proof. It follows from Lemma 3.5. □

Corollary 3.16. *Let R be a ring, $S' \subseteq S$ be multiplicative subsets of R and S^* be the saturation of S. Then u-S-gl.dim$(R) \leq u$-S'-gl.dim(R) and u-S-gl.dim$(R) = u$-S^*-gl.dim(R).*

Proof. It follows from Lemma 3.7 and Proposition 3.8. □

Proposition 3.17. *Let R_i be a ring and S_i be a multiplicative subset of R_i $(i = 1, \ldots, n)$. Set $R = R_1 \times \cdots \times R_n$ and $S = S_1 \times \cdots \times S_n$ a multiplicative subset of R_i. Then u-S-gl.dim$(R) = \sup\limits_{1\leq i\leq n} \{u$-$S_i$-gl.dim$(R_i)\}$.*

Proof. It follows from Proposition 3.9. □

The following example shows that the global dimension of rings and the u-S-global dimension of rings can be wildly different.

Example 3.18. Let R_1 be a ring with gl.dim$(R_1) = n$ and R_2 be a ring with gl.dim$(R_2) = m$. Set $R = R_1 \times R_2$ and $S = \{(1,1),(1,0)\}$. Then gl.dim$(R) = \max\{m,n\}$. But u-S-gl.dim$(R) = n$ by Proposition 3.17.

Let $\mathfrak{p}$ be a prime ideal of a ring R and u-$\mathfrak{p}$-gl.dim(R) denote u-$(R \setminus \mathfrak{p})$-gl.dim$(R)$ briefly. By Proposition 3.12, we have a new characterization of global dimensions of commutative rings.

Corollary 3.19. *Let R be a ring. Then*

$$\begin{aligned} \text{gl.dim}(R) &= \sup\{u\text{-}\mathfrak{p}\text{-gl.dim}(R) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &= \sup\{u\text{-}\mathfrak{m}\text{-gl.dim}(R) \mid \mathfrak{m} \in \mathrm{Max}(R)\}. \end{aligned}$$

Recall from [7] that an R-module M is called u-S-semisimple provided that any u-S-short exact sequence $0 \to A \to M \to C \to 0$ is u-S-split. And R is called a u-S-semisimple ring provided that any free R-module is u-S-semisimple. Thus by [7, Theorem 3.5], the following result holds.

Corollary 3.20. *Let R be a ring and S a multiplicative subset of R. The following statements are equivalent*:

(1) *R is a u-S-semisimple ring*;
(2) *every R-module is u-S-semisimple*;
(3) *every R-module is u-S-projective*;
(4) *every R-module is u-S-injective*;
(5) u-S-gl.dim$(R) = 0$.

4. The change of rings theorems for u-S-global dimensions

The author in [5] investigated the u-S-weak global dimensions of factor rings and polynomial rings. In this section, we mainly consider u-S-global dimensions of factor rings and polynomial rings. Although the research approaches and proof ideas are very similar, we will still present the proof in its entirety to show some differences in their proofs.

We will give an inequality of u-S-global dimensions for ring homomorphisms. Let $\theta : R \to T$ be a ring homomorphism. Suppose S is a multiplicative subset of R, then $\theta(S) = \{\theta(s) \mid s \in S\}$ is a multiplicative subset of T.

Proposition 4.1. *Let $\theta : R \to T$ be a ring homomorphism, S a multiplicative subset of R. Suppose M is a T-module. Then*

$$u\text{-}S\text{-pd}_R(M) \leq u\text{-}\theta(S)\text{-pd}_T(M) + u\text{-}S\text{-pd}_R(T).$$

Proof. Assume u-$\theta(S)$-pd$_T(M) = n < \infty$. If $n = 0$, then M is u-$\theta(S)$-projective over T. Then there exists u-$\theta(S)$-split short exact sequence $0 \to A \to F \to M \to 0$ with F a free R-module of rank at least 1. By Proposition 3.11, we have u-$\theta(S)$-pd$_T(F) \geq u$-$\theta(S)$-pd$_T(M)$. So u-S-pd$_R(M) \leq u$-S-pd$_R(F) = u$-S-pd$_R(T) \leq n + u$-S-pd$_R(T)$.

Now we assume $n > 0$. Let $0 \to A \to F \to M \to 0$ be an exact sequence of T-modules, where F is a free T-module of rank at least 1. Then u-$\theta(S)$-pd$_T(A) = n - 1$ by Corollary 2.3 and Proposition 3.3. By induction, u-S-pd$_R(A) \leq n - 1 + u$-S-pd$_R(T)$. Note that u-S-pd$_R(T) = u$-S-pd$_R(F)$. By Proposition 3.10, we have

$$\begin{aligned} u\text{-}S\text{-}pd_R(M) &\leq 1 + \max\{u\text{-}S\text{-}pd_R(F), u\text{-}S\text{-}pd_R(A)\} \\ &\leq 1 + n - 1 + u\text{-}S\text{-}pd_R(T) \\ &= u\text{-}\theta(S)\text{-}pd_T(M) + u\text{-}S\text{-}pd_R(T). \end{aligned}$$ □

Let R be a ring, I an ideal of R and S a multiplicative subset of R. Then $\pi : R \to R/I$ is a ring epimorphism and $\pi(S) := \overline{S} = \{s + I \in R/I \mid s \in S\}$ is naturally a multiplicative subset of R/I.

Proposition 4.2. *Let R be a ring, S a multiplicative subset of R. Let a be a non-zero-divisor in R which does not divide any element in S. Written $\overline{R} = R/aR$ and $\overline{S} = \{s + aR \in \overline{R} \mid s \in S\}$. Then the following statements hold.*

(1) *Let M be a nonzero $\overline{R}$-module. If u-$\overline{S}$-pd$_{\overline{R}}(M) < \infty$, then*

$$u\text{-}S\text{-pd}_R(M) = u\text{-}\overline{S}\text{-pd}_{\overline{R}}(M) + 1.$$

(2) *If u-$\overline{S}$-gl.dim$(\overline{R}) < \infty$, then*

$$u\text{-}S\text{-gl.dim}(R) \geq u\text{-}\overline{S}\text{-gl.dim}(\overline{R}) + 1.$$

Proof. (1) Set u-$\overline{S}$-$\mathrm{pd}_{\overline{R}}(M) = n$. Since a is a non-zero-divisor which does not divide any element in S, the exact sequence $0 \to aR \to R \to R/aR \to 0$ is not u-S-split. Indeed, suppose $g : R \to aR$ is an R-homomorphism such that $g(a) = sa$. Then $ag(1) = sa$ and thus $g(1) = s \in aR$ since a is a non-zero-divisor. So $s = ar$ for some $r \in R$, which is a contradiction since a does not divide any element in S. Thus u-S-$\mathrm{pd}_R(\overline{R}) = 1$ by [7, Corollary 2.10]. By Proposition 4.1, we have u-S-$\mathrm{pd}_R(M) \leq u$-$\overline{S}$-$\mathrm{pd}_{\overline{R}}(M) + 1 = n + 1$. Since u-$\overline{S}$-$\mathrm{pd}_{\overline{R}}(M) = n$, there is an injective $\overline{R}$-module C such that $\mathrm{Ext}^n_{\overline{R}}(M, C)$ is not u-$\overline{S}$-torsion. By [3, Theorem 2.4.22], there is an injective R-module E such that $0 \to C \to E \to E \to 0$ is exact. By [3, Proposition 3.8.12(4)], $\mathrm{Ext}^{n+1}_R(M, E) \cong \mathrm{Ext}^n_{\overline{R}}(M, C)$. Thus $\mathrm{Ext}^{n+1}_R(M, E)$ is not u-S-torsion. So u-S-$\mathrm{pd}_R(M) = u$-$\overline{S}$-$\mathrm{pd}_{\overline{R}}(M) + 1$.

(2) Let $n = u$-$\overline{S}$-gl.dim$(\overline{R})$. Then there is a nonzero $\overline{R}$-module M such that u-$\overline{S}$-$\mathrm{pd}_{\overline{R}}(M) = n$. Thus u-S-$\mathrm{pd}_R(M) = n+1$ by (1). So u-S-gl.dim$(R) \geq u$-$\overline{S}$-gl.dim$(\overline{R}) + 1$. □

Let R be a ring and $R[x]$ denotes the polynomial ring with one indeterminate, where all coefficients are in R. The well-known Hilbert syzygies Theorem states that gl.dim$(R[x])$ =gl.dim(R) for any ring R (see [3, Theorem 3.8.23]). In the rest of this section, we will give a u-S-analogue of this result. Let S be a multiplicative subset of R, then S is a multiplicative subset of $R[x]$ naturally.

Lemma 4.3. *Let R be a ring, S a multiplicative subset of R. Suppose T is an R-module and F is an $R[x]$-module. If P is u-S-projective over $R[x]$, then P is u-S-projective over R.*

Proof. Suppose P is a u-S-projective $R[x]$-module. Then there exist a free $R[x]$-module F and a u-S-split $R[x]$-short exact sequence $0 \to K \to F \xrightarrow{\pi} P \to 0$. Thus we have an $R[x]$-homomorphism $\pi' : P \to F$ such that $\pi \circ \pi' = s\mathrm{Id}_P$ for some $s \in S$. Note that π' is also an R-homomorphism. So $0 \to K \to F \xrightarrow{\pi} P \to 0$ is also u-S-split over R. Note that F is also a free R-module. So P is u-S-projective over R by [7, Proposition 2.8]. □

Let M be an R-module. Set $M[x] = M \otimes_R R[x]$. Then $M[x]$ can be seen as an $R[x]$-module naturally.

Proposition 4.4. *Let R be a ring, S a multiplicative subset of R and M an R-module. Then u-S-$\mathrm{pd}_{R[x]}(M[x]) = u$-$S$-$\mathrm{pd}_R(M)$.*

Proof. Assume that u-S-$\mathrm{pd}_R(M) \leq n$. Then M has a u-S-projective resolution over R:

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Since $R[x]$ is free over R, $R[x]$ is a u-S-flat R-module by [7, Proposition 2.7]. Thus the natural sequence

$$0 \to P_n[x] \to \cdots \to P_1[x] \to P_0[x] \to M[x] \to 0$$

is u-S-exact over $R[x]$. Consequently, u-S-$\mathrm{pd}_{R[x]}(M[x]) \leq n$ by Proposition 3.3.

Let $0 \to F_n \to \cdots \to F_1 \to F_0 \to M[x] \to 0$ be an exact sequence with each F_i u-S-projective over $R[x]$ $(1 \leq i \leq n)$. Then it is also a u-S-projective resolution of $M[x]$ over R by Lemma 4.3. Thus $\mathrm{Ext}_R^{n+1}(M[x], N)$ is u-S-torsion for any R-module N by Proposition 3.3. It follows that $s\mathrm{Ext}_R^{n+1}(M[x], N) \cong s\prod_{i=1}^{\infty} \mathrm{Ext}_R^{n+1}(M, N) = 0$. Thus $\mathrm{Ext}_R^{n+1}(M, N)$ is u-S-torsion. Consequently, u-S-$\mathrm{pd}_R(M) \leq u$-S-$\mathrm{pd}_{R[x]}(M[x])$ by Proposition 3.3 again. □

Let M be an $R[x]$-module. Then M can be naturally viewed as an R-module. Define $\psi : M[x] \to M$ by

$$\psi(\sum_{i=0}^{n} x^i \otimes m_i) = \sum_{i=0}^{n} x^i m_i, \qquad m_i \in M.$$

And define $\varphi : M[x] \to M[x]$ by

$$\varphi(\sum_{i=0}^{n} x^i \otimes m_i) = \sum_{i=0}^{n} x^{i+1} \otimes m_i - \sum_{i=0}^{n} x^i \otimes x m_i, \qquad m_i \in M.$$

Lemma 4.5 ([3, Theorem 3.8.22]). *Let R be a ring. For any $R[x]$-module M,*

$$0 \to M[x] \xrightarrow{\varphi} M[x] \xrightarrow{\psi} M \to 0$$

is exact.

Theorem 4.6. *Let R be a ring and S a multiplicative subset of R. Then*

$$u\text{-}S\text{-gl.dim}(R[x]) = \begin{cases} u\text{-}S\text{-gl.dim}(R) + 1, & 0 \notin S, \\ 0, & 0 \in S. \end{cases}$$

Proof. Suppose $0 \in S$. Then every $R[x]$-module is u-S-projective, and so u-S-gl.dim$(R[x]) = 0$.

Now, suppose $0 \notin S$. Let M be an $R[x]$-module. Then, by Lemma 4.5, there is an exact sequence over $R[x]$:

$$0 \to M[x] \to M[x] \to M \to 0.$$

By Proposition 3.10, Proposition 4.1 and Proposition 4.4,

$$(*) \quad u\text{-}S\text{-pd}_R(M) \leq u\text{-}S\text{-pd}_{R[x]}(M) \leq 1 + u\text{-}S\text{-pd}_{R[x]}(M[x]) = 1 + u\text{-}S\text{-pd}_R(M).$$

Thus if u-S-gl.dim$(R) < \infty$, then u-S-gl.dim$(R[x]) < \infty$.

Conversely, if u-S-gl.dim$(R[x]) < \infty$, then for any R-module M,

$$u\text{-}S\text{-pd}_R(M) = u\text{-}S\text{-pd}_{R[x]}(M[x]) < \infty$$

by Proposition 4.4. Therefore we have u-S-gl.dim$(R) < \infty$ if and only if u-S-gl.dim$(R[x]) < \infty$. Now we assume that both of these are finite. Then u-S-gl.dim$(R[x]) \leq u$-S-gl.dim$(R) + 1$ by $(*)$. Trivially, x is a non-zero-divisor of $R[x]$. Since $0 \notin S$, it is easy to check x does not divide any element in S.

Since $R \cong R[x]/xR[x]$, u-S-gl.dim$(R[x]) \geq u$-S-gl.dim$(R) + 1$ by Proposition 4.2. Consequently, we have u-S-gl.dim$(R[x]) = u$-S-gl.dim$(R) + 1$. $\square$

Remark 4.7. It was proved in [5, Theorem 4.7] that if u-S-fd$_{R[x]}(R) = 1$, then u-S-w.gl.dim$(R[x]) = u$-S-w.gl.dim$(R) + 1$. But we do not require u-S-pd$_{R[x]}(R) = 1$ in Theorem 4.6.

Corollary 4.8. *Let R be a ring and S a multiplicative subset of R. Then for any $n \geq 1$ we have*

$$u\text{-}S\text{-gl.dim}(R[x_1, \ldots, x_n]) = \begin{cases} u\text{-}S\text{-gl.dim}(R) + n, & 0 \notin S, \\ 0, & 0 \in S. \end{cases}$$

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