VOLUME DENSITY ASYMPTOTICS OF
CENTRAL HARMONIC SPACES

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Abstract. We show the asymptotics of the volume density function in the class of central harmonic manifolds can be specified arbitrarily and do not determine the geometry.

1. Introduction

Let $\mathbb{M}_\psi := (M, \psi^2 g)$ be the conformal deformation of a connected Riemannian manifold $\mathbb{M} := (M, g)$ of dimension $m \geq 4$, where ψ is a smooth positive function on M. If P is a point of M, let $r_{\mathbb{M},P}(Q)$ be the geodesic distance from P to Q, let $\iota_{\mathbb{M},P}$ be the injectivity radius at P, let $B_{\mathbb{M},P} := \{Q \in M : r_{\mathbb{M},P}(Q) < \iota_{\mathbb{M},P}\}$ be the open ball about P of radius $\iota_{\mathbb{M},P}$, let $\vec{e} = (e_1, \ldots, e_m)$ be an orthonormal basis for $T_P M$, and let $\vec{x} = (x^1, \ldots, x^m) := \exp_P(x^1 e_1 + \cdots + x^m e_m)$ be geodesic coordinates centered at P. We then have that

$$r_{\mathbb{M},P}(\vec{x}) = \|\vec{x}\| = \left\{ (x^1)^2 + \cdots + (x^m)^2 \right\}^{1/2}.$$

Let $\mathrm{dvol}_{\mathbb{M}}$ be the Riemannian measure on $\mathbb{M}$, let $g_{ij} := g(\partial_{x^i}, \partial_{x^j})$, and let $d\vec{x} = dx^1 \cdots dx^m$ be the Euclidean measure on $T_P M$. Then

$$\mathrm{dvol}_{\mathbb{M}} = \tilde{\Theta}_{\mathbb{M},P} dx^1 \cdots dx^m, \quad \text{where} \quad \tilde{\Theta}_{\mathbb{M},P} := \det(g_{ij})^{1/2},$$

where $\tilde{\Theta}_{\mathbb{M},P}$ is the *volume density function*. Let $S_P^{m-1} := \{\vec{\theta} \in T_P M : \|\vec{\theta}\| = 1\}$ be the unit sphere in $T_P M$ and let $\mathbb{S}_P^{m-1} := (S_P^{m-1}, g_S)$, where g_S is the metric on S_P^{m-1} induced from Euclidean metric on $T_P M$ defined by g. Introduce geodesic polar coordinates $(r, \vec{\theta})$ on $B_{\mathbb{M},P} - \{P\}$ to express

$$\vec{x} = r(\vec{x})\vec{\theta}(\vec{x}) \quad \text{for} \quad 0 < r(\vec{x}) := \|\vec{x}\| < \iota_{\mathbb{M},P} \quad \text{and} \quad \vec{\theta}(\vec{x}) = \|\vec{x}\|^{-1}\vec{x} \in S_P^{m-1}.$$

Note that $\vec{\theta}(\vec{x})$ is not defined when $x = 0$. We may also express

$$\mathrm{dvol}_{\mathbb{M}} = \Theta_{\mathbb{M},P} dr \, \mathrm{dvol}_{\mathbb{S}_P^{m-1}}, \quad \text{where} \quad \Theta_{\mathbb{M},P} := r^{m-1} \tilde{\Theta}_{\mathbb{M},P}.$$

We say that a smooth function f, which is defined near P, is *radial* if there exists a smooth function η_1 of one real variable so $f(\vec{x}) = \eta_1(\|\vec{x}\|)$; f is smooth at P if and only if we can write $f(\vec{x}) = \eta_2(\|\vec{x}\|^2)$ or, equivalently, η_1 is an even function of $\|\vec{x}\|$. We say that $\mathbb{M}$ is *central harmonic* at P if $\tilde{\Theta}_{\mathbb{M},P}$ is a radial function on $B_{\mathbb{M},P}$. We say that $\mathbb{M}$ is a *harmonic space* if $\mathbb{M}$ is central harmonic about every point.

There is a vast literature on this subject; we refer to $[2, 3, 5, 12–14]$ and the references cited therein for further details. Note that if $\mathbb{M}$ is a harmonic space, then we can rescale the metric to replace g by c^2g for any $c > 0$ to obtain another harmonic space $\mathbb{M}_c := (M, c^2g)$. Similarly, we shall show in Corollary 2.2 that if $\mathbb{M}$ is central harmonic at P and if ψ is a smooth positive radial function, then the radial conformal deformation $\mathbb{M}_\psi := (M, \psi^2g)$ is again central harmonic at P.

1.1. The asymptotic expansion of the volume density function

We can expand the volume density function in geodesic polar coordinates in a formal power-series

$$\tilde{\Theta}_{\mathbb{M},P}(r,\vec{\theta}) \sim 1 + \sum_{\nu=2}^{\infty} \mathcal{H}_\nu(\mathbb{M}, P, \vec{\theta})r^\nu.$$

If $\xi \in T_PM$, let $\mathcal{J}(\xi) := \mathcal{J}_0(\xi)$ be the Jacobi operator and $\mathcal{J}_k(\xi) := \nabla_\xi^k \mathcal{J}(\xi)$; $\mathcal{J}_k(\xi)$ is a self-adjoint endomorphism of T_PM which is characterized by the relationship:

$$g(\mathcal{J}_k(\xi)\xi_1, \xi_2) = (\nabla^k R)(\xi_1, \xi, \xi, \xi_2; \xi, \ldots, \xi);$$

We refer to Gray [10] for the computation of $\mathcal{H}_\nu(\mathbb{M}, P, \vec{\theta})$ for $2 \le \nu \le 6$ and to Gilkey and Park [7] for the computation of $\mathcal{H}_\nu(\mathbb{M}, P, \vec{\theta})$ when $\nu = 7, 8$.

Theorem 1.1. *Let P be a point of a Riemannian manifold $\mathbb{M}$ and let $\vec{\theta} \in S_{\mathbb{M}}^P$.*

(1) $\mathcal{H}_2(\mathbb{M}, P, \vec{\theta}) = -\frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}}{6}$.

(2) $\mathcal{H}_3(\mathbb{M}, P, \vec{\theta}) = -\frac{\text{Tr}\{\mathcal{J}_1(\vec{\theta})\}}{12}$.

(3) $\mathcal{H}_4(\mathbb{M}, P, \vec{\theta}) = \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}^2}{72} - \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})^2\}}{180} - \frac{\text{Tr}\{\mathcal{J}_2(\vec{\theta})\}}{40}$.

(4) $\mathcal{H}_5(\mathbb{M}, P, \vec{\theta}) = \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}\text{Tr}\{\mathcal{J}_1(\vec{\theta})\}}{72} - \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\mathcal{J}_1(\vec{\theta})\}}{180} - \frac{\text{Tr}\{\mathcal{J}_3(\vec{\theta})\}}{180}$.

(5) $\mathcal{H}_6(\mathbb{M}, P, \vec{\theta}) = -\frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}^3}{1296} + \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}\text{Tr}\{\mathcal{J}(\vec{\theta})^2\}}{1080} + \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}\text{Tr}\{\mathcal{J}_2(\vec{\theta})\}}{240}$

$\qquad -\frac{\text{Tr}\{\mathcal{J}(\vec{\theta})^3\}}{2835} - \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\mathcal{J}_2(\vec{\theta})\}}{630} + \frac{\text{Tr}\{\mathcal{J}_1(\vec{\theta})\}^2}{288}$

$\qquad -\frac{\text{Tr}\{\mathcal{J}_1(\vec{\theta})^2\}}{672} - \frac{\text{Tr}\{\mathcal{J}_4(\vec{\theta})\}}{1008}$.

(6) $\mathcal{H}_7(\mathbb{M}, P, \vec{\theta}) = \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}\text{Tr}\{\mathcal{J}(\vec{\theta})\mathcal{J}_1(\vec{\theta})\}}{1080} - \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}^2\text{Tr}\{\mathcal{J}_1(\vec{\theta})\}}{864} - \frac{\text{Tr}\{\mathcal{J}_5(\vec{\theta})\}}{6720}$

$\qquad +\frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\}\text{Tr}\{\mathcal{J}_3(\vec{\theta})\}}{1080} + \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})^2\}\text{Tr}\{\mathcal{J}_1(\vec{\theta})\}}{2160} - \frac{\text{Tr}\{\mathcal{J}(\vec{\theta})^2\mathcal{J}_1(\vec{\theta})\}}{1890}$

$\qquad -\frac{\text{Tr}\{\mathcal{J}(\vec{\theta})\mathcal{J}_3(\vec{\theta})\}}{3024} + \frac{\text{Tr}\{\mathcal{J}_1(\vec{\theta})\}\text{Tr}\{\mathcal{J}_2(\vec{\theta})\}}{480} - \frac{\text{Tr}\{\mathcal{J}_1(\vec{\theta})\mathcal{J}_2(\vec{\theta})\}}{1120}$.

$$(7)\quad \mathcal{H}_8(\mathbb{M},P,\vec{\theta}) = \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\}^4}{31104} - \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\}^2\operatorname{Tr}\{\mathcal{J}(\vec{\theta})^2\}}{12960} - \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\}^2\operatorname{Tr}\{\mathcal{J}_2(\vec{\theta})\}}{2880}$$

$$+ \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\}\operatorname{Tr}\{\mathcal{J}(\vec{\theta})^3\}}{17010} + \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\}\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\mathcal{J}_2(\vec{\theta})\}}{3780} - \frac{\operatorname{Tr}\{\mathcal{J}_6(\vec{\theta})\}}{51840}$$

$$- \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\}\operatorname{Tr}\{\mathcal{J}_1(\vec{\theta})\}^2}{1728} + \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\}\operatorname{Tr}\{\mathcal{J}_1(\vec{\theta})^2\}}{4032}$$

$$- \frac{\operatorname{Tr}\{\mathcal{J}_2(\vec{\theta})^2\}}{7200} + \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\}\operatorname{Tr}\{\mathcal{J}_4(\vec{\theta})\}}{6048} + \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})^2\}\operatorname{Tr}\{\mathcal{J}_2(\vec{\theta})\}}{7200}$$

$$+ \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})^2\}^2}{64800} - \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})^4\}}{37800} - \frac{17\operatorname{Tr}\{\mathcal{J}(\vec{\theta})^2\mathcal{J}_2(\vec{\theta})\}}{113400}$$

$$+ \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\mathcal{J}_1(\vec{\theta})\}\operatorname{Tr}\{\mathcal{J}_1(\vec{\theta})\}}{2160} - \frac{5\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\mathcal{J}_1(\vec{\theta})^2\}}{18144}$$

$$- \frac{\operatorname{Tr}\{\mathcal{J}(\vec{\theta})\mathcal{J}_4(\vec{\theta})\}}{18144} + \frac{\operatorname{Tr}\{\mathcal{J}_1(\vec{\theta})\}\operatorname{Tr}\{\mathcal{J}_3(\vec{\theta})\}}{2160} - \frac{\operatorname{Tr}\{\mathcal{J}_1(\vec{\theta})\mathcal{J}_3(\vec{\theta})\}}{5184}$$

$$+ \frac{\operatorname{Tr}\{\mathcal{J}_2(\vec{\theta})\}^2}{3200}.$$

If $\mathbb{M}$ is central harmonic at P, then $\mathcal{H}_\nu(\mathbb{M},P,\vec{\theta})$ is independent of $\vec{\theta}$; we set $\mathcal{H}_\nu(\mathbb{M},P) := \mathcal{H}_\nu(\mathbb{M},P,\vec{\theta})$ for any $\vec{\theta} \in S_P^{m-1}$. Since $\tilde{\Theta}_{\mathbb{M},P}(\vec{x}) = \tilde{\Theta}_{\mathbb{M},P}(-\vec{x})$, we may conclude that $\mathcal{H}_\nu(\mathbb{M},P) = 0$ if ν is odd. If M is a harmonic space, then one can show that the value is independent of P and we set $\mathcal{H}_\nu(\mathbb{M}) = \mathcal{H}_\nu(\mathbb{M},P)$ for any P.

1.2. Specifying the volume density function

In Section 2, we will use an argument shown to us by Professor J. Álvarez-López [1] to establish the following result.

Theorem 1.2.

(1) *Let $\mathbb{M}$ be the germ of a Riemannian manifold which is central harmonic at P. Let Ξ be the germ of a smooth positive function of one real variable. There exists the germ of a smooth positive radial function ψ defined on M near P so that $\tilde{\Theta}_{\mathbb{M}_\psi,P} = \Xi(r_{\mathbb{M}_\psi})$.*

(2) *If $\Xi \equiv 1$, then ψ can be defined on all of $B_{\mathbb{M},P}$.*

(3) *If $\mathbb{M}$ and Ξ are real analytic, then ψ is real analytic.*

1.3. Specifying the volume density asymptotics in even dimensions

Let $\mathcal{W}_\mathbb{M}$ be the *Weyl curvature operator*. We refer to Section 3.3 for more details. Let Q_i be points of Riemannian manifolds $\mathbb{M}_i$. We say that $(\mathbb{M}_1,Q_1)$ is *Weyl curvature operator isomorphic* to $(\mathbb{M}_2,Q_2)$ if there exists a linear isomorphism Φ from $T_{Q_1}(M_1)$ to $T_{Q_2}(M_2)$ so that $\mathcal{W}_{\mathbb{M}_1}(Q_1) = \Phi^*\mathcal{W}_{\mathbb{M}_2}(Q_2)$. We say that $\mathbb{M}_1$ is *nowhere Weyl curvature operator isomorphic* to $\mathbb{M}_2$ if $(\mathbb{M}_1,Q_1)$ is not Weyl curvature operator isomorphic to $(\mathbb{M}_2,Q_2)$ for any points $Q_1 \in M_1$ and $Q_2 \in M_2$. This is a conformal condition since $\mathcal{W}_{\mathbb{M}_\psi} = \mathcal{W}_\mathbb{M}$. We introduce the following spaces for certain values of (i,m); they are not defined for all values of (i,m).

Definition 1.3. Let $\mathbb{M}_{1,m}$ be the complex projective space $\mathbb{CP}^k$ if $m = 2k$, let $\mathbb{M}_{2,m}$ be the quaternionic projective space $\mathbb{HP}^k$ if $m = 4k$, and let $\mathbb{M}_{3,m}$ be

the Cayley projective plane $\mathbb{OP}^2$ if $m = 16$; we refer to Section 1.6 for details. Let $\mathbb{M}_{4,m} := \widetilde{\mathbb{CP}}^k$, $\mathbb{M}_{5,m} := \widetilde{\mathbb{HP}}^k$, and $\mathbb{M}_{6,m} := \widetilde{\mathbb{OP}}^2$ be the negative curvature duals in the appropriate dimensions; we refer to Section 1.7 for further details. These are rank 1 symmetric spaces. Note that $\mathbb{M}_{2,m}$ and $\mathbb{M}_{5,m}$ are not defined unless $m = 4k$, and $\mathbb{M}_{3,m}$ and $\mathbb{M}_{6,m}$ are not defined if $m \neq 16$. Let $\mathbb{M}_{7,m} = \mathbb{R}^m$ be the flat Euclidean space. Let $P_{i,m} \in \mathbb{M}_{i,m}$; the particular point in question is irrelevant as $\mathbb{M}_{i,m}$ is a homogeneous space. Let $Q_{i,m}$ be arbitrary points of $M_{i,m}$. Let $\psi_{i,m}$ be a positive function on $M_{i,m}$ with $\psi_{i,m}(P_{i,m}) = 1$ which is radial on $B_{\mathbb{M}_{i,m},P_{i,m}}$ and which satisfies $\psi_{i,m}(Q_{i,m}) = 1$ if $r_{i,m}(Q_{i,m}) \geq \varepsilon_{i,m}$ for some $0 < \varepsilon_{i,m} < \frac{1}{2}\iota_{i,m}$; if $\iota_{i,m} = \infty$ this means $\varepsilon_{i,m}$ is finite. Finally, let $\vec{\mathfrak{H}} := \{\mathfrak{H}_0, \mathfrak{H}_1, \dots\}$ be a sequence of real numbers with $\mathfrak{H}_0 = 1$ and $\mathfrak{H}_\nu = 0$ for ν odd.

The asymptotic coefficients $\mathcal{H}_\nu$ have been used to obtain constraints on the possible geometries of harmonic spaces. Examining $\mathcal{H}_2(\mathbb{M}, P)$ implies, for example, that $\mathbb{M}$ is Einstein at P. However, the following result, which we will establish in Section 3, shows that they do not determine the local geometry of a central harmonic manifold.

Theorem 1.4. *Adopt the notation of Definition 1.3. Let $m \geq 4$ be even. There exist radial functions $\psi_{i,m,\vec{\mathfrak{H}}}$ on $\mathbb{M}_{i,m}$ so $\mathbb{M}_{i,m,\vec{\mathfrak{H}}} := (M_{i,m}, \psi_{i,m,\vec{\mathfrak{H}}}^2 g_{i,m})$ satisfies*:

(1) *If $i \leq 3$, $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ is compact.*
(2) *If $4 \leq i$, $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ is diffeomorphic to $\mathbb{R}^m$ and geodesically complete.*
(3) *$\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ is central harmonic at $P_{i,m}$.*
(4) *$\mathcal{H}_\nu(\mathbb{M}_{i,m,\vec{\mathfrak{H}}}, P_{i,m}) = \mathfrak{H}_\nu$ for all ν.*
(5) *If $i \neq j$, then $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ is nowhere Weyl curvature operator isomorphic to $\mathbb{M}_{j,m,\vec{\mathfrak{H}}}$; the local geometries of $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ and $\mathbb{M}_{j,m,\vec{\mathfrak{H}}}$ are different everywhere.*

1.4. Specifying the volume density asymptotics in odd dimensions

Any odd dimensional rank 1 symmetric space is conformally flat and thus the rank 1 symmetric spaces can not be used to extend Theorem 1.4 to odd dimensions. Damek–Ricci spaces are non-symmetric Hadamard manifolds which are harmonic, but they do not exist in all dimensions and thus are not adapted to our purposes. Instead, we use Theorem 1.2 to construct odd dimensional examples as follows.

Definition 1.5. If $m \geq 5$ is *odd*, let $\mathbb{M}_{i,m-1} = (M_{i,m-1}, g_{i,m-1})$ be the *even* dimensional Riemannian symmetric space which was specified in Definition 1.3. Let $B_{i,m-1}$ be a small open geodesic ball about the basepoint of $M_{i,m-1}$. By Theorem 1.2, we can choose a radial warping function $\phi_{i,m-1}$ on $B_{i,m-1}$ so that the Riemannian manifold $\mathbb{N}_{i,m-1} := (B_{i,m-1}, \phi_{i,m-1}^2 g_{i,m-1})$ satisfies $\tilde{\Theta}_{\mathbb{N}_{i,m-1},0} \equiv 1$. Let dt^2 be the Euclidean metric on $\mathbb{R}$. Give $B_{i,m-1} \times \mathbb{R}$ the product metric $g_{i,m} := \phi_{i,m-1}^2 g_{i,m-1} \oplus dt^2$ and let $B_{i,m}$ be a small geodesic ball in

the resulting Riemannian manifold. Set $\mathbb{M}_{i,m} := (B_{i,m}, g_{i,m})$. Since the Weyl conformal curvature operator is a conformal invariant, $\mathcal{W}_{\mathbb{N}_{i,m-1}} = \mathcal{W}_{\mathbb{M}_{i,m-1}}$. Since we are considering a product metric, we have

$$(1.1) \qquad \begin{aligned} \tilde{\Theta}_{\mathbb{M}_{i,m}} &= \tilde{\Theta}_{\mathbb{N}_{i,m-1}} \cdot \tilde{\Theta}_{\mathbb{R}^1} \equiv 1, \quad \text{and} \\ \mathcal{W}_{\mathbb{M}_{i,m}} &= \mathcal{W}_{\mathbb{N}_{i,m-1}} \oplus \mathcal{W}_{\mathbb{R}^1} = \mathcal{W}_{\mathbb{M}_{i,m-1}} \oplus 0. \end{aligned}$$

Theorem 1.6. *Adopt the notation of Definitions 1.3 and 1.5. Let $m \geq 5$ be odd. Then there exist radial functions $\psi_{i,m,\vec{\mathfrak{H}}}$ on $\mathbb{M}_{i,m}$ so*

$$\mathbb{M}_{i,m,\vec{\mathfrak{H}}} := (\mathbb{M}_{i,m}, \psi^2_{i,m,\vec{\mathfrak{H}}} g_{i,m})$$

satisfies:

(1) $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ *is geodesically complete.*
(2) $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ *is central harmonic at 0 and $\mathcal{H}_\nu(\mathbb{M}_{i,m,\vec{\mathfrak{H}}}, 0) = \mathfrak{H}_\nu$ for all ν.*
(3) *If $i \neq j$, then $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ is nowhere Weyl curvature operator isomorphic to $\mathbb{M}_{j,m,\vec{\mathfrak{H}}}$; the local geometries of $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ and $\mathbb{M}_{j,m,\vec{\mathfrak{H}}}$ are different everywhere.*

1.5. A 5 dimensional example

By Lemma 2.1, any radial conformal deformation of a central harmonic space is again central harmonic. There are, however, central harmonic spaces which do not arise in this fashion. We will establish the following result in Section 2.4.

Lemma 1.7. $\mathbb{M}_{1,5}$ *a central harmonic space which is nowhere Weyl curvature isomorphic to a conformal deformation of a harmonic space.*

1.6. The rank 1 symmetric spaces with positive curvature

Let $\mathbb{S}^m$ be the unit sphere in $\mathbb{R}^{m+1}$, let $\mathbb{CP}^k$ be the complex projective space, let $\mathbb{HP}^k$ be the quaternionic projective space, and let $\mathbb{OP}^2$ be the Cayley projective plane. We give these spaces the standard metrics normalized so:

(1.2)

$\mathbb{M}$	dimension	diameter	$\Theta_{\mathbb{M},P}$
$\mathbb{S}^m$	m	π	$\sin(r)^{m-1}$
$\mathbb{CP}^k$	$2k$	$\frac{1}{2}\pi$	$\sin(r)^{2k-1}\cos(r)$
$\mathbb{HP}^k$	$4k$	$\frac{1}{2}\pi$	$\sin(r)^{4k-1}\cos^3(r)$
$\mathbb{OP}^2$	16	$\frac{1}{2}\pi$	$\sin(r)^{15}\cos^7(r)$

The metric on $\mathbb{S}^m$ is the standard metric inherited from Euclidean space, the metric on $\mathbb{CP}^k$ is the suitably normalized Fubini-Study metric, and so forth. The rank 1 symmetric spaces in positive curvature are compact 2 point homogeneous spaces with $B_{\mathbb{M},P} = M - \mathcal{C}_{\mathbb{M},P}$ where $\mathcal{C}_{\mathbb{M},P}$ is the cut-locus:

$$\mathcal{C}_{\mathbb{S}^m,P} = \{-P\}, \quad \mathcal{C}_{\mathbb{CP}^k,P} = \mathbb{CP}^{k-1}, \quad \mathcal{C}_{\mathbb{HP}^k,P} = \mathbb{HP}^{k-1}, \quad \mathcal{C}_{\mathbb{OP}^2,P} = \mathbb{OP}^1 = S^8.$$

1.7. The rank 1 symmetric spaces of negative curvature

There are negative curvature duals of the spaces discussed in Section 1.6 that we shall denote by $\widetilde{\mathbb{S}}^m$ (hyperbolic space), $\widetilde{\mathbb{CP}}^k$ (complex hyperbolic space), $\widetilde{\mathbb{HP}}^k$ (quaternionic hyperbolic space), and $\widetilde{\mathbb{OP}}^2$ (Cayley hyperbolic plane). These are the rank 1 symmetric spaces of negative curvature; they are all 2-point homogeneous spaces and are geodesically complete. The curvature tensor of these spaces is obtained by reversing the sign of the curvature tensor of the corresponding positive curvature example. We note that any simply-connected 2-point homogeneous space is either flat or a rank 1 symmetric space.

If $\mathbb{M}$ is a rank 1 symmetric space with negative curvature, then the exponential map is a global diffeomorphism so the underlying topology of all these spaces is Euclidean space; the cut locus is empty. We adopt the same normalizations as those used to normalize the positive curvature examples. We replace sin by sinh and cos by cosh in Equation (1.2) to obtain:

$\mathbb{M}$	dimension	$\Theta_{\mathbb{M},P}$
$\widetilde{\mathbb{S}}^m$	m	$\sinh(r)^{m-1}$
$\widetilde{\mathbb{CP}}^k$	$2k$	$\sinh(r)^{2k-1}\cosh(r)$
$\widetilde{\mathbb{HP}}^k$	$4k$	$\sinh(r)^{4k-1}\cosh^3(r)$
$\widetilde{\mathbb{OP}}^2$	16	$\sinh(r)^{15}\cosh^7(r)$

1.8. Outline of the paper

In Section 2, we construct radial conformal deformations of any central harmonic space realizing any sequence of asymptotic coefficients $\vec{\mathfrak{H}}$ with $\mathfrak{H}_0 = 1$ and $\mathfrak{H}_\nu = 0$ if ν is odd. We also show that a radial conformal deformation of a central harmonic space is again central harmonic. In Section 3, we use the Weyl conformal curvature to construct conformal invariants of the curvature tensor to distinguish the spaces $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ of Theorems 1.4 and 1.6.

2. Prescribing the volume density function: The proof of Theorem 1.2

Let P be a point of a central harmonic Riemannian manifold $\mathbb{M}$. In Section 2.1, we show that a radial conformal deformation of $\mathbb{M}$ is again central harmonic and we determine the resulting volume density function. In Section 2.2, we solve the ODE relating the volume density function of a radial conformal deformation to the original volume density function; we use this solution in Section 2.3 to complete the proof of Theorem 1.2. In Section 2.4, we establish Lemma 1.7 and determine the warping function $\phi_{1,4}$ on $\mathbb{CP}^2$ to ensure $\tilde{\Theta} \equiv 1$.

2.1. Radial conformal deformations

Let $\eta(r)$ be a smooth odd function of a single variable with $\dot{\eta}(0) = 1$ and $\dot{\eta} > 0$. Set

$$\eta_{\mathbb{M}} := \eta \circ r_{\mathbb{M}}, \quad \psi_{\mathbb{M}} := \dot{\eta} \circ r_{\mathbb{M}}, \quad g_\eta := \psi_{\mathbb{M}}^2 g, \quad \text{and} \quad \mathbb{M}_\eta := (B_{\mathbb{M},P}, g_\eta).$$

We restrict to $B_{\mathbb{M},P}$ to ensure $r_{\mathbb{M}}^2$ is smooth. Consequently, since $\dot{\eta}$ is an even function of $r_{\mathbb{M}}$, $\psi_{\mathbb{M}}$ is a smooth radial function on $B_{\mathbb{M},P}$ and $\mathbb{M}_\eta$ is a smooth radial conformal deformation of $\mathbb{M}$. We use an argument introduced previously in Gilkey and Park [8] to establish the following result.

Lemma 2.1. *Assume that $\mathbb{M}$ is central harmonic at P.*

(1) $r_{\mathbb{M}_\eta} = \eta_{\mathbb{M}}$.
(2) $\tilde{\Theta}_{\mathbb{M}_\eta} = \eta_{\mathbb{M}}^{1-m} r_{\mathbb{M}}^{m-1} \psi_{\mathbb{M}}^{m-1} \tilde{\Theta}_{\mathbb{M}}$.
(3) $\mathbb{M}_\eta$ *is central harmonic at P.*

Proof. Introduce a system of local coordinates $\vec{\theta} = (\theta^1, \ldots, \theta^{m-1})$ on S^{m-1} and let $h_{ij}(r, \vec{\theta}) := g(\partial_{\theta^i}, \partial_{\theta^j})$. We have $d\eta_{\mathbb{M}} = d(\eta \circ r_{\mathbb{M}}) = \{\dot{\eta} \circ r_{\mathbb{M}}\} dr_{\mathbb{M}} = \psi_{\mathbb{M}} dr_{\mathbb{M}}$ so:

$$g = dr_{\mathbb{M}} \otimes dr_{\mathbb{M}} + h_{ij}(r, \vec{\theta}) d\theta^i \otimes d\theta^j,$$

(2.1) $$g_\eta = \psi_{\mathbb{M}} dr_{\mathbb{M}} \otimes \psi_{\mathbb{M}} dr_{\mathbb{M}} + (\psi_{\mathbb{M}})^2 h_{ij}(r, \vec{\theta}) d\theta^i \otimes d\theta^j$$

$$= d\eta_{\mathbb{M}} \otimes d\eta_{\mathbb{M}} + (\psi_{\mathbb{M}})^2 h_{ij}(r, \vec{\theta}) d\theta^i \otimes d\theta^j.$$

Since $\dot{\eta} > 0$, the map $Q \to (\eta_{\mathbb{M}}(Q), \vec{\theta}(Q))$ introduces new coordinates which, by Equation (2.1), are geodesic polar coordinates centered at P for the metric g_η. We have reparametrized the radial parameter to ensure it has unit length and left the angular parameter unchanged. Assertion (1) now follows.

Let $\varepsilon(\vec{\theta})$ be defined by the identity $\varepsilon(\vec{\theta}) \, \mathrm{dvol}_{\mathbb{S}^{m-1}}(\vec{\theta}) = d\theta^1 \cdots d\theta^{m-1}$; ε is independent of the radial parameter. We may express

(2.2) $$\mathrm{dvol}_{\mathbb{M}} = \det(h_{ij})^{1/2} dr_{\mathbb{M}} d\theta^1 \cdots d\theta^{m-1} = \det(h_{ij})^{1/2} \varepsilon(\vec{\theta}) dr_{\mathbb{M}} \, \mathrm{dvol}_{\mathbb{S}^{m-1}},$$
$$\Theta_{\mathbb{M}} = \det(h_{ij})^{1/2} \varepsilon(\vec{\theta}), \quad \text{and} \quad \tilde{\Theta}_{\mathbb{M}} = r_{\mathbb{M}}^{1-m} \det(h_{ij})^{1/2} \varepsilon(\vec{\theta}).$$

The angular variable $\vec{\theta}$ is the same for both systems of geodesic polar coordinates. We use Equation (2.1) and Equation (2.2) to complete the proof by showing:

$$\tilde{\Theta}_{\mathbb{M}_\eta} = \eta_{\mathbb{M}}^{1-m} \psi_{\mathbb{M}}^{m-1} \det(h_{ij}(r_{\mathbb{M}}, \vec{\theta}))^{1/2} \varepsilon(\vec{\theta}) = \eta_{\mathbb{M}}^{1-m} r_{\mathbb{M}}^{m-1} \psi_{\mathbb{M}}^{m-1} \tilde{\Theta}_{\mathbb{M}}. \qquad \square$$

We have chosen to start with η, which is the new radial distance function. However, if we start with a deformation Ψ which is radial on $B_{\mathbb{M},P}$, then we have $\Psi(\vec{x}) = \psi(\|\vec{x}\|)$ for some smooth even function ψ of 1-variable if $r_{\mathbb{M}} < \iota_{\mathbb{M}}$. We set

$$\eta_\psi(r) := \int_{t=0}^r \psi(t) dt.$$

We then have $\Psi = \psi_{\mathbb{M}}$ on $B_{\mathbb{M},P}$ so the two formalisms are equivalent. The following observation, which proves Assertion (3) of Theorem 1.4, is now immediate; we also refer to the discussion in [9] for a different treatment.

Corollary 2.2. *Let $\mathbb{M}$ be a Riemannian manifold which is central harmonic at P. Let Ψ be a smooth positive function on $\mathbb{M}$ which is radial on $B_{\mathbb{M},P}$. Then the conformal deformation $(M, \Psi^2 g)$ is central harmonic at P.*

2.2. Solving an ODE

The proof of the following result was shown to us by J. Álvarez-López [1].

Lemma 2.3. *Let $f_i(r)$ be positive smooth even functions of one variable which are defined for $0 \leq r \leq \varepsilon$ and which satisfy $f_i(0) = 1$. Then there exist $0 < \delta \leq \varepsilon$ and a smooth odd function η which is defined for $0 \leq r \leq \delta$ so that $\dot{\eta}(0) = 1$ and so that*

$$(2.3) \qquad f_1(\eta(r)) = \eta(r)^{1-m} r^{m-1} \dot{\eta}(r)^{m-1} f_2(r) \quad for \quad 0 \leq r \leq \delta.$$

Proof. Set $\phi_i := f_i^{\frac{1}{1-m}}$. Then Equation (2.3) is equivalent to

$$(2.4) \qquad \frac{1}{\phi_1(\eta(r))} = \frac{\dot{\eta}(r) r}{\eta(r) \phi_2(r)}, \quad \text{i.e.,} \quad \frac{\phi_2(r)}{r} = \frac{\dot{\eta}(r) \phi_1(\eta(r))}{\eta(r)}.$$

Multiplying Equation (2.4) by dr and noting $\dot{\eta} dr = d\eta$ yields the equivalent relation

$$(2.5) \qquad \frac{\phi_2(r) dr}{r} = \frac{\phi_1(\eta) d\eta}{\eta}, \quad \text{i.e.,} \quad \int \frac{\phi_2(r) dr}{r} = \int \frac{\phi_1(\eta) d\eta}{\eta} + C.$$

Because ϕ_i are even functions with $\phi_i(0) = 1$, we may express $\phi_i(r) = 1 + r^2 \Phi_i(r)$ to rewrite Equation (2.5) in the form

$$(2.6) \qquad \begin{aligned} &\int \frac{(1 + r^2 \Phi_2(r)) dr}{r} = \int \frac{(1 + \eta^2 \Phi_1(\eta)) d\eta}{\eta} + C, \quad \text{i.e.,} \\ &\ln|r| + \int r \Phi_2(r) dr = \ln(|\eta|) + \int \eta \Phi_1(\eta) d\eta + C. \end{aligned}$$

We set

$$\alpha_i(r) := \int_{t=0}^{r} t \Phi_i(t) dt \quad \text{and} \quad \eta(r) = r \beta(r).$$

We then have that α_i is a smooth even function with $\alpha_i(0) = 0$. We may rewrite Equation (2.6) in the form

$$(2.7) \qquad \ln(|r|) + \alpha_2(r) = \ln(|r|) + \ln(|\beta(r)|) + \alpha_1(r\beta(r)).$$

Equation (2.7) is then equivalent to the relation $G(r, \beta(r)) = 0$, where

$$(2.8) \qquad G(r, \beta) := \alpha_2(r) - \ln(|\beta|) - \alpha_1(r\beta).$$

Equation (2.8) is solved when $r = 0$ and $\beta = 1$. We compute

$$\partial_\beta G(r, \beta) \Big|_{r=0, \beta=1} = \left\{ -\frac{1}{\beta} - r \dot{\alpha}_1(r\beta) \right\} \Big|_{r=0, \beta=1} \neq 0.$$

Thus we may use the implicit function theorem to solve Equation (2.8) near the point $(r = 0, \beta = 1)$; the solution is unique and a smooth function of r; if the data is real analytic, then β is real analytic. Since the functions α_i are even functions of r, it follows β is an even function of r and hence η is an odd function of r with $\dot{\eta}(0) = 1$. $\square$

2.3. The proof of Theorem 1.2

Assertions (1) and (3) of Theorem 1.2 follow immediately from Lemma 2.3. If $\Xi = 1$, it is not necessary to localize. If we set $f_1 \equiv 1$ in Lemma 2.3, then $\Phi_1 \equiv 0$, $\alpha_1 \equiv 0$, and Equation (2.7) simplifies to become $\beta(r) = e^{\alpha_2(r)}$; since $\alpha_2(0) = 0$, $\beta(0) = 1$. Thus we can find ψ which is defined on all of $B_{\mathbb{M},P}$ so that $\tilde{\Theta}_{\mathbb{M}_\psi,0} \equiv 1$; it is not necessary to invoke the implicit function theorem and work locally. $\square$

2.4. The proof of Lemma 1.7

We have by definition that $\mathbb{M}_{1,5}$ is a small geodesic ball in $\mathbb{N}_{1,4} \times \mathbb{R}$. Equation (1.1) shows $\mathbb{M}_{1,5}$ is central harmonic about the origin and that $\mathcal{W}_{\mathbb{M}_{1,5}}$ is nowhere vanishing. Nikolayevsky [12] has shown that every harmonic space of dimension 5 is a space form and hence conformally flat. Thus $\mathbb{M}_{1,5}$ is nowhere Weyl curvature isomorphic to a radial conformal deformation of a harmonic space; consequently not all central harmonic spaces arise as radial conformal deformations of harmonic spaces. $\square$

$\mathbb{N}_{1,4}$ is defined by a conformal radial deformation $\phi_{1,4}$ of the Fubini-Study metric $B_{\mathbb{CP}^2,P}$ which is described as follows.

Lemma 2.4. *Let* $\mathbb{M} = \mathbb{CP}^2$ *and let*

$$\phi_{1,4}(r) := 3^{\frac{3}{4}} e^{\frac{\pi}{2\sqrt{3}}} \sin(r) \exp\left(-\frac{1}{2}\sqrt{3}\tan^{-1}\left(\frac{2\cos^{\frac{2}{3}}(r) + 1}{\sqrt{3}}\right)\right)$$

$$\cdot \left(\sqrt{1 - \cos^{\frac{2}{3}}(r)} \sqrt[3]{\cos(r)} \left(\cos^{\frac{2}{3}}(r) + \cos^{\frac{4}{3}}(r) + 1\right)^{5/4}\right)^{-1}$$

for $r < \frac{\pi}{2}$. *Then* $\phi_{1,4}(0) = 1$ *and* $\tilde{\Theta}_{\mathbb{M}_{\phi_{1,4}}} = 1$ *on* $B_{\mathbb{CP}^2,P}$.

Proof. By Equation (1.2), $r^3 \tilde{\Theta}_{\mathbb{CP}^2} = \Theta_{\mathbb{CP}^2} = \sin^3(r)\cos(r)$. Consequently, Equation (2.3) becomes $1 = \dot{\eta}^3 \eta^{-3} \sin^3(r)\cos(r)$. Mathematica solves this equation to yield

$$\eta(r) = c_1 \exp\left(\frac{1}{2}\log\left(1 - \cos^{\frac{2}{3}}(r)\right)\right)$$

$$\cdot \exp\left(-\frac{1}{4}\log\left(\cos^{\frac{2}{3}}(r) + \cos^{\frac{4}{3}}(r) + 1\right)\right)$$

$$\cdot \exp\left(-\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2\cos^{\frac{2}{3}}(r) + 1}{\sqrt{3}}\right)\right)$$

and consequently

$$\phi_{1,4}(r) = c_1 \sin(r) \exp\left(-\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2\cos^{\frac{2}{3}}(r)+1}{\sqrt{3}}\right)\right)$$
$$\cdot\left(\sqrt{1-\cos^{\frac{2}{3}}(r)}\sqrt[3]{\cos(r)}\left(\cos^{\frac{2}{3}}(r)+\cos^{\frac{4}{3}}(r)+1\right)^{5/4}\right)^{-1}.$$

This is defined for $0 < r < \frac{\pi}{2}$; there is an apparent singularity at $r = 0$ which we ignore for the moment. We set $c_1 = 3^{3/4}e^{\frac{\pi}{2\sqrt{3}}}$ and expand $\phi_{1,4}(r)$ for $r > 0$:

$$\phi_{1,4}(r) = 1 + \frac{1}{2}r^2 + \frac{13}{72}r^4 + \frac{1177}{19440}r^6 + \frac{7369}{362880}r^8 + \frac{681907}{97977600}r^{10} + O(r^{12}).$$

We conclude that $\phi_{1,4}$ is regular at 0 with $\phi_{1,4}(0) = 1$ and thus this is the radial conformal deformation given by Theorem 1.2. $\qquad\square$

Remark 2.5. The injectivity radius of $\mathbb{CP}^2$ is $\frac{\pi}{2}$. Since $\lim_{r\to\frac{\pi}{2}}\psi(r) = \infty$, ψ does not extend to all of $\mathbb{CP}^2$. Since

$$\psi(r) \sim \frac{3^{3/4}e^{\frac{\pi}{4\sqrt{3}}}}{\{\frac{\pi}{2}-r\}^{\frac{1}{3}}} + O(1) \quad\text{as}\quad r \to \frac{\pi}{2},$$

ψ is integrable on $[0,\frac{\pi}{2}]$ so the deformed metric on $B_{\mathbb{CP}^2,P}$ is geodesically incomplete.

3. Prescribing the volume density asymptotics

In Section 3.1 we establish the first four Assertions of Theorem 1.4 and in Section 3.2, we establish the first two Assertions of Theorem 1.6. The heart of the matter, of course, is to distinguish the manifolds $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$. In Section 3.3, we review some facts concerning the Weyl conformal curvature operator. In Section 3.4, we complete the proof of Theorem 1.4 and in Section 3.5, we complete the proof of Theorem 1.6.

3.1. The proof of Assertions (1)–(4) of Theorem 1.4

Since the underlying manifold is unchanged, and since $\mathbb{CP}^k$, $\mathbb{HP}^k$, and $\mathbb{OP}^2$ are compact, it is immediate that $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ is compact for $1 \le i \le 3$. If $i \ge 4$, then $\mathbb{M}_{i,m}$ is a homogeneous space and hence geodesically complete. Since the warping function ψ is 1 outside a compact set, it is follows that $\mathbb{M}_{i,m,\vec{\mathfrak{H}}}$ is geodesically complete for $4 \le i \le 7$. This establishes Assertions (1) and (2) of Theorem 1.4.

Assertion (3) of Theorem 1.4 follows from Corollary 2.2; a radial conformal deformation of a central harmonic space is again central harmonic. Let $\vec{\mathfrak{H}} = \{\mathfrak{H}_0,\dots\}$, where $\mathfrak{H}_0 = 1$ and $\mathfrak{H}_\nu = 0$ if ν is odd. Any formal Taylor series can be realized. Thus, we can find the germ of a smooth positive even function Ξ of 1-real variable so that $\Xi(t) \sim \sum_{\nu=0}^{\infty}\mathfrak{H}_\nu t^\nu$. We apply Theorem 1.2 to find the germ of a radial function ψ so $\tilde{\Theta}_{\mathbb{M}_\psi}(r_{\mathbb{M}_\psi}) = \Xi(r_{\mathbb{M}_\psi})$ and thus $\tilde{\Theta}_{\mathbb{M}_\psi}$ has the right

asymptotic coefficients. By using a partition of unity, we may assume $\psi(r) = 1$ for $r \geq \varepsilon$. Assertion (4) of Theorem 1.4 then follows.

3.2. The proof of Assertions (1), (2) of Theorem 1.6

Let $m \geq 5$ be odd and let $\mathbb{M} = (M, g) := \mathbb{M}_{i,m}$. It is obvious from the definition that $\mathbb{M}$ is central harmonic at the center 0 of the small geodesic ball defining M. We argue as above to find the germ of a radial function ψ so $\tilde{\Theta}_{\mathbb{M}_\psi}$ has the right asymptotic coefficients. By using a partition of unity, we can suppose that ψ grows sufficiently rapidly at the boundary of M and hence $\mathbb{M}_\psi$ is geodesically complete.

3.3. The Weyl tensor

Let ρ be the Ricci tensor and let τ be the scalar curvature. Let

$$W(x, y, z, w) := R(x, y, z, w) + \tau \frac{g(x, w)g(y, z) - g(x, z)g(y, w)}{(m - 1)(m - 2)}$$
$$+ \frac{\rho(x, z)g(y, w) + \rho(y, w)g(x, z) - \rho(y, z)g(x, w) - \rho(x, w)g(y, z)}{m - 2}$$

be the *Weyl conformal curvature tensor*. The *Weyl conformal curvature operator* $\mathcal{W}$ is the skew-symmetric operator which is characterized by the relation

$$g(\mathcal{W}(x, y)z, w) = W(x, y, z, w).$$

The *Weyl Jacobi operator* $\mathcal{J}^W$ is the self-adjoint operator defined by

$$\mathcal{J}^W(x)y := \mathcal{W}(y, x)x.$$

We say that $\mathbb{M}$ is *conformally flat* if $\mathbb{M}$ is isometric to $\mathbb{R}_\psi^m$ for some ψ. The following result is well known.

Lemma 3.1. *Let $\mathbb{M}$ be a Riemannian manifold of dimension $m \geq 4$. Then*

$$W_{\mathbb{M}_\psi} = \psi^2 W_{\mathbb{M}}, \quad \mathcal{W}_{\mathbb{M}_\psi} = \mathcal{W}_{\mathbb{M}}, \quad and \quad \mathcal{J}^W_{\mathbb{M}_\psi} = \mathcal{J}^W_{\mathbb{M}}.$$

Furthermore, $\mathbb{M}$ is conformally flat if and only if $\mathcal{W}_{\mathbb{M}}$ vanishes identically.

3.4. The proof of Theorem 1.4(5)

We examine the eigenvalue structure of the Jacobi operator $\mathcal{J}$ and the conformal Jacobi operator $\mathcal{J}^W$ of the spaces $\mathbb{M}_{i,m}$ for m even.

Lemma 3.2. *Let $\mathbb{M} = (M, g)$ be a rank 1 symmetric space with positive curvature and let $x \in T_P(M)$ and $y \in T_P(M)$ be unit tangent vectors. Let $k \geq 2$.*

(1) *$\mathcal{J}(x)$ is a self-adjoint operator with eigenvalues $\{0, 1, 4\}$ and corresponding eigenspace decomposition of $T_P(M) = E_0(x) \oplus E_1(x) \oplus E_4(x)$.*

Let y be a unit tangent vector. We have:

$\mathbb{M}$	$\dim\{E_0(x)\}$	$\dim\{E_1(x)\}$	$\dim\{E_4(x)\}$	$\rho(y,y)$
$\mathbb{R}^m$	m	0	0	0
$\mathbb{CP}^k$	1	$2k-2$	1	$2k+2$
$\mathbb{HP}^k$	1	$4k-4$	3	$4k+8$
$\mathbb{OP}^2$	1	8	7	36

(2) *The decomposition of Assertion* (1) *gives the eigenspace decomposition of $\mathcal{J}^W(x)$. The corresponding eigenvalues λ_i are given by:*

$\mathbb{M}$	λ_0	λ_1	λ_4
$\mathbb{R}^m$	0	0	0
$\mathbb{CP}^k$	0	$1 - \frac{2k+2}{2k-1}$	$4 - \frac{2k+2}{2k-1}$
$\mathbb{HP}^k$	0	$1 - \frac{4k+8}{4k-1}$	$4 - \frac{4k+8}{4k-1}$
$\mathbb{OP}^2$	0	$1 - \frac{36}{15}$	$4 - \frac{36}{15}$

Proof. The eigenvalue and eigenspace structure of the Jacobi operator for $\mathbb{S}^m$, $\mathbb{CP}^k$, and $\mathbb{HP}^k$ is well known. The curvature tensor of $\mathbb{OP}^2$ was computed by Brown and Gray [4] (see Theorem 6.1); the corresponding eigenvalue decomposition of the Jacobi operator computed by Nikolayevsky [11] (see the discussion on p. 510). The rank 1 symmetric spaces are Einstein. For such spaces, the Weyl conformal Jacobi operator is defined by subtracting a suitable multiple κ of the Jacobi operator for the sphere where we set $\kappa := \frac{\rho_{\mathbb{M}}(x,x)}{m-1}$ to ensure $\mathrm{Tr}\{\mathcal{J}_{\mathbb{M}}^W\}(x) = 0$ for all unit vectors x. Assertion (2) now follows. $\qquad\square$

We reverse the sign of the curvature tensor to compute for the negative curvature duals. The following result is now immediate from Lemma 3.1 and from Lemma 3.2.

Lemma 3.3. *Let $m \geq 4$ be even, let $\mathbb{M} = \mathbb{M}_{i,m,\vec{\mathfrak{H}}}$, let Q be a point of M, and let $0 \neq x \in T_Q(M_{i,m})$.*

(1) *If $i < 7$, then 0 is an eigenvalue of multiplicity 1 of $\mathcal{J}_{\mathbb{M}}^W(x)$. If $i = 7$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ vanishes identically.*

(2) *If $i = 1$, so $M_{i,m} = \mathbb{CP}^k$ for $m = 2k$ and $k \geq 2$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a negative eigenvalue of multiplicity $m - 2$ and a positive eigenvalue of multiplicity 1.*

(3) *If $i = 2$, so $M_{i,m} = \mathbb{HP}^k$ for $m = 4k$ and $k \geq 2$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a negative eigenvalue of multiplicity $m - 4$ and a positive eigenvalue of multiplicity 3.*

(4) *If $i = 3$, so $M_{i,m} = \mathbb{OP}^2$ for $m = 16$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a negative eigenvalue of multiplicity 8 and a positive eigenvalue of multiplicity 7.*

(5) *If $i = 4$, so $M_{i,m} = \widetilde{\mathbb{CP}}^k$ for $m = 2k$ and $k \geq 2$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a positive eigenvalue of multiplicity $m - 2$ and a negative eigenvalue of multiplicity 1.*

(6) *If $i = 5$, so $M_{i,m} = \widetilde{\mathbb{HP}}^k$ for $m = 4k$ and $k \geq 2$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a positive eigenvalue of multiplicity $m - 4$ and a negative eigenvalue of multiplicity 3.*

(7) *If $i = 6$, so $M_{i,m} = \widetilde{\mathbb{OP}}^2$ for $m = 16$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a positive eigenvalue of multiplicity 8 and a negative eigenvalue of multiplicity 7.*

Assertion (5) of Theorem 1.4 now follows from Lemma 3.3; this completes the proof of Theorem 1.4 □

3.5. The proof of Theorem 1.6(3)

Let $m \geq 5$ be odd. Let $y = (x, t)$ be a tangent vector of $M_{i,m}$ where y is a tangent vector to $M_{i,m-1}$ and t is a tangent vector to $\mathbb{R}$. Since $\mathcal{J}$ is conformal, we may use Equation (1.1) to see:

$$(3.1) \qquad \mathcal{J}_{\mathbb{M}_{i,m,\vec{5}}}^W(x) = \mathcal{J}_{\mathbb{M}_{i,m}}^W(x) = \mathcal{J}_{\mathbb{M}_{i,m-1}}^W(y).$$

The following result now follows from Equation (3.1) and from Lemma 3.3.

Lemma 3.4. *Let $m \geq 5$ be odd, let $\mathbb{M} = \mathbb{M}_{i,m,\vec{5}}$, let Q be a point of $M_{i,m}$. Choose $x \in T_P(M_{i,m})$ so $\mathcal{J}_{\mathbb{M}}^W(x)$ has maximal rank.*

(1) *If $i < 7$, then 0 is an eigenvalue of multiplicity 2 of $\mathcal{J}_{\mathbb{M}}^W(x)$. If $i = 7$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ vanishes identically.*

(2) *If $i = 1$ so $M_{i,m-1} = \mathbb{CP}^k$ for $m - 1 = 2k$ and $k \geq 2$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a negative eigenvalue of multiplicity $m - 3$ and a positive eigenvalue of multiplicity 1.*

(3) *If $i = 2$ so $M_{i,m-1} = \mathbb{HP}^k$ for $m - 1 = 4k$ and $k \geq 2$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a negative eigenvalue of multiplicity $m - 5$ and a positive eigenvalue of multiplicity 3.*

(4) *If $i = 3$ so $M_{i,m-1} = \mathbb{OP}^2$ for $m - 1 = 16$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a negative eigenvalue of multiplicity 8 and a positive eigenvalue of multiplicity 7.*

(5) *If $i = 4$ so $M_{i,m-1} = \widetilde{\mathbb{CP}}^k$ for $m - 1 = 2k$ and $k \geq 2$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a positive eigenvalue of multiplicity $m - 3$ and a negative eigenvalue of multiplicity 1.*

(6) *If $i = 5$ so $M_{i,m-1} = \widetilde{\mathbb{HP}}^k$ for $m - 1 = 4k$ and $k \geq 2$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a positive eigenvalue of multiplicity $m - 5$ and a negative eigenvalue of multiplicity 3.*

(7) *If $i = 6$ so $M_{i,m-1} = \widetilde{\mathbb{OP}}^2$ for $m - 1 = 16$, then $\mathcal{J}_{\mathbb{M}}^W(x)$ has a positive eigenvalue of multiplicity 8 and a negative eigenvalue of multiplicity 7.*

Assertion (3) of Theorem 1.6 now follows; this completes the proof of Theorem 1.6 and thereby of all of the results of this paper.

funded by the Korea government (MSIT) (NRF-2019R1A2C1083957). Helpful suggestions and comments were provided by our colleagues and friends J. Álvarez-López, M. van den Berg, and E. García-Río.

Dedication

This paper is being written during one of the worst outbreaks of war in Europe since World War II. We dedicate this paper to show our solidarity with the innocent victims in Ukraine.

References

[1] J. Álvarez-López, private communication.

[2] J. Berndt, F. Tricerri, and L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Mathematics, 1598, Springer, Berlin, 1995. `https://doi.org/10.1007/BFb0076902`

[3] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 93, Springer, Berlin, 1978.

[4] R. B. Brown and A. Gray, *Riemannian manifolds with holonomy group* Spin (9), in Differential geometry (in honor of Kentaro Yano), 41–59, Kinokuniya, Tokyo, 1972.

[5] E. T. Copson and H. S. Ruse, *Harmonic Riemannian spaces*, Proc. Roy. Soc. Edinburgh **60** (1940), 117–133.

[6] J. Díaz-Ramos, B. Fiedler, E. García-Río, and P. Gilkey, *The structure of algebraic covariant derivative curvature tensors*, Int. J. Geom. Methods Mod. Phys. **1** (2004), no. 6, 711–720. `https://doi.org/10.1142/S0219887804000356`

[7] P. Gilkey and J. H. Park, *Harmonic spaces and density functions*, Results Math. **75** (2020), no. 3, Paper No. 121, 18 pp. `https://doi.org/10.1007/s00025-020-01248-7`

[8] P. Gilkey and J. H. Park, *Harmonic radial vector fields on harmonic spaces*, J. Math. Anal. Appl. **504** (2021), no. 2, Paper No. 125405, 9 pp. `https://doi.org/10.1016/j.jmaa.2021.125405`

[9] P. Gilkey and J. H. Park, *Centrally harmonic manifolds*, Periodica Mathematica Hungarica **86** (2023), no. 1, 12p.

[10] A. Gray, *The volume of a small geodesic ball of a Riemannian manifold*, Michigan Math. J. **20** (1973), 329–344 (1974). `http://projecteuclid.org/euclid.mmj/1029001150`

[11] Y. A. Nikolayevsky, *Osserman conjecture in dimension* $n \neq 8, 16$, Math. Ann. **331** (2005), no. 3, 505–522. `https://doi.org/10.1007/s00208-004-0580-8`

[12] Y. A. Nikolayevsky, *Two theorems on harmonic manifolds*, Comment. Math. Helv. **80** (2005), no. 1, 29–50. `https://doi.org/10.4171/CMH/2`

[13] H. S. Ruse, *On the elementary solution of Laplace's equation*, Proc. Edinburgh Math. Soc. **2** (1931), 135–139.

[14] H. S. Ruse, *General solutions of Laplace's equation in a simply harmonic manifold*, Quart. J. Math. Oxford Ser. (2) **14** (1963), 181–192. `https://doi.org/10.1093/qmath/14.1.181`

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