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OPERATORS A, B FOR WHICH THE ALUTHGE TRANSFORM $\widetilde{AB}$ IS A GENERALISED n-PROJECTION

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Abstract. A Hilbert space operator $A \in \mathcal{B}(H)$ is a generalised n-projection, denoted $A \in (G-n-P)$, if $A^{*n} = A$. $(G-n-P)$-operators A are normal operators with finitely countable spectra $\sigma(A)$, subsets of the set $\{0\} \cup \{\sqrt[n+1]{1}\}$. The Aluthge transform $\tilde{A}$ of $A \in \mathcal{B}(H)$ may be $(G-n-P)$ without A being $(G-n-P)$. For doubly commuting operators $A, B \in \mathcal{B}(H)$ such that $\sigma(AB) = \sigma(A)\sigma(B)$ and $\|A\|\|B\| \leq \left\|\widetilde{AB}\right\|$, $\widetilde{AB} \in (G-n-P)$ if and only if $A = \left\|\tilde{A}\right\|(A_{00} \oplus (A_0 \oplus A_u))$ and $B = \left\|\tilde{B}\right\|(B_0 \oplus B_u)$, where A_{00} and B_0, and $A_0 \oplus A_u$ and B_u, doubly commute, $A_{00}B_0$ and A_0 are 2 nilpotent, A_u and B_u are unitaries, $A_u^{*n} = A_u$ and $B_u^{*n} = B_u$. Furthermore, a necessary and sufficient condition for the operators αA, βB, $\alpha\tilde{A}$ and $\beta\tilde{B}$, $\alpha = \frac{1}{\|\tilde{A}\|}$ and $\beta = \frac{1}{\|\tilde{B}\|}$, to be $(G-n-P)$ is that A and B are spectrally normaloid at 0.

1. Introduction

Let $\mathcal{B}(H)$ denote the algebra of operators, i.e., bounded linear transformations, on an infinite dimensional complex Hilbert space $\mathcal{H}$ into itself. An operator $A \in \mathcal{B}(H)$, with adjoint A^*, is *a generalised n-projection*, denoted $A \in (G-n-P)$, if $A^{*n} = A$. Ever since the introduction of the concept of a generalised 2-projection on a finite dimensional Hilbert space by Gross and Trenkler [7], generalised n-projections have been studied by a number of authors, amongst them Baksalary and Liu [1], Du and Li [5], Lebtahi and Thome [9], and Duggal and Kim [6]. It is immediate from the definition that $(G-n-P)$-operators A are normal with spectra $\sigma(A)$, subsets of the set $\{0\} \cup \{\sqrt[n+1]{1}\}$.

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Given a commuting pair of operators $A, B \in \mathcal{B}(H)$ such that $A, B \in (G-n-P)$, it is straightforward to see that $AB \in (G-n-P)$. The reverse implication: does A, B commute and $AB \in (G-n-P)$ imply A and B, or a multiple thereof, in $(G-n-P)$ was considered in [6], where it is shown that if $\|AB\| = \|A\|\|B\|$ and $\sigma(A)$, $\sigma(B)$ are finitely countable, then there exist direct sum decompositions $\frac{A}{\|A\|} = E_1 \oplus E_2$ and $\frac{B}{\|B\|} = F_1 \oplus F_2$ such that $E_iF_i = F_iE_i$ $(i = 1, 2)$, E_1 (or, F_1) is unitary and F_1 (respectively, E_1) is normal, E_2 (or, F_2) is quasinilpotent and $E_2F_2 = 0$. The Aluthge transform of an operator $A \in \mathcal{B}(H)$ with polar decomposition $A = UP$ is the operator $\tilde{A} = P^{\frac{1}{2}}UP^{\frac{1}{2}}$. Evidently, $A \in (G-n-P)$ implies $\tilde{A} \in (G-n-P)$. The converse fails. Thus, if we let $A = A_1 \oplus A_2 \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, $A_1^{*n} = A_1$ and $A_2^2 = 0$, then A is not normal, hence it can not be $(G-n-P)$ for any value of n. However, if A_2 has the polar decomposition $A_2 = U_2P_2$, then, since $A_2^2 = U_2P_2^{\frac{1}{2}}\left[P_2^{\frac{1}{2}}U_2P_2^{\frac{1}{2}}\right]P_2^{\frac{1}{2}} = 0$ if and only if $\tilde{A}_2 = 0$, $\tilde{A} = \tilde{A}_1 \oplus \tilde{A}_2 = \tilde{A}_1 \oplus 0 \in (G-n-P)$.

Given operators $A, B \in \mathcal{B}(H)$ such that the Aluthge transform $\widetilde{AB}$ of AB is $(G-n-P)$, we consider in the following the problem of determining the structure of the operators $A, B, \tilde{A}$ and $\tilde{B}$. For this, an important first step is the ensuring of a reasonable relationship between the polar forms of $\widetilde{AB}$ and (the Aluthge transforms) $\tilde{A}$, $\tilde{B}$ of A , B, respectively. In general, there is little relationship between the product of the Aluthge transforms of A and B and the Aluthge transform of the product AB. For example, if $A, B \in \mathcal{B}(H)$ are defined by $Ax = \left(0, \frac{1}{2}x_1, 2x_2, \frac{1}{2}x_3, 2x_4, \dots\right)$ and $Bx = (0, a_1x_1, a_2x_2, a_3x_3, \dots)$, where $x = (x_1, x_2, x_3, \dots) \in \mathcal{H}$ and $a_j = e^{i\theta_j}|a_j|$, then $\tilde{A}x = (0, x_1, x_2, x_3, \dots)$, $\tilde{B}x = \left(0, e^{i\theta_1}|a_1a_2|x_1, e^{i\theta_2}|a_2a_3|x_2, \dots\right)$ and $\widetilde{AB} \neq \tilde{A}\tilde{B} \neq \tilde{B}\tilde{A}$. A simple commutativity hypothesis on A and B is not enough: what one needs here is the double commutativity hypothesis $AB - BA = AB^* - B^*A = 0$. Such a doubly commutative hypothesis ensures that if A, B have the polar forms $A = UP$ and $B = VQ$, then $\widetilde{AB} = \tilde{A}\tilde{B} = \tilde{B}\tilde{A}$. We prove that if A, B doubly commute, the spectrum of AB is the product of the spectra of A and B and $\|A\|\|B\| \leq \left\|\widetilde{AB}\right\|$, then $\widetilde{AB} \in (G-n-P)$ if and only if there exist decompositions $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{H}_0 \oplus (\mathcal{H}_{00} \oplus \mathcal{H}_u)$ such that $A = \left\|\tilde{A}\right\|(A_{00} \oplus (A_0 \oplus A_u)) \in B(\mathcal{H}_0 \oplus (\mathcal{H}_{00} \oplus \mathcal{H}_u))$ and $B = \left\|\tilde{B}\right\|(B_0 \oplus B_u) \in B(\mathcal{H}_0 \oplus \mathcal{H}_1)$, where $A_{00}B_0$ and A_0 are 2-nilpotents, A_u and B_u are unitaries, $A_u^{*n} = A_u$ and $B_u^{*n} = B_u$. (Here, either of the components A_0, B_0 and A_{00} may be missing, i.e., act on the 0 space.) It is seen that a necessary and sufficient condition for the operators αA, βB, $\alpha\tilde{A}$ and $\beta\tilde{B}$, $\alpha = \frac{1}{\|\tilde{A}\|}$ and $\beta = \frac{1}{\|\tilde{B}\|}$, to be $(G-n-P)$ is that A and B are spectrally normaloid at 0. Tensor products $A \otimes B$ such that $\widetilde{A \otimes B} \in (G-n-P)$ are considered.

In the following, we shall denote the commutator $AB - BA$ of A and B by $[A, B]$. The spectrum, the approximate point spectrum, the surjectivity spectrum, the spectral radius $\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ and the peripheral spectrum $\{\lambda \in \sigma(A) : |\lambda| = r(A)\}$ [8, p. 225] will be denoted by $\sigma(A)$, $\sigma_a(A)$, $\sigma_s(A)$, $r(A)$ and $\sigma_\pi(A)$, respectively. Recall that the isolated points of the spectrum of a normal operator are (poles of the resolvent of the operator, hence) reducing eigenvalues of the operator.

2. Preliminaries

We start by recalling some facts from [1, 6, 7, 9]. The hypothesis $A \in (G - n - P)$, i.e., $A^{*n} = A$, implies

$$A^*A = A^{*n+1} = A^{*n}A^* = AA^*,\ A^{*n+1} = (A^*A) = A^{n+1},$$

hence A is normal and A^{n+1} is self-adjoint. Consequently,

$$\sigma(A) = \sigma_a(A) = \sigma_s(A) \subseteq \{0\} \cup \left\{ \sqrt[n+1]{1} \right\},\ \|A\| = 1.$$

The spectrum of (the normal operator) A being a finite set consists of normal eigenvalues of A (i.e., the corresponding eigenspaces are reducing) and A has a direct sum representation of type

$$A = \bigoplus_{i=1}^{n+1} A\mid_{\mathcal{H}_i} \oplus A\mid_{\mathcal{H}_0} = \bigoplus_{i=1}^{n+1} \lambda_i I_i \oplus 0 = \mathcal{A}_1 \oplus 0,$$

where $\mathcal{H}_i = (A - \lambda_i I)^{-1}(0),\ \lambda_0 = 0,\ \lambda_i,\ 1 \le i \le n+1$, are the $(n+1)$th roots of unity, I_i is the unity of $\mathcal{B}(\mathcal{H}_i)$ and the operator $\mathcal{A}_1$ is unitary. (Here some of the components $A\mid_{\mathcal{H}_i}, i = 0, 1, \ldots, n+1$, may be missing.)

If we let (QP), (PL) and (N) denote, respectively, the classes of operators $A \in \mathcal{B}(H)$ such that

$$A \in (QP) \Longleftrightarrow A^{n+2} = A,$$
$$A \in (PL) \Longleftrightarrow A \text{ is a partial isometry (i.e., } \mathrm{AA^*A = A}) \text{ and}$$
$$A \in (N) \Longleftrightarrow [A, A^*] = 0, \text{ i.e., A is normal},$$

then operators $A \in (G - n - P)$ have the following structural properties.

Proposition 2.1 ([6]). *The following statements are mutually equivalent.*
(i) $A \in (G - n - P)$.
(ii) $A \in (QP) \wedge (PL) \wedge (N)$.
(iii) $A \in (QP) \wedge (N)$.
(iv) $A \in (QP) \wedge (PL)$.

The eigenvalues λ of a contraction operator $A \in \mathcal{B}(H)$ of length one (i.e., such that $|\lambda| = 1$) are normal eigenvalues of the operator: if $(A - \lambda I)x = 0$ for an $x \in \mathcal{H}$, then

$$\|(A - \lambda I)^*x\|^2 \le \|A^*x\|^2 - 2\|A^*x\|\|\overline{\lambda}x\| + \|\overline{\lambda}x\|^2 \le 0.$$

The ascent (resp., descent) of $A \in \mathcal{B}(H)$, asc(A) (resp., dsc(A)), is the least positive integer n such that $A^n(0) = A^{n+1}(0)$ (resp., $A^n(\mathcal{H}) = A^{n+1}(\mathcal{H})$); if no such integer n exists, then asc$(A) = \infty$ (resp., dsc$(A) = \infty$). An isolated pointed λ of the spectrum of A, $\lambda \in$ iso(A), is a pole (of the resolvent) of A of order m if asc$(A - \lambda I) =$ dsc$(A - \lambda I) = m < \infty$. The deficiency indices $\alpha(A - \lambda I)$ and $\beta(A - \lambda I)$ are the integers $\alpha(A - \lambda I) = \dim(A - \lambda I)^{-1}(0)$ and $\beta(A - \lambda I) = \dim(A^* - \overline{\lambda} I)^{-1}(0)$. The operator A is normaloid if $r(A) = \|A\|$. Recall from [8, Proposition 54.2] that if a non-trivial operator $A \in \mathcal{B}(H)$ is normaloid and $\lambda \in \sigma_\pi(A)$ (thus, $|\lambda| = \|A\|$), then asc$(A - \lambda I) \leq 1$ and $\beta(A - \lambda I) > 0$.

Given an operator $A \in \mathcal{B}(H)$ with polar decomposition $A = UP$, the Aluthge transform $\tilde{A} = P^{\frac{1}{2}} U P^{\frac{1}{2}}$ preserves, often improves upon, many spectral properties of the operator A. If the product $AB \in \mathcal{B}(H)$ of $A, B \in \mathcal{B}(H)$ has the polar form $AB = W|AB|$, then $\widetilde{AB} = |AB|^{\frac{1}{2}} W |AB|^{\frac{1}{2}}$. How is the Aluthge transform $\widetilde{AB}$ of the product AB related to the product of the Aluthge transforms of A and B? Ensuring a reasonable relationship requires the assumption of certain commutativity hypotheses on A and B. It is not enough to assume that $[A, B] = 0$, and a more reasonable hypothesis here is that of doubly commutative. $A, B \in \mathcal{B}(H)$ doubly commute if $[A, B] = [A, B^*] = 0$. If A, B doubly commute, and if B has the polar decomposition $B = VQ$, then a straightforward argument (depending almost entirely upon the facts that $\ker U = \ker P$, $\ker V = \ker Q$ and $\overline{P(\mathcal{H})} \oplus \ker P = \overline{Q(\mathcal{H})} \oplus \ker V = \mathcal{H}$) proves that

$$[P, B] = [P, B^*] = [U, B] = [U, B^*] = [Q, A] = [Q, A^*] = [V, A] = [V^*, A] = 0$$

and hence that

$$[P, Q] = [U, V] = [P, V] = [Q, U] = [U^*, V] = 0.$$

Thus if AB has the polar decomposition $AB = W|AB|$, see above, then

$$AB = W|AB| = W|A||B| = UV|A||B| = UVPQ$$

and

$$\begin{aligned} \widetilde{AB} &= |AB|^{\frac{1}{2}} W |AB|^{\frac{1}{2}} = |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} UV |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \\ &= |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} V |B|^{\frac{1}{2}} = \tilde{A}\tilde{B} = \tilde{B}\tilde{A}. \end{aligned}$$

(Indeed, $\tilde{A}$ and $\tilde{B}$ doubly commute.)

The operation of taking Aluthge transforms preserves the spectrum, the ascent and the descent of the operator [2,4]. Hence, an operator and its Aluthge transform have the same poles. Observe that for an operator $A \in \mathcal{B}(H)$ with polar decomposition $A = UP$, $A^n = UP^{\frac{1}{2}} \tilde{A}^{n-1} P^{\frac{1}{2}}$. Hence, A is an n- nilpotent, $n > 1$, if and only if $\tilde{A}$ is $(n-1)$-nilpotent.

3. Results

Recall from [6, Theorem 3.1] that if the operators $C, D \in \mathcal{B}(H)$, (as always, non-trivial) are such that $[C, D] = 0$, $\|CD\| = \|C\|\|D\|$, $\sigma(CD) = \sigma(C)\sigma(D)$ and $CD \in (G - n - P)$, then there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$, and decompositions $C = C_1 \oplus C_2$ and $D = D_1 \oplus D_2$ of C and D, such that $[C_1, D_1] = 0$, $\frac{1}{\|D\|}D_1$ (or, $\frac{1}{\|C\|}C_1$) is unitary, $\frac{1}{\|C\|}C_1$ (resp., $\frac{1}{\|D\|}D_1$) is normal, $[C_2, D_2] = 0$, D_2 (or, C_2) is quasinilpotent and $C_2D_2 = 0$. Here, if both the components C_2 and D_2 are absent (i.e., act on the 0 space), then $\frac{1}{\|C\|}C$ and $\frac{1}{\|D\|}D$ are unitaries; if, instead, one of the components C_2 and D_2 is missing then the other component is the 0 operator. Replacing operators C, D and CD by $\tilde{A}$, $\tilde{B}$ and $\tilde{A}\tilde{B}$, respectively, this gives us information about the structure of the operators $\tilde{A}$ and $\tilde{B}$, and hence possibly operators A and B. What if we replace C, D and CD by $\tilde{A}, \tilde{B}$ and $\widetilde{AB}$? The following theorem, our main result, considers this situation.

Theorem 3.1. *Given non-trivial doubly commuting operators $A, B \in \mathcal{B}(H)$ satisfying*

$$\sigma(AB) = \sigma(A)\sigma(B) \text{ and } \|A\|\|B\| \leq \left\|\widetilde{AB}\right\|,$$

$\widetilde{AB} \in (G - n - P)$ if and only if there exist decompositions $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{H}_0 \oplus (\mathcal{H}_{00} \oplus \mathcal{H}_u)$ such that $A = \left\|\tilde{A}\right\| (A_{00} \oplus (A_0 \oplus A_u)) \in B(\mathcal{H}_0 \oplus (\mathcal{H}_{00} \oplus \mathcal{H}_u))$ and $B = \left\|\tilde{B}\right\| (B_0 \oplus B_u) \in B(\mathcal{H}_0 \oplus \mathcal{H}_1)$, where $A_{00}B_0$ and A_0 are 2-nilpotents, A_u and B_u are unitaries, $A_u^{*n} = A_u$ and $B_u^{*n} = B_u$. Here, either of the components A_0, B_0 and A_{00} may be missing (i.e., act on the 0 space).

Proof. The proof of the theorem consists of two parts: in the first part we determine the structure of the Aluthge transforms $\tilde{A}$ and $\tilde{B}$, and in the second part we translate this into what it means for the operators A and B.

The doubly commutative hypothesis on A, B implies

$$\widetilde{AB} = \tilde{A}\tilde{B}, \; \left[\tilde{A}, \tilde{B}\right] = \left[\tilde{A}, \tilde{B}^*\right] = 0.$$

The hypothesis $\widetilde{AB} \in (G - n - P)$ implies $\widetilde{AB}$ is normal and $\sigma\left(\widetilde{AB}\right) \subseteq \{0\} \cup \left\{ \sqrt[n+1]{1} \right\}$ and (since Aluthge transforms preserve spectrum)

$$r\left(\widetilde{AB}\right) = \left\|\widetilde{AB}\right\| = \left\|\tilde{A}\tilde{B}\right\| = 1 = r(AB).$$

Since

$$\left\|\tilde{A}\right\| = \left\|P^{\frac{1}{2}}UP^{\frac{1}{2}}\right\| \leq \|PUP\|^{\frac{1}{2}} \leq \|A\|$$

and similarly $\left\|\tilde{B}\right\| \leq \|B\|$, the hypothesis $\|A\|\|B\| \leq \left\|\widetilde{AB}\right\|$ implies

$$1 = \left\|\widetilde{AB}\right\| = \left\|\tilde{A}\tilde{B}\right\| \leq \left\|\tilde{A}\right\| \left\|\tilde{B}\right\| \leq \|A\|\|B\| \leq \left\|\widetilde{AB}\right\|,$$

i.e.,

$$\left\|\widetilde{AB}\right\| = \left\|\tilde{A}\right\| \left\|\tilde{B}\right\| = \|A\| \|B\| = 1.$$

Define contractions $E, F \in \mathcal{B}(H)$ by

$$E = \alpha\tilde{A},\ F = \beta\tilde{B};\ \alpha = \frac{1}{\left\|\tilde{A}\right\|},\ \beta = \frac{1}{\left\|\tilde{B}\right\|},\ \alpha\beta = 1.$$

Then

$$[E, F] = 0,\ \|EF\| = 1 = \|E\|\|F\| \text{ and } \sigma(EF) = \sigma(E)\sigma(F).$$

The hypothesis $\widetilde{AB} \in (G-n-P)$ implies $EF \in (G-n-P)$, hence

$$\sigma(EF)) \subseteq \{0\} \cup \left\{\sqrt[n+1]{1}\right\},$$

and $\sigma(E)$, $\sigma(F)$ are subsets of the set $\{0\} \cup \left\{\sqrt[n+1]{1}\right\}$. We have the following four possibilities:

(a) $\sigma(E) = S_1 = \cup_{i=1}^{k}\{\lambda_i\} \subseteq \{\sqrt[n+1]{1}\}$ and $\sigma(F) = S_2 = \cup_{j=1}^{t}\{\mu_j\} \subseteq \{\sqrt[n+1]{1}\}$, $|\lambda_i| = |\mu_j| = 1$ for all $1 \leq i \leq k \leq n+1$ and $1 \leq j \leq t \leq n+1$;
(b) $\sigma(E) = \{0\} \cup S_1$ and $\sigma(F) = S_2$;
(c) $\sigma(E) = S_1$ and $\sigma(F) = \{0\} \cup S_2$;
(d) $\sigma(E) = \{0\} \cup S_1$ and $\sigma(F) = \{0\} \cup S_2$.

If (a) holds, then $\|E\| = r(E) = 1 = r(F) = \|F\|$, E and F are normaloid operators with spectrum consisting of the peripheral spectrum. Hence, see [8, Proposition 54.2],

$$\operatorname{asc}(E - \lambda_i I) \leq 1,\ \operatorname{asc}(F - \mu_j I) \leq 1,\ \beta(E - \lambda_i I) > 0 \text{ and } \beta(F - \mu_j I) > 0$$

for all $1 \leq i \leq k$ and $1 \leq j \leq t$. E^* and F^* being contractions, $\overline{\lambda_i}$ and $\overline{\mu_j}$ are eigenvalues of E^* and F^* respectively. The eigenvalues in the peripheral spectrum of a contraction being normal eigenvalues of the contraction, λ_i and μ_j are simple (i.e., mulptiplicity one) eigenvalues of E and F respectively. Furthermore,

$$E = \oplus_{i=1}^{k}\lambda_i I|_{\mathcal{H}_{\lambda_i}} = \oplus_{i=1}^{k} E_i \text{ and } F = \oplus_{j=1}^{t}\mu_j I|_{\mathcal{H}_{\mu_j}} = \oplus_{j=1}^{t} F_j,$$

where $\mathcal{H}_{\lambda_i} = (E - \lambda_i I)^{-1}(0)$ and $\mathcal{H}_{\mu_j} = (F - \mu_j I)^{-1}(0)$ for all $1 \leq i \leq k$ and $1 \leq j \leq t$. Thus E and F are unitaries such that $\tilde{A} = \alpha E$ and $\tilde{B} = \beta F$; scalars α and β defined as above.

If (b) holds, then an argument similar to the one above implies

$$E = E_0 \oplus \lambda_i I|_{\mathcal{H}_{\lambda_i}} = \oplus_{i=0}^{k} E_i \text{ and } F = \oplus_{j=1}^{t}\mu_j I|_{\mathcal{H}_{\mu_j}} = \oplus_{j=1}^{t} F_j,$$

where $\sigma(E_0) = \{0\}$ (thus, E_0 is a quasinilpotent operator in $B(\mathcal{H}_0) = B(\mathcal{H} \ominus \oplus_{i=1}^{k}\mathcal{H}_{\lambda_i}))$. The eigenvalues λ_i and μ_j are simple, normal eigenvalues. Let $F \in B\left(\mathcal{H}_0 \oplus_{i=1}^{k} \mathcal{H}_{\lambda_i}\right)$ have the matrix representation $F = [F_{ij}]_{i,j=0}^{k}$. The commutativity E and F then implies

$$E_i F_{ij} - F_{ij} E_j = (\lambda_i - \lambda_j) F_{ij} = 0,\ 0 \leq i, j \leq k.$$

Since $\lambda_i \neq \lambda_j$ for all $i \neq j$, $F_{ij} = 0$ for all $0 \leq i \neq j \leq k$ and

$$F = \oplus_{i=0}^{k} F_{ii},\ F_{ii} \text{ unitary for all } 0 \leq i \leq k.$$

The operator E_0 being quasinilpotent, E_0F_{00} is quasinilpotent; the normality of EF implies that $E_0F_{00} = 0$, and this in view of the fact that F_{00} is unitary implies $E_0 = 0$. In conclusion,

$$E = 0 \oplus_{i=1}^{k} E_i,\ F = \oplus_{i=0}^{k} F_{ii};\ F_{00},\ E_i \text{ and } F_{ii} \text{ unitary for all } 1 \leq i \leq k.$$

The case in which (c) holds is similarly dealt with: we have

$$E = \oplus_{j=0}^{t} E_{jj},\ F = 0 \oplus_{j=1}^{t} F_j;\ E_{00},\ E_{jj} \text{ and } F_j \text{ unitary for all } 1 \leq j \leq t.$$

This brings us to case (d). If (d) holds, then

$$E = E_0 \oplus_{i=1}^{k} \lambda_i I|_{\mathcal{H}_{\lambda_i}} = \oplus_{i=0}^{k} E_i,\ F = F_0 \oplus_{j=1}^{t} \mu_j|_{\mathcal{H}_{\mu_j I}} = \oplus_{j=0}^{t} F_j,$$

where E_0 and F_0 are quasinilpotents. Letting $E \in B\left(\oplus_{j=0}^{t} \mathcal{H}_{\mu_j}\right)$ have the matrix representation $E = [E_{ij}]_{i,j=0}^{t}$, it is seen that $E_{ij} = 0$ for all $0 \leq i \neq j \leq t$ and $E = \oplus_{i=0}^{t} E_{ii}$. The operator F_0 being quasinilpotent, the commutativity of E, F taken along with the normality of EF (hence, $E_{00}F_0$) implies

$$E_{00}F_0 = 0 = [E_{00}, F_0].$$

Furthermore, if $0 \in \sigma(E_{ii})$ for some $1 \leq i \leq t$, then E_{ii} is a direct sum $E_{ii} = L_0 \oplus L_1 \in B\left(E_{ii}^{-1}(0) \oplus (H_{\mu_i} \ominus E_{ii}^{-1}(0))\right)$ of a quasinilpotent operator L_0 and a unitary operator L_1; since $E_{ii}F_i$ is normal (because EF is), L_0 is the 0 operator and $E_{ii} = 0 \oplus L_1$. Thus we conclude:

$$E = E_{00} \oplus (0 \oplus E_u)$$

for some unitary E_u with $\sigma(E_u) = S_1$.

To conclude what the above translates into for operators A and B, we start by proving that αA and βB are contractions. (Recall: $\alpha = \frac{1}{\|\tilde{A}\|}$, $\beta = \frac{1}{\|\tilde{B}\|}$ and $\alpha\beta = 1$.) As seen above $\|A\|\|B\| = \left\|\tilde{A}\right\| \left\|\tilde{B}\right\|$; hence $\|\alpha A\|\|\beta B\| = 1$. Since Aluthge transforms preserve spectrum, $\sigma(\alpha A) = \sigma(\alpha\tilde{A}) \subseteq \{0\} \cup S_1 \subseteq \{0\} \cup \partial(\mathbb{D})$ and $\sigma(\beta B) = \sigma(\beta\tilde{B}) = \{0\} \cup S_2 \subseteq \{0\} \cup \partial(\mathbb{D})$. Consequently,

$$r(\alpha A) = r(\beta B) = 1.$$

If αA and βB are normaloid, then there is nothing to prove. So assume one of them, say αA, has norm 1. (Observe that $\|A\|\|B\| = \left\|\tilde{A}\right\| \left\|\tilde{B}\right\|$ rules out both αA and αB having norm greater than one.) Then $\|A\| = \left\|\tilde{A}\right\|$, and hence $\|A\|\|B\| = \left\|\tilde{A}\right\| \left\|\tilde{B}\right\|$ forces $\|B\| = \left\|\tilde{B}\right\|$ and $\|\beta B\| = \left\|\beta\tilde{B}\right\| = 1$.

Aluthge transforms preserve both the ascent and the descent at non-zero points of the spectrum of an operator [2, 4]. Hence all non-zero points of the spectrum of αA and βB are poles (of the resolvent), and therefore eigenvalues, of the operators. Since all these eigenvalues lie in $\partial(\mathbb{D})$, and the operators are contractions, all non-zero points of the spectra of αA and βB are normal eigenvalues of the operators. In conclusion,

$$A = \alpha\left(A_0 \oplus_{i=1}^{k} \lambda_i I_i\right) \text{ and } B = \beta\left(B_0 \oplus_{j=1}^{t} \mu_j \mathbf{I}_j\right),$$

where $I_i = I|_{(\alpha A - \lambda_i I)^{-1}(0)}$, $\mathbf{I}_j = I|_{(\beta B - \mu_j I)^{-1}(0)}$ and the operators A_0, B_0 are quasinilpotent. For the cases (a) to (d) this translates into the following.

(a) If $\sigma(A) = \sigma\left(\tilde{A}\right) = \left\|\tilde{A}\right\| S_1$ and $\sigma(B) = \sigma\left(\tilde{B}\right) = \left\|\tilde{B}\right\| S_2$, then

$$A = \left\|\tilde{A}\right\| \left(\oplus_{i=1}^{k} \lambda_i I_i\right) = \left\|\tilde{A}\right\| A_u \text{ and } B = \left\|\tilde{B}\right\| \left(\oplus_{j=1}^{t} \mu_j \mathbf{I}_j\right) = \left\|\tilde{B}\right\| B_u.$$

Since $\tilde{A} = \left\|\tilde{A}\right\| A_u = \left\|\tilde{A}\right\| E$ and $\tilde{B} = \left\|\tilde{B}\right\| B_u = \left\|\tilde{B}\right\| F$, the unitaries A_u and B_u satisfy $A_u = E$ and $B_u = F$. Evidently, $AB \in (G - n - P)$.

(b) and (c) If $\sigma(A) = \sigma\left(\tilde{A}\right) = \left\|\tilde{A}\right\| S_1$ and $\sigma(B) = \sigma\left(\tilde{B}\right) = \left\|\tilde{B}\right\| S_2$, then

$$A = \left\|\tilde{A}\right\| (A_0 \oplus_{i=1}^{k} \lambda_i I_i) = \left\|\tilde{A}\right\| (A_0 \oplus A_u)$$

and

$$B = \left\|\tilde{B}\right\| (\oplus_{j=1}^{t} \mu_j \mathbf{I}_j) = \left\|\tilde{B}\right\| B_u$$

(with respect to $\mathcal{H} = \left(\mathcal{H} \ominus_{i=1}^{k} (\alpha A - \lambda_i I)^{-1}(0)\right) \oplus_{i=1}^{k} (\alpha A - \lambda_i I)^{-1}(0))$, where A_0 is quasinilpotent. Since

$$\widetilde{(\alpha A)} = \alpha\left(\tilde{A}_0 \oplus A_u\right) = E = 0 \oplus_{i=1}^{k} E_i \text{ and } \beta B_u = F = \oplus_{i=0}^{k} F_{ii},$$

the operator A_0 is 2-nilpotent. A similar argument shows that if $\sigma(A) = \sigma\left(\tilde{A}\right) = \left\|\tilde{A}\right\| S_1$ and $\sigma(B) = \sigma\left(\tilde{B}\right) = \left\|\tilde{B}\right\| S_2$, then

$$A = \left\|\tilde{A}\right\| \left(\oplus_{j=0}^{t} E_{jj}\right) = \left\|\tilde{A}\right\| A_u \text{ and } B = \left\|\tilde{B}\right\| \left(B_0 \oplus_{j=1}^{t} F_j\right) = \left\|\tilde{B}\right\| (B_0 \oplus B_u),$$

where B_0 is 2-nilpotent and A_u, B_u are unitary. (Evidently, $AB \notin (G - n - P)$ in either of the cases, unless A_0, respectively B_0, is the 0 operator.)

(d) Finally, if ($\sigma(A) = \sigma(\tilde{A}) = \|\tilde{A}\| S_1$ and) $\sigma(B) = \sigma(\tilde{B}) = \|\tilde{B}\| S_2$, then $B = \|\tilde{B}\|(B_0 \oplus B_u)$, $\sigma(B_0) = \{0\}$ and $B_u = \oplus \mu_j \mathbf{I}_j = \oplus_{j=1}^{t} B_j$ unitary. Letting αA have the matrix representation $[A_{ij}]_{i,j=0}^{t}$ with respect to the decomposition of $\mathcal{H}$ enforced by $B_0 \oplus B_u$, the commutative property of A and B implies

$$A_{ij} B_j = B_i A_{ij}, \ 0 \le i, j \le t.$$

Hence, since $B_i - B_j = (\mu_i \mathbf{I}_i - \mu_j \mathbf{I}_j)$ for all $i \neq j$, and the operators $(B_i - B_0)$ for $i \neq 0$ and $(B_0 - B_j)$ for $j \neq 0$ are invertible, $A_{ij} = 0$ for all $0 \le i \neq j \le t$. Consequently

$$A = \left\|\tilde{A}\right\| \left(\oplus_{j=0}^{t} A_{jj}\right), \ [A_{00}, B_0] = 0 \text{ and } A_{00} B_0 \text{ is quasinilpotent.}$$

Furthermore, if $0 \in \sigma\left(\oplus_{j=1}^{t} A_{jj}\right)$, then $\oplus_{j=1}^{t} A_{jj} = A_0 \oplus A_u$, where A_0 is quasinilpotent and A_u is unitary. Thus we conclude:

$$A = \left\|\tilde{A}\right\| (A_{00} \oplus (A_0 \oplus A_u)) \text{ and } \tilde{B} = \left\|\tilde{B}\right\| (B_0 \oplus B_u).$$

Taking Aluthge transforms

$$\tilde{A} = \left\|\tilde{A}\right\| \left(\widetilde{A_{00}} \oplus \left(\tilde{A_0} \oplus A_u\right)\right) \text{ and } B = \left\|\tilde{B}\right\| \left(\widetilde{B_0} \oplus B_u\right).$$

Since $\tilde{A}, \tilde{B}$ doubly commute and $\tilde{A}\tilde{B}$ is normal,

$$\widetilde{A_{00}}\widetilde{B_0} = 0 \Longleftrightarrow A_{00}B_0 \text{ is } 2-\text{nilpotent}.$$

To determine $\tilde{A_0}$, let B_u have the matrix representation $[B_{ij}]_{i,j=1}^2$ (with respect to the decomposition enforced by $\tilde{A_0} \oplus A_u$). Then

$$\tilde{A_0}B_{12} = B_{12}A_u \text{ and } \tilde{A_0}^* B_{12} = B_{12}A_u^*;$$
$$A_u B_{21} = B_{21}\tilde{A_0} \text{ and } A_u^* B_{21} = B_{21}\tilde{A_0}^*.$$

This implies $B_{12} = B_{21} = 0$ and hence (by the normality of $\tilde{A_0}B_{11}$ and the fact that $\sigma\left(\tilde{A_0}\right) = \{0\}$)

$$\tilde{A_0}B_{11} = 0 \Longleftrightarrow \tilde{A_0} = 0 \Longleftrightarrow A_0 \text{ is 2-nilpotent}.$$

Summarising, if (d) holds, then

$$A = \left\|\tilde{A}\right\| (A_{00} \oplus (A_0 \oplus A_u)) \text{ and } B = \left\|\tilde{B}\right\| (B_0 \oplus B_u),$$

where A_u, B_u are unitary, $[A_{00}, B_0] = 0$, and $A_{00}B_0$ and A_0 are 2-nilpotent. (Here, as pointed out in the statement of the theorem, either of the components may be absent.)

To complete the proof of the theorem, we prove that if A, B are as in the general case (d) above, then $\widetilde{AB} \in (G-n-P)$. We have

$$\widetilde{AB} = \tilde{A}\tilde{B} = \left\|\tilde{A}\right\| \left\|\tilde{B}\right\| \left(\widetilde{A_{00}}\widetilde{B_0} \oplus \left(\widetilde{A_0} \oplus A_u\right) B_u\right) = \widetilde{A_{00}B_0} \oplus \left(\widetilde{A_0} \oplus A_u\right) B_u,$$

where $\widetilde{A_{00}B_0} = \widetilde{A_0} = 0$ (since $A_{00}B_0$ and A_0 are 2-nilpotent). Thus

$$\widetilde{AB} = 0 \oplus (0 \oplus A_u)B_u, \; A_u \text{ and } B_u \text{ unitary}.$$

A straightforward computation (similar to our earlier ones) using the commutativity of $0 \oplus A_u$ and B_u shows that $(0 \oplus A_u)B_u = (0 \oplus A_u)(B_{u1} \oplus B_{u2}) = 0 \oplus A_u B_{u2}$; B_{u1} and B_{u2} unitaries. Hence

$$\begin{aligned} \widetilde{AB}^{*n} &= 0 \oplus (0 \oplus A_u^{*n} B_{u2}^{*n}) \\ &= 0 \oplus (0 \oplus A_u B_{u2}) = \widetilde{AB}, \end{aligned}$$

since $A_u^{*n} = A_u$ and $B_{u2}^{*n} = B_{u2}$. ☐

Remark 3.2. There is nothing sacrosanct about our choice of the operator B to have the representation $B = \left\|\tilde{B}\right\|(B_{00} \oplus B_u)$. We could have chosen $A = \left\|\tilde{A}\right\|(A_{00} \oplus A_u)$, which would have then forced $B = \left\|\tilde{B}\right\|(B_{00} \oplus (B_0 \oplus B_u))$.

The hypotheses of the theorem are not sufficient to guarantee the normality, much less the property of being $(G-n-P)$, of either of the operator A, B, AB, $\tilde{A}$ and $\tilde{B}$. Indeed, if $0 \in \sigma(A) \cap \sigma(B)$, then $AB \in (G-n-P)$, hence is normal, if and only if $A_0 = A_{00}B_0 = 0$. A necessary and sufficient condition for suitable multiples of $\tilde{A}$, $\tilde{B}$, A and B to be $(G-n-P)$ may be given as follows.

For a Banach space operator $T \in B(\mathcal{X})$ with a spectral set σ, let P_σ denote the spectral projection associated with σ [8, p. 204]. The operator T is said to be spectrally normaloid if $T|_{P_\sigma(\mathcal{X})}$ is normaloid for every spectral set σ of $\sigma(T)$ [8, p. 227]. The proof of Theorem 3.1 implies the following corollary.

Corollary 3.3. *A necessary and sufficient condition for the operators* $\alpha\tilde{A}$, αA, $\beta\tilde{B}$ *and* βB *of Theorem 3.1 to be* $(G-n-P)$ *is that* A, B *are spectrally normaloid at* 0.

The spectrally normaloid at 0 hypothesis of the theorem ensures that the quasinilpotent parts of the operators $A, B, \tilde{A}$ and $\tilde{B}$ are the 0 operator. We observe here that the spectrally normaloid property at 0 for A (resp., B) is vacuously satisfied if $0 \notin \sigma(A)$ (resp., $0 \notin \sigma(B)$).

A particular case of Theorem 3.1, where a number of the hypotheses of the theorem are inbuilt into the operators being considered, is that of the tensor products of operators satisfying the $(G-n-P)$ property.

Let $\mathcal{H}\bar{\otimes}\mathcal{H}$ denote the completion, endowed with a reasonable uniform cross norm, of the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$. For $S, T \in \mathcal{B}(H)$, let $S \otimes T \in \mathcal{B}(\mathcal{H}\bar{\otimes}\mathcal{H})$ denote the tensor product of S and T. Define operators $A, B \in \mathcal{B}(\mathcal{H}\bar{\otimes}\mathcal{H})$ by $A = S \otimes I$ and $B = I \otimes T$, and let S, T, A, B have the polar decompositions

$$S = U_1|S|, \; T = V_1|T|, \; A = UP \text{ and } B = VQ.$$

Then A and B doubly commute,

$$\begin{aligned}
&UP = (U_1 \otimes I)(|S| \otimes I), \; VQ = (I \otimes V_1)(I \otimes |T|),\\
&\tilde{A} = P^{\frac{1}{2}}UP^{\frac{1}{2}} = \left(|S|^{\frac{1}{2}} \otimes I\right)(U_1 \otimes I)\left(|S|^{\frac{1}{2}} \otimes I\right),\\
&\tilde{B} = Q^{\frac{1}{2}}VPQ\frac{1}{2} = \left(I \otimes |T|^{\frac{1}{2}}\right)(I \otimes V_1)\left(I \otimes |T|^{\frac{1}{2}}\right) \text{ and}\\
&\widetilde{AB} = \tilde{A}\tilde{B} = \tilde{B}\tilde{A} = \left(|S|^{\frac{1}{2}} \otimes |T|^{\frac{1}{2}}\right)(U_1 \otimes V_1)\left(|S|^{\frac{1}{2}} \otimes |T|^{\frac{1}{2}}\right).
\end{aligned}$$

Furthermore,

$$\left\|\widetilde{AB}\right\| = \left\|\tilde{A}\right\|\left\|\tilde{B}\right\|,$$

$$\sigma\left(\widetilde{AB}\right) = \sigma\left(\tilde{S} \otimes \tilde{T}\right) = \sigma\left(\tilde{S}\right)\sigma\left(\tilde{T}\right) = \sigma\left(\tilde{A}\right)\sigma\left(\tilde{B}\right) = \sigma(S)\sigma(T) = \sigma(A)\sigma(B).$$

Evidently,

$$\begin{aligned} \tilde{A} \in (G-n-P) &\Longleftrightarrow \tilde{S} \in (G-n-P), \\ \tilde{B} \in (G-n-P) &\Longleftrightarrow \tilde{T} \in (G-n-P) \text{ and} \\ \widetilde{AB} \in (G-n-P) &\Longleftrightarrow \tilde{S}\tilde{T} \in (G-n-P). \end{aligned}$$

Combining this information, we have:

Corollary 3.4. *Given operators $S, T \in \mathcal{B}(H)$, if $\widetilde{S \otimes T} \in (G-n-P)$ and $\|S \otimes T\| \leq \left\|\widetilde{S \otimes T}\right\|$, then $\frac{S}{\|\tilde{S}\|}, \frac{T}{\|\tilde{T}\|} \in (G-n-P)$ if and only if S, T are spectrally normaloid at* 0.

The extension of Corollary 3.4 to the Hilbert-Schmidt class $\mathcal{B}(\mathcal{C}_2(\mathcal{H}))$ is almost automatic for the reason that the tensor product $S \otimes T$ can be identified with the restriction $\mathcal{E}_{S,T^*}|_{\mathcal{B}(\mathcal{C}_2(\mathcal{H}))}$ of the elementary operator $\mathcal{E}_{S,T^*}(X) = SXT^*$, $X \in \mathcal{B}(\mathcal{C}_2(\mathcal{H}))$ [3].

Corollary 3.5. *Given operators $S, T \in \mathcal{B}(H)$ such that $\left\|\widetilde{S \otimes T}\right\| \leq \|S \otimes T\|$ and $\mathcal{E}_{\tilde{S},\tilde{T^*}} \in (G-n-P)$, $\frac{S}{\|\tilde{S}\|}$ and $\frac{T}{\|\tilde{T}\|}$ are $G-n-P$ if and only if S and T are spectrally normaloid at* 0.

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