Bull. Korean Math. Soc. **60** (2023), No. 6, pp. 1567–1605
https://doi.org/10.4134/BKMS.b220752
pISSN: 1015-8634 / eISSN: 2234-3016

TWO-WEIGHT NORM ESTIMATES FOR SQUARE FUNCTIONS ASSOCIATED TO FRACTIONAL SCHRÖDINGER OPERATORS WITH HARDY POTENTIAL

Tongxin Kang and Yang Zou

Abstract. Let $d \in \mathbb{N}$ and $\alpha \in (0, \min\{2, d\})$. For any $a \in [a^*, \infty)$, the fractional Schrödinger operator $\mathcal{L}_a$ is defined by

$$\mathcal{L}_a := (-\Delta)^{\alpha/2} + a|x|^{-\alpha},$$

where $a^* := -\frac{2^\alpha \Gamma((d+\alpha)/4)^2}{\Gamma((d-\alpha)/4)^2}$. In this paper, we study two-weight Sobolev inequalities associated with $\mathcal{L}_a$ and two-weight norm estimates for several square functions associated with $\mathcal{L}_a$.

1. Introduction

Let $d \in \mathbb{N}$ and $\alpha \in (0, \min\{2, d\})$. For any $a \in [a^*, \infty)$, the fractional Schrödinger operator $\mathcal{L}_a$ is defined by

$$\mathcal{L}_a := (-\Delta)^{\alpha/2} + a|x|^{-\alpha}, \tag{1.1}$$

where

$$a^* := -\frac{2^\alpha \Gamma((d+\alpha)/4)^2}{\Gamma((d-\alpha)/4)^2}. \tag{1.2}$$

Here, the constant a^* is derived from the sharp constant in the following Hardy-type inequality

$$\int_{\mathbb{R}^d} |x|^{-\alpha} |u(x)|^2 dx \leq -\frac{1}{a^*} \int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{u}(\xi)|^2 dx, \quad u \in C_c^\infty(\mathbb{R}^d),$$

where $\widehat{u}$ denotes the Fourier transform of u. Here and thereafter, $C_c^\infty(\mathbb{R}^d)$ denotes the set of *all infinitely differentiable functions on* $\mathbb{R}^d$ *with compact*

Received October 29, 2022; Accepted April 21, 2023.
2020 *Mathematics Subject Classification.* 35R11, 42B35, 42B20, 47B38, 35B65.
Key words and phrases. Sobolev inequality, generalized Hardy operator, fractional Schrödinger operator, square function, weighted norm inequality.
This work is partially supported by the National Natural Science Foundation of China (Grant Nos. 11871254 and 12071431), the Key Project of Gansu Provincial National Science Foundation (Grant No. 23JRRA1022), the Fundamental Research Funds for the Central Universities (Grant No. lzujbky-2021-ey18) and the Innovative Groups of Basic Research in Gansu Province (Grant No. 22JR5RA391).

support. Hence, the restriction $a \in [a^*, \infty)$ guarantees the nonnegativity of the operator $\mathcal{L}_a$. Assume that $p \in (1, \infty)$, $\omega \in A_q(\mathbb{R}^n)$ with some $q \in [1, \infty)$. Here and thereafter, $A_q(\mathbb{R}^n)$ denotes the Muckenhoupt weight class (see, for instance, [19, Chapter 7] or [30, Chapter V] for its definition). We now recall the following parameterization in [17],

$$\Psi_{\alpha,d}(\delta) := -2^{\alpha} \frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{d-\delta}{2})}{\Gamma(\frac{d-\delta-\alpha}{2})\Gamma(\frac{\delta}{2})}, \quad \delta \in (-\alpha, (d-\alpha)/2]\backslash\{0\},$$

and $\Psi_{\alpha,d}(0) := 0$. Indeed, it was proved in [17] that the function $\Psi_{\alpha,d}$ is continuous and strictly decreasing in $\delta \in (-\alpha, (d-\alpha)/2]$ with

$$\lim_{\delta \to -\alpha} \Psi_{\alpha,\, d}(\delta) = \infty \quad \text{and} \quad \Psi_{\alpha,\, d}\left(\frac{d-\alpha}{2}\right) = a^*.$$

Therefore, for any $a \in [a^*, \infty)$, we define

$$\sigma := \Psi^{-1}_{\alpha,d}(a), \tag{1.3}$$

so that $\sigma \in (-\alpha, (d-\alpha)/2]$. Moreover, for a given constant $\theta \in \mathbb{R}$, we define

$$d_\theta := \frac{d}{\min\{\theta, 0\}}, \tag{1.4}$$

in particular, $d/0 := \infty$.

The operator $-\mathcal{L}_a$ generates a semigroup which is denoted by $\{e^{-t\mathcal{L}_a}\}_{t>0}$, we also consider the semigroup $\{e^{-t\sqrt{\mathcal{L}_a}}\}_{t>0}$ defined via the subordination formula

$$e^{-t\sqrt{\mathcal{L}_a}} f(y) = \frac{1}{\sqrt{\pi}} \int_0^\infty u^{\frac{1}{2}} e^{-u} e^{-\frac{t^2}{4u}\mathcal{L}_a} f(y) \frac{\mathrm{d}u}{u}. \tag{1.5}$$

For any m, $K \in [0, \infty)$, we define several square functions associated with $\mathcal{L}_a$ by setting, for any $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\mathbf{s}_{m,P} f(y) := \left[\int_0^\infty \left|\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t\sqrt{\mathcal{L}_a}} f(y)\right|^2 \frac{dt}{t}\right]^{\frac{1}{2}}, \tag{1.6}$$

$$\mathbf{s}_{m,H} f(y) := \left[\int_0^\infty \left|\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t^2\mathcal{L}_a} f(y)\right|^2 \frac{dt}{t}\right]^{\frac{1}{2}}, \tag{1.7}$$

$$\mathbf{S}_{m,P} f(x) := \left[\int_0^\infty \int_{B(x,t^{\frac{2}{\alpha}})} \left|\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t\sqrt{\mathcal{L}_a}} f(y)\right|^2 \frac{dy\,dt}{t^{1+\frac{2d}{\alpha}}}\right]^{\frac{1}{2}}, \tag{1.8}$$

$$\mathbf{S}_{m,H} f(x) := \left[\int_0^\infty \int_{B(x,t^{\frac{2}{\alpha}})} \left|\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t^2\mathcal{L}_a} f(y)\right|^2 \frac{dy\,dt}{t^{1+\frac{2d}{\alpha}}}\right]^{\frac{1}{2}}, \tag{1.9}$$

$$\mathbf{g}_{K,P} f(y) := \left[\int_0^\infty \left|t^2 (-\Delta)^{\frac{\alpha}{2}} \left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t\sqrt{\mathcal{L}_a}} f(y)\right|^2 \frac{dt}{t}\right]^{\frac{1}{2}}, \tag{1.10}$$

$$\mathbf{g}_{K,H}f(y) := \left[\int_0^\infty \left|t^2\,(-\Delta)^{\frac{\alpha}{2}}\left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t^2\mathcal{L}_a}f(y)\right|^2\frac{dt}{t}\right]^{\frac12}, \tag{1.11}$$

$$\mathbf{G}_{K,P}f(x) := \left[\int_0^\infty\int_{B(x,t^{\frac{2}{\alpha}})}\left|t^2\,(-\Delta)^{\frac{\alpha}{2}}\left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t\sqrt{\mathcal{L}_a}}f(y)\right|^2\frac{dy\,dt}{t^{1+\frac{2d}{\alpha}}}\right]^{\frac12}, \tag{1.12}$$

and

$$\mathbf{G}_{K,H}f(x) := \left[\int_0^\infty\int_{B(x,t^{\frac{2}{\alpha}})}\left|t^2\,(-\Delta)^{\frac{\alpha}{2}}\left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t^2\mathcal{L}_a}f(y)\right|^2\frac{dy\,dt}{t^{1+\frac{2d}{\alpha}}}\right]^{\frac12}. \tag{1.13}$$

In this article, motivated by the recent work of Bui and Bui [5] on the weighted Sobolev inequality and the weighted boundedness of the square function associated to $\mathcal{L}_a$, we study two-weight Sobolev inequalities associated with $\mathcal{L}_a$ and two-weight norm estimates for several square functions associated with $\mathcal{L}_a$.

Let $p \in [1,\infty)$ and $\omega \in A_q(\mathbb{R}^n)$ with some $q \in [1,\infty)$. Recall that the *weighted Lebesgue space* $L^p_\omega(\mathbb{R}^d)$ is defined by setting

$$L^p_\omega(\mathbb{R}^d) := \left\{f \text{ is measurable on } \mathbb{R}^d : \|f\|_{L^p_\omega(\mathbb{R}^d)} < \infty\right\},$$

where

$$\|f\|_{L^p_\omega(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d}|f|^p\omega\,dx\right)^{\frac1p}.$$

In particular, when $\omega \equiv 1$, the weighted spaces $L^p_\omega(\mathbb{R}^d)$ is just, respectively, the classical Lebesgue space $L^p(\mathbb{R}^d)$. For any given $x \in \mathbb{R}^d$ and $r \in (0,\infty)$, let $B(x,r) := \{y \in \mathbb{R}^d : |y-x| < r\}$.

For any given $a, b \in \mathbb{R}$, we use the notations $a\wedge b := \min\{a,b\}$ and $a\vee b := \max\{a,b\}$. For any given $x \in \mathbb{R}^d$ and any given measurable subset $E \subset \mathbb{R}^d$, let $\operatorname{dist}(x,E) := \inf\{|x-y| : y \in E\}$. Meanwhile, for any measurable subsets $E,\ F \subset \mathbb{R}^d$, let

$$\operatorname{dist}(E,F) := \inf\{|x-y| : x \in E, y \in F\}$$

and $\operatorname{diam}(E) := \sup\{|x-y| : x, y \in E\}$.

Now, we state the main results of this article.

Theorem 1.1. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2\wedge d)$, and $s \in (0,2]$. Assume further that $a \in [a^*,\infty)$ with a^* being as in* (1.2) *and σ is defined by* (1.3).

(i) *Let $d'_\sigma < p_0 < p < q_0 < d_{\sigma+\alpha s/2}$ with d_σ and $d_{\sigma+\alpha s/2}$ being as in* (1.4), *$\omega \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{s_0}(\mathbb{R}^d)$ with $s_0 \in ((\frac{q_0}{p})',\infty]$, and*

$$\left[\omega, v^{1-(\frac{p}{p_0})'}\right]_{A_{\frac{p}{p_0}}(\mathbb{R}^d)} := \sup_{B\subseteq\mathbb{R}^d}\left[\fint_B \omega\,dx\right]\left[\fint_B v^{1-(\frac{p}{p_0})'}\,dx\right]^{\frac{p}{p_0}-1} < \infty,$$

where the surpremum is taken over all balls B of $\mathbb{R}^d$. Then there exists a positive constant C, depending on d, p, s, $[\omega]_{A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)}$, and $[\omega]_{RH_{s_0}(\mathbb{R}^d)}$, such that, for any $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$,

$$\left\|(-\Delta)^{\alpha s/4} f\right\|_{L^p_\omega(\mathbb{R}^d)} \leq C\left\|\mathcal{L}_a^{s/2} f\right\|_{L^p_v(\mathbb{R}^d)}. \tag{1.14}$$

(ii) *Let $d'_\sigma < p_0 < p < q_0 < \frac{d}{(\alpha s/2)\vee\sigma} := \tilde{d}$ with d_σ being as in* (1.4)*, $\omega \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{\tilde{d}}{p})'}(\mathbb{R}^d)$, $\omega^{1-p'} \in RH_{s_0}(\mathbb{R}^d)$ with $s_0 \in ((\frac{p'_0}{p'})', \infty]$, and*

$$\left[\omega^{1-p'}, v^{(1-p')[1-(\frac{p}{q'_0})']}\right]_{A_{\frac{p}{q'_0}}(\mathbb{R}^d)} < \infty.$$

Then there exists a positive constant C, depending on d, p, s, $[\omega]_{A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)}$, $[\omega]_{RH_{(\frac{\tilde{d}}{p})'}(\mathbb{R}^d)}$, and $[\omega^{1-p'}]_{RH_{s_0}(\mathbb{R}^d)}$, such that, for any $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$,

$$\left\|\mathcal{L}_a^{s/2} f\right\|_{L^p_v(\mathbb{R}^d)} \leq C\left\|(-\Delta)^{\alpha s/4} f\right\|_{L^p_\omega(\mathbb{R}^d)}. \tag{1.15}$$

We point out that the special case of Theorem 1.1 has been proved in [24] for the case $\alpha := 2$, in [18] for the case $p = 2$, in [28] for the case general p but with $a \geq 0$, and in [6] for the case $a \geq a^*$. Furthermore, the case of $w = v$ for Theorem 1.1 has been proved in [5]. Therefore, Theorem 1.1 is an extension of those results obtained in [5, 6, 18, 28] to the case of two-weight.

We prove Theorem 1.1 by borrowing some ideas from [6, Theorem 1.1] and [31, Theorem 2.14].

Theorem 1.2. *Let $m \in [0, \infty)$, $d \in \mathbb{N}$, and σ be as in* (1.3)*. Then, for any given $p \in (d'_\sigma, d_\sigma)$, and any $v \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_\sigma}{p})'}(\mathbb{R}^d)$, where d_σ is as in* (1.4)*, there exists a positive constant C such that, for any $f \in L^p_v(\mathbb{R}^d)$,*

(i) $\|\mathbf{S}_{m,H} f\|_{L^p_v(\mathbb{R}^d)} \leq C \|\mathbf{s}_{m,H} f\|_{L^p_v(\mathbb{R}^d)}$;
(ii) $\|\mathbf{S}_{m,P} f\|_{L^p_v(\mathbb{R}^d)} \leq C \|\mathbf{s}_{m,P} f\|_{L^p_v(\mathbb{R}^d)}$;
(iii) $\|\mathbf{s}_{m,P} f\|_{L^p_v(\mathbb{R}^d)} \leq C \|\mathbf{s}_{m,H} f\|_{L^p_v(\mathbb{R}^d)}$.

Theorem 1.3. *Let m, $K \in [0, \infty)$, $d \in \mathbb{N}$, $\alpha \in (0, 2\wedge d)$, and σ be as in* (1.3)*. Then, for given any $p \in (d'_\sigma, d_{\sigma+\alpha})$ and any $v \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+\alpha}}{p})'}(\mathbb{R}^d)$ with $d_{\sigma+\alpha} \in (2, \infty)$, where $d_{\sigma+\alpha}$ and d_σ are as in* (1.4)*, there exists a positive constant C such that, for any $f \in L^p_v(\mathbb{R}^d)$,*

(i) $C^{-1} \|\mathbf{s}_{m+2,H} f\|_{L^p_v(\mathbb{R}^d)} \leq \|\mathbf{g}_{m,H} f\|_{L^p_v(\mathbb{R}^d)} \leq C \|\mathbf{s}_{m+2,H} f\|_{L^p_v(\mathbb{R}^d)}$;
(ii) $C^{-1} \|\mathbf{s}_{m+2,P} f\|_{L^p_v(\mathbb{R}^d)} \leq \|\mathbf{g}_{m,P} f\|_{L^p_v(\mathbb{R}^d)} \leq C \|\mathbf{s}_{m+2,P} f\|_{L^p_v(\mathbb{R}^d)}$;
(iii) $\|\mathbf{G}_{K,H} f\|_{L^p_v(\mathbb{R}^d)} \leq C \|\mathbf{g}_{K,H} f\|_{L^p_v(\mathbb{R}^d)}$;
(iv) $\|\mathbf{G}_{K,P} f\|_{L^p_v(\mathbb{R}^d)} \leq C \|\mathbf{s}_{K,P} f\|_{L^p_v(\mathbb{R}^d)}$.

We prove Theorems 1.2 and 1.3 by borrowing some ideas from [29].

The organization of this article is as follows.

In Section 2, we present the notions of the Muckenhoupt weight class and the reverse Hölder class, some properties of the Muckenhoupt weight class and the reverse Hölder class, the definition of the Hardy–Littlewood maximal function, the weighted estimates of the maximal functions, a criteria for singular integrals to be bounded on Lebesgue spaces, some elementary estimates and kernel estimates, the Hardy inequality for the operator $\mathcal{L}_a$, and the boundedness of square functions involving the difference $t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}}$.

In Section 3, we give the proof of Theorem 1.1. First, to prove this theorem, we need to prove some subtle ingredients such as the weighted Hardy inequality related to the operator $\mathcal{L}_a$ (see Lemma 3.1 below) and the weighted norm inequalities for the square functions (see Lemma 3.2 below). Next, we present the weighted good-λ inequality for a pair of functions, (F, f), on $\mathbb{R}^d$ satisfying the assumptions (3.11) and (3.12) (see Lemma 3.4 below) and the two-weight boundedness criterion for a pair of functions, (F, f), on $\mathbb{R}^d$ satisfying the assumptions (3.11) and (3.12) in the scale of weighted Lebesgue spaces (see Lemma 3.5 below). Then we are almost ready to establish the two-weight boundedness for $T_{\mathcal{L}_a,s}$ in the scale of weighted Lebesgue spaces (see Theorem 3.3 below) and the two-weight boundedness for $S_{\mathcal{L}_a,\gamma}$ in the scale of weighted Lebesgue spaces (see Theorem 3.6 below). Finally, we summarize what we have proved to complete the proof of Theorem 1.1.

In Section 4, as applications of Lemma 2.9, we obtain the weighted norm estimates related to the square functions associated with $\mathcal{L}_a$ (see Theorems 4.3, 4.5, 4.6 and 4.8 below). In order to prove these estimates, we subtly use the extrapolation theorem (see Lemma 4.1 below) and the change of angle formulas (see Lemma 4.2 below). Moreover, we give an application of these estimates of square functions to the Hardy space associated with $\mathcal{L}_a$.

Finally, we make some conventions on notation. Throughout the whole article, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. For any ball $B := B(x_B, r_B)$ in $\mathbb{R}^n$, with some $x_B \in \mathbb{R}^n$, $r_B \in (0,\infty)$, and $\alpha \in (0,\infty)$, let $\alpha B := B(x_B, \alpha r_B)$; furthermore, denote the set $B(x,r) \cap \Omega$ by $B_\Omega(x,r)$ and the set $(\alpha B) \cap \Omega$ by αB_Ω. For any subset E of $\mathbb{R}^n$, we denote the *set* $\mathbb{R}^n \setminus E$ by $E^{\complement}$ and its *characteristic function* by $\mathbf{1}_E$. For any given $q \in [1,\infty]$, we denote by q' its *conjugate exponent*, namely, $\frac{1}{q} + \frac{1}{q'} = 1$. For any $\omega \in A_p(\mathbb{R}^n)$ with some $p \in [1,\infty)$ and any measurable set $E \subset \mathbb{R}^n$, let $\omega(E) := \int_\Omega \omega(x)\,dx$. In addition, for any $f \in L^1(E)$, we denote the integral $\int_E |f(x)|\omega(x)\,dx$ simply by $\int_E |f|\omega\,dx$ and, when $|E| < \infty$, we use the notation $⨍_E f\,dx := \frac{1}{|E|}\int_E f(x)\,dx$.

2. Preliminaries

In this section, we recall the notions of the Muckenhoupt weight class and the reverse Hölder class, some properties of the Muckenhoupt weight class and the reverse Hölder class, the definition of the Hardy–Littlewood maximal function, a boundedness criteria for singular integrals on Lebesgue spaces, the Hardy inequality for the operator $\mathcal{L}_a$, and the boundedness of square functions involving the difference $t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}}$.

We first recall the concepts of both the Muckenhoupt weight class and the reverse Hölder class as follows (see, for instance, [14, 19, 30]).

Definition 2.1. Let $q \in [1, \infty)$. A non-negative and locally integrable function ω on $\mathbb{R}^d$ is called an $A_q(\mathbb{R}^d)$ *weight*, denoted by $\omega \in A_q(\mathbb{R}^n)$, if, when $q \in (1, \infty)$,

$$[\omega]_{A_q(\mathbb{R}^d)} := \sup_{B\subset\mathbb{R}^n} \left(\fint_B \omega\, dx \right) \left(\fint_B \omega^{-\frac{1}{q-1}}\, dx \right)^{q-1} < \infty,$$

and

$$[\omega]_{A_1(\mathbb{R}^d)} := \sup_{B\subset\mathbb{R}^d} \left(\fint_B \omega\, dx \right) \left\{ \operatorname*{ess\,sup}_{y\in B} [\omega(y)]^{-1} \right\} < \infty,$$

where the suprema are taken over all balls B of $\mathbb{R}^d$. Moreover, let

$$A_\infty(\mathbb{R}^d) := \bigcup_{p\in[1,\infty)} A_p(\mathbb{R}^d).$$

Let $s \in (1, \infty]$. A non-negative and locally integrable function ω on $\mathbb{R}^d$ is said to belong to the *reverse Hölder class* $RH_s(\mathbb{R}^d)$, denoted by $\omega \in RH_s(\mathbb{R}^d)$, if, when $s \in (1, \infty)$,

$$[\omega]_{RH_s(\mathbb{R}^d)} := \sup_{B\subset\mathbb{R}^d} \left(\fint_B \omega^s\, dx \right)^{\frac{1}{s}} \left(\fint_B \omega\, dx \right)^{-1} < \infty,$$

and

$$[\omega]_{RH_\infty(\mathbb{R}^d)} := \sup_{B\subset\mathbb{R}^d} \left[\operatorname*{ess\,sup}_{y\in B} \omega(y) \right] \left(\fint_B \omega\, dx \right)^{-1} < \infty,$$

where the suprema are taken over all balls B of $\mathbb{R}^d$.

For the Muckenhoupt weight class and the reverse Hölder class, we have the following properties which are well known (see, for instance, [14, 19, 30]).

Lemma 2.2. (i) *$\omega \in A_p(\mathbb{R}^d)$ if and only if $\omega^{1-p'} \in A_{p'}(\mathbb{R}^d)$.*
(ii) *$A_1(\mathbb{R}^d) \subseteq A_p(\mathbb{R}^d) \subseteq A_q(\mathbb{R}^d)$ for any given $1 \le p \le q \le \infty$.*
(iii) *$RH_\infty(\mathbb{R}^d) \subseteq RH_q(\mathbb{R}^d) \subseteq RH_p(\mathbb{R}^d)$ for any given $1 < p \le q \le \infty$.*
(iv) *If $\omega \in A_p(\mathbb{R}^d)$ with $p \in (1, \infty)$, then there exists a $q \in (1, p)$ such that $\omega \in A_q(\mathbb{R}^d)$.*
(v) *If $\omega \in RH_q(\mathbb{R}^d)$ for some $q \in (1, \infty)$, then there exists a $p \in (q, \infty)$ such that $\omega \in RH_p(\mathbb{R}^d)$.*
(vi) *$A_\infty(\mathbb{R}^d) = \bigcup_{p\in[1,\infty)} A_p(\mathbb{R}^d) = \bigcup_{q\in(1,\infty]} RH_q(\mathbb{R}^d)$.*

(vii) *Let* $1 < p_0 < p < q_0 < \infty$. *Then*

$$\omega \in A_{\frac{p}{p_0}}(\mathbb{R}^d) \cap RH_{(\frac{q_0}{p})'}(\mathbb{R}^d) \Longleftrightarrow \omega^{1-p'} \in A_{\frac{p'}{q_0'}}(\mathbb{R}^d) \cap RH_{(\frac{p_0'}{p'})'}(\mathbb{R}^d).$$

(viii) *Let* $p \in [1, \infty)$ *and* $q \in (1, \infty]$. *If* $\omega \in A_p(\mathbb{R}^d) \cap RH_q(\mathbb{R}^d)$, *then there exists a positive constant* C *such that, for any ball* $B \subset \mathbb{R}^d$ *and any measurable subset* E *of* B,

$$C^{-1} \left[\frac{|E|}{|B|}\right]^p \leq \frac{\omega(E)}{\omega(B)} \leq C \left[\frac{|E|}{|B|}\right]^{\frac{q-1}{q}},$$

where, for any measurable subset $E \subset \mathbb{R}^d$,

$$\omega(E) := \int_E \omega(x)\, dx.$$

In particular, Lemma 2.2(viii) implies that, if $\omega \in A_p(\mathbb{R}^d)$ for some $p \in [1, \infty)$, then, for any ball $B \subset \mathbb{R}^d$ and $\lambda \in (1, \infty)$,

$$\omega(\lambda B) \leq [\omega]_{A_p(\mathbb{R}^d)} \lambda^{dp} \omega(B). \tag{2.1}$$

Moreover, we observe that, under some assumptions, we can compare the average of any given function f with respect to the measure given by a weight $v \in A_\infty(\mathbb{R}^d)$.

Remark 2.3. Let $0 < \widetilde{p} \leq \widetilde{q} < \infty$. Assume that $v \in A_{\frac{\widetilde{q}}{\widetilde{p}}}(\mathbb{R}^d)$. Then there exists a positive constant C such that, for any given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, a ball $B \subset \mathbb{R}^d$, and $j \in \mathbb{N}$,

$$\left(\fint_{S_j(B)} |f(x)|^{\widetilde{p}}\, dx\right)^{\frac{1}{\widetilde{p}}} \leq C \left(\fint_{S_j(B)} |f(x)|^{\widetilde{q}} v(x)\, dx\right)^{\frac{1}{\widetilde{q}}}.$$

Here and thereafter, $S_j(B) := (2^{j+1}B)\backslash(2^jB)$ for any $j \in \mathbb{N}$ and $S_0(B) := 2B$.

Meanwhile, for any given $r \in (0, \infty)$, the Hardy–Littlewood maximal function $\mathcal{M}_r$ is defined by, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\mathcal{M}_r f(x) := \sup_{B \ni x} \left(\frac{1}{|B|} \int_B |f(y)|^r dy\right)^{1/r},$$

where the supremum is taken over all balls B that contain the given point x. When $r := 1$, we simply write $\mathcal{M}$ instead of $\mathcal{M}_1$.

Then, we have the following weighted boundedness for the Hardy–Littlewood maximal function $\mathcal{M}_r$ (see, for instance, [14, 19]).

Lemma 2.4. *Let* $r \in (0, \infty)$, $p \in (r, \infty)$, *and* $\omega \in A_{p/r}(\mathbb{R}^d)$. *Then there exists a positive constant* C, *depending only on* r, p, *and* $[\omega]_{A_{p/r}(\mathbb{R}^d)}$, *such that, for any* $f \in L^p_\omega(\mathbb{R}^d)$,

$$\|\mathcal{M}_r f\|_{L^p_\omega(\mathbb{R}^d)} \leq C \|f\|_{L^p_\omega(\mathbb{R}^d)}.$$

Next, we recall a criteria for singular integrals to be bounded on the space $L^p(\mathbb{R}^d)$ with $p \in (1,2)$, which plays an important role in the proof of the boundedness of the square functions (see, for instance, [2, 5, 6]).

Proposition 2.5. *Let $1 \le p_0 < q_0 < \infty$ and T be a linearizable operator. Suppose that T is bounded on $L^{q_0}(\mathbb{R}^d)$. Suppose further that there exists a family of operators $\{\mathcal{A}_t\}_{t>0}$ satisfying that, for any $j \ge 2$, any ball $B := (x_B, r_B) \subset \mathbb{R}^d$, and any function f supported in B,*

$$\left(\fint_{S_j(B)} |T(I-\mathcal{A}_{r_B})f|^{q_0}\,dx\right)^{1/q_0} \le \alpha(j)\left(\fint_B |f|^{q_0}\,dx\right)^{1/q_0}$$

and

$$\left(\fint_{S_j(B)} |A_{r_B}f|^{q_0}\,dx\right)^{1/q_0} \le \alpha(j)\left(\fint_B |f|^{p_0}\,dx\right)^{1/p_0}.$$

If $\sum_{j=2}^{\infty}\alpha(j)2^{jd} < \infty$, then T is bounded on $L^p_\omega(\mathbb{R}^d)$ for any $p \in (p_0, q_0)$ and $\omega \in A_{\frac{p}{p_0}}(\mathbb{R}^d) \cap RH_{(\frac{q_0}{p})'}(\mathbb{R}^d)$.

The elementary estimates stated in Lemmas 2.6, 2.7, and 2.8 below can be found in [5, 6]. We omit the details here.

Lemma 2.6. *Let $d \in \mathbb{N}$ and $\kappa \in (-\infty, d)$. Then there exists a positive constant C such that, for any $t \in (0,\infty)$,*

$$\int_{B(0,t)} \left(\frac{t}{|x|}\right)^{\kappa} dx \le Ct^d.$$

Lemma 2.7. *Let $d \in \mathbb{N}$ and $\alpha \in (0, 2\wedge d)$. For any given $\epsilon \in (0,\infty)$, there exists a positive constant C such that, for any $t \in (0,\infty)$ and $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \frac{1}{t^{d/\alpha}}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\epsilon} dy \le C.$$

Lemma 2.8. *Let $d \in \mathbb{N}$ and $\alpha \in (0, 2\wedge d)$. For any given $\epsilon \in (0,\infty)$, there exists a positive constant C such that, for any $t \in (0,\infty)$ and $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \frac{1}{t^{d/\alpha}}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\epsilon} |f(y)|\,dy \le C\mathcal{M}f(x).$$

In what follows, for a given constant $\theta \in \mathbb{R}$, we use $D_\theta(x,t)$ to denote $(1+\frac{t^{1/\alpha}}{|x|})^\theta$ for any $t \in (0,\infty)$ and $x \in \mathbb{R}^d$; namely,

$$D_\theta(x,t) := \left(1+\frac{t^{1/\alpha}}{|x|}\right)^{\theta}. \tag{2.2}$$

Then we have the following conclusion which was established in [5, 6].

Lemma 2.9. *Let $\{T_t\}_{t>0}$ be a family of linear operators on $L^2(\mathbb{R}^d)$. Assume that, for any $t \in (0,\infty)$, T_t is defined by, for any $f \in L^2(\mathbb{R}^d)$ and almost every $x \in \mathbb{R}^d$,*

$$T_t f(x) = \int_{\mathbb{R}^d} T_t(x,y) f(y)dy.$$

Here, the kernel function $T_t(\cdot,\cdot)$ satisfies the following condition: there exist positive constants C, c and θ, $\eta \in \mathbb{R}$ such that, for any $t \in (0,\infty)$ and x, $y \in \mathbb{R}^d \backslash \{0\}$,

$$|T_t(x,y)| \leq Ct^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} D_\theta(x,t) D_\eta(y,t),$$

where D_θ and D_δ are as in (2.2). Assume further that $d'_\eta < p \leq q < d_\theta$, where d_η and d_θ are as in (1.4). Then, for any ball $B := B(x_B, r_B) \subset \mathbb{R}^d$, any $t \in (0,\infty)$, any $j \in \mathbb{N}$, and any $f \in L^p(\mathbb{R}^d)$ supported in B,

$$\begin{aligned}\left(\fint_{S_j(B)} |T_t f|^q dx \right)^{1/q} &\leq C \left[\left(\frac{r_B}{t^{1/\alpha}} \right)^{d/p} \vee \left(\frac{r_B}{t^{1/\alpha}} \right)^{d} \right] \\ &\quad \times \left(1 + \frac{t^{1/\alpha}}{2^j r_B} \right)^{d/q} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_B |f|^p dx \right)^{1/p} \end{aligned} \tag{2.3}$$

and, for any $f \in L^p(\mathbb{R}^d)$ supported in $S_j(B)$,

$$\begin{aligned}\left(\fint_{B} |T_t f|^q dx \right)^{1/q} &\leq C \left[\left(\frac{2^j r_B}{t^{1/\alpha}} \right)^{d/p} \vee \left(\frac{2^j r_B}{t^{1/\alpha}} \right)^{d} \right] \\ &\quad \times \left(1 + \frac{t^{1/\alpha}}{r_B} \right)^{d/q} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_{S_j(B)} |f|^p dx \right)^{1/p}. \end{aligned} \tag{2.4}$$

Moreover, we have the following pointwise estimates for the heat kernels of the fractional Schrödinger operator $\mathcal{L}_a$ (see, for instance, [3, 4, 12, 23, 28]).

Lemma 2.10. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2 \wedge d)$, $a \in [a^*, \infty)$ with a^* being as in (1.2), and let σ be as in (1.3). Assume that $\{p_t\}_{t>0}$ are the kernels associated to the heat semigroup $\{e^{-t\mathcal{L}_a}\}_{t>0}$. Then there exist positive constants C and c such that, for any $t \in (0,\infty)$ and x, $y \in \mathbb{R}^d \backslash \{0\}$,*

$$p_t(x,y) \leq Ct^{-d/\alpha} D_\sigma(x,t) D_\sigma(y,t) \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha},$$

where D_σ is as in (2.2).

Proposition 2.11. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2 \wedge d)$, $a \in [a^*, \infty)$ with a^* being as in (1.2), and σ be as in (1.3). Then, for any $s \in (0,\infty)$ and $p \in (d'_\sigma, d_\sigma)$ with d_σ being as in (1.4), $(t\mathcal{L}_a)^s e^{-t\mathcal{L}_a}$ is uniformly bounded on $L^p(\mathbb{R}^d)$ for any $t \in (0,\infty)$.*

Proposition 2.11 was established by Bui and D'Ancona in [6].

Proposition 2.12. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2 \wedge d)$, $a \in [a^*, \infty)$ with a^* being as in* (1.2)*, and σ be as in* (1.3)*. For any $s \in \mathbb{R}$ with $s \in (1 + 2[0 \vee (\sigma/\alpha)], \infty)$, there exists a positive constant $C_{(s)}$, depending on s, such that, for any $t \in (0, \infty)$ and x, $y \in \mathbb{R}^d \setminus \{0\}$,*

$$|p_{t,s}(x, y)| \le C_{(s)} t^{-(s+d/\alpha)} D_\sigma(x, t) D_\sigma(y, t) \left(\frac{t^{1/\alpha} + |x - y|}{t^{1/\alpha}} \right)^{-d-\alpha},$$

where $p_{t,\,s}(\cdot, \cdot)$ denotes the associated kernel of $\mathcal{L}_a^s e^{-t\mathcal{L}_a}$, and D_σ is as in (2.2).

Next, we recall the Hardy inequality associated with the operator $\mathcal{L}_a$ established in [28, Proposition 1.2].

Lemma 2.13. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2 \wedge d)$, $a \in [a^*, \infty)$ with a^* being as in* (1.2)*, and σ be as in* (1.3)*. Suppose that $s\alpha/2 \in (0, d)$. Then, for any given $p \in (d'_\sigma, d_{\sigma+s\alpha/2})$, there exists a positive constant C such that, for any $f \in C_{\rm c}^\infty(\mathbb{R}^d)$,*

$$\left\| |\cdot|^{-\alpha s/2} f \right\|_{L^p(\mathbb{R}^d)} \le C \left\| \mathcal{L}_a^{s/2} f \right\|_{L^p(\mathbb{R}^d)}.$$

We also need the following boundedness of square functions involving the difference $t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2} e^{-t(-\Delta)^{\alpha/2}}$, which was first proved by Merz [28, Proposition 5.2] (see also [5, 6]).

Lemma 2.14. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2 \wedge d)$, $a \in [a^*, \infty)$ with a^* being as in* (1.2)*, and σ be as in* (1.3)*. Assume that $p \in (d'_\sigma, d_{\sigma+s\alpha/2})$ and $\omega \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(d_{(\sigma/p)})'}(\mathbb{R}^d)$, where d_σ and $d_{\sigma+s\alpha/2}$ are as in* (1.4)*. Then there exists a positive constant C, depending only on p, $[\omega]_{A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)}$, and $[\omega]_{RH_{(\frac{d_\sigma}{p})'}(\mathbb{R}^d)}$, such that, for any $f \in C_{\rm c}^\infty(\mathbb{R}^d)$,*

$$\left\| \left\{ \int_0^\infty t^{-s} \left| \left(t\mathcal{L}_a e^{-t\mathcal{L}_a} - t(-\Delta)^{\alpha/2} e^{-t(-\Delta)^{\alpha/2}} \right) f \right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^p_\omega(\mathbb{R}^d)}$$

$$\le C \left\| \frac{f}{|\cdot|^{s\alpha/2}} \right\|_{L^p_\omega(\mathbb{R}^d)}. \tag{2.5}$$

Moreover, we recall that the special case of Theorem 1.1 has been proved in [24] for the case $\alpha = 2$, in [18] for the case $p = 2$, in [28] for the case general p but with $a \ge 0$, and in [6] for the case $a \ge a^*$ (see Lemma 2.15 below).

Lemma 2.15. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2 \wedge d)$ and $s \in (0, 2]$. Assume that $a \in [a^*, \infty)$ with a^* being as in* (1.2) *and σ is as in* (1.3)*. Then there exists a positive constant C, depending on p and d, such that, for any $f \in C_{\rm c}^\infty(\mathbb{R}^d)$,*

(i) *if $\frac{d}{d - \sigma \vee 0} < p < \frac{d}{(\sigma + \alpha s/2) \vee 0}$ with $\frac{d}{0} := \infty$, then*

$$\left\| (-\Delta)^{\alpha s/4} f \right\|_{L^p(\mathbb{R}^d)} \le C \left\| \mathcal{L}_a^{s/2} f \right\|_{L^p(\mathbb{R}^d)};$$

(ii) *if* $1 < p < \infty$ *with* $\frac{d}{d-\sigma\vee 0} < p < \frac{d}{(\alpha s/2)\vee 0}$, *then*

$$\left\|\mathcal{L}_a^{s/2} f\right\|_{L^p(\mathbb{R}^d)} \le C\left\|(-\Delta)^{\alpha s/4} f\right\|_{L^p(\mathbb{R}^d)}.$$

We point out that the main reason for the restriction $a \ge 0$ in [28] is the essential use of the spectral multiplier theorem from [20], which requires a suitable polynomial decay on the heat kernel. We note indeed that when $a < 0$, the kernel of $\mathcal{L}_a$ fails to enjoy the Poisson upper bound, which would ensure a polynomial decay. In order to overcome the weak decay of the kernel, T. A. Bui and P. D'Ancona employ a new approach in [6]. This approach is quite similar to the method in [7]. The method in [7] was built upon the following heat kernel estimate,

$$\left\|e^{-t\mathcal{L}_a} f\right\|_{L^q(F)} \le Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} e^{-\frac{d(E,F)^2}{ct}} \|f\|_{L^p(E)}$$

for any measurable subsets $E, F \subset \mathbb{R}^d$, any $f \in L^p(E)$, and suitable $1 \le p \le q \le \infty$. For both approaches in [7] and [24], the exponential term plays an essential role in the above estimate. However, this type of estimate fails to be true in the case that $\alpha < 2$ (see [5, 6]). To deal with this obstructions, Bui and D'Ancona [6] proved the $L^p - L^q$ off-diagonal estimates on balls and their corresponding annuli.

Furthermore, the case of $w = v$ for Theorem 1.1 has been proved in [5]; see Lemma 2.16 below.

Lemma 2.16. *Let* $d \in \mathbb{N}$, $\alpha \in (0, 2\wedge d)$, *and* $s \in (0, 2]$. *Assume that* $a \in [a^*, \infty)$ *with* a^* *being as in* (1.2) *and* σ *is as in* (1.3).

(i) *Assume further that* $p \in (d'_\sigma, d_{\sigma+\alpha s/2})$ *and*

$$\omega \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \bigcap RH_{(\frac{d_{\sigma+\alpha s/2}}{p})'}(\mathbb{R}^d),$$

where d_σ *and* $d_{\sigma+s\alpha/2}$ *are as in* (1.4). *Then there exists a positive constant* C, *depending on* d, p, $[\omega]_{A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)}$, *and* $[\omega]_{RH_{(\frac{d_{\sigma+\alpha s/2}}{p})'}(\mathbb{R}^d)}$, *such that, for any* $f \in C_c^\infty(\mathbb{R}^d)$,

$$\left\|(-\Delta)^{\alpha s/4} f\right\|_{L^p_\omega(\mathbb{R}^d)} \le C\left\|\mathcal{L}_a^{s/2} f\right\|_{L^p_\omega(\mathbb{R}^d)}.$$

(ii) *Let* $\tilde{d} := \frac{d}{(\alpha s/2)\vee\sigma}$. *Assume that* $p \in (d'_\sigma, \tilde{d})$ *and*

$$\omega \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \bigcap RH_{(\frac{\tilde{d}}{p})'}(\mathbb{R}^d)$$

with d_σ *being as in* (1.4). *Then there exists a positive constant* C, *depending on* d, p, $[\omega]_{A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)}$, *and* $[\omega]_{RH_{(\frac{\tilde{d}}{p})'}(\mathbb{R}^d)}$, *such that, for any* $f \in C_c^\infty(\mathbb{R}^d)$,

$$\left\|\mathcal{L}_a^{s/2} f\right\|_{L^p_\omega(\mathbb{R}^d)} \le C\left\|(-\Delta)^{\alpha s/4} f\right\|_{L^p_\omega(\mathbb{R}^d)}.$$

Therefore, Theorem 1.1 can be considered as extensions of those obtained in [5, 6, 18, 28] to the case of two-weight.

As in [5], using the $L^p - L^q$ off-diagonal estimates on balls and their corresponding annuli obtained by T. A. Bui and P. D'Ancona in [6], together with some tools from harmonic analysis, such as the properties of Muckenhoupt weights, and the Minkowski integral inequality, we can obtain the weighted Sobolev norm estimates related to the generalized Hardy operator.

3. Two-weight Sobolev inequalities associated with $\mathcal{L}_a$

In this section, we give the proof of Theorem 1.1. The proof of Theorem 1.1 relies on some subtle ingredients, such as the weighted Hardy inequality related to the operator $\mathcal{L}_a$ and the weighted inequalities for the square function. We first establish the following weighted Hardy inequality.

Lemma 3.1. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2 \wedge d)$, $a \in [a^*, \infty)$ with a^* being as in* (1.2), *and σ be as in* (1.3). *Suppose further that $s \in (0, \infty)$ satisfies $s\alpha/2 \in (0, d)$, $p \in (d'_\sigma, d_{\sigma+s\alpha/2})$, and $\omega \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+s\alpha/2}}{p})'}(\mathbb{R}^d)$, where d_σ and $d_{\sigma+s\alpha/2}$ are as in* (1.4). *Then there exists a positive constant C, depending only on p, s, d, $[\omega]_{A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)}$, and $[\omega]_{RH_{(\frac{d_{\sigma+s\alpha/2}}{p})'}(\mathbb{R}^d)}$, such that, for any $f \in C_c^\infty(\mathbb{R}^d)$,*

$$\left\| |\cdot|^{-\alpha s/2} f \right\|_{L^p_\omega(\mathbb{R}^d)} \le C \left\| \mathcal{L}_a^{s/2} f \right\|_{L^p_\omega(\mathbb{R}^d)}.$$

Proof. Let $s\alpha/2 \in (0, d)$, $p \in (d'_\sigma, d_{\sigma+s\alpha/2})$, and $\omega \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+s\alpha/2}}{p})'}(\mathbb{R}^d)$. To show this lemma, it suffices to prove that, for any $g \in C_c^\infty(\mathbb{R}^d)$,

$$\left\| |\cdot|^{-\alpha s/2} \mathcal{L}_a^{-s/2} g \right\|_{L^p_\omega(\mathbb{R}^d)} \lesssim \|g\|_{L^p_\omega(\mathbb{R}^d)}. \tag{3.1}$$

Define the linear operator $T_{\mathcal{L}_a, s}$ by setting, for any $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$T_{\mathcal{L}_a, s} f(x) := |x|^{-\alpha s/2} \, \mathcal{L}_a^{-s/2} f(x). \tag{3.2}$$

By (iv) and (v) of Lemma 2.2, we find that there exist $d'_\sigma < p_0 < p < q_0 < d_{\sigma+s\alpha/2}$ such that $\omega \in A_{\frac{p}{p_0}}(\mathbb{R}^d) \cap RH_{(\frac{q_0}{p})'}(\mathbb{R}^d)$. Fix a ball $B := B(x_B, r_B) \subseteq \mathbb{R}^d$ and $m \in (d/\alpha + 1, +\infty)$, and let

$$\mathcal{A}_{r_B} := I - \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m.$$

To show (3.1) via applying Proposition 2.5, it suffices to prove that, for any $j \ge 2$ and any function f supported in B,

$$\left(\fint_{S_j(B)} |T_{\mathcal{L}_a, s} (I - \mathcal{A}_{r_B}) f|^{q_0} \, dx \right)^{1/q_0} \lesssim 2^{-(d+\alpha)j} \left(\fint_B |f|^{q_0} \, dx \right)^{1/q_0} \tag{3.3}$$

and

$$\left(\fint_{S_j(B)} |\mathcal{A}_{r_B} f|^{q_0}\, dx\right)^{1/q_0} \lesssim 2^{-(d+\alpha)j} \left(\fint_B |f|^{p_0}\, dx\right)^{1/p_0}. \tag{3.4}$$

We first prove the inequality (3.3). From the formula

$$\mathcal{L}_a^{-s/2} f = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2} e^{-t\mathcal{L}_a} f\, \frac{dt}{t},$$

and the fact that

$$I - \mathcal{A}_{r_B} = \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m,$$

it follows that

$$T_{\mathcal{L}_a,\, s}\,(I - \mathcal{A}_{r_B})\, f = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2} |\cdot|^{-\alpha s} e^{-t\mathcal{L}_a} \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f\, \frac{dt}{t}. \tag{3.5}$$

Then, applying the Minkowski inequality to (3.5), we obtain that

$$\begin{aligned}
&\left\| T_{\mathcal{L}_a,\, s} \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f \right\|_{L^{q_0}(S_j(B))} \\
&\lesssim \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2} \left\| |\cdot|^{-\alpha s} e^{-t\mathcal{L}_a} \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f \right\|_{L^{q_0}(S_j(B))} \frac{dt}{t} \\
&\lesssim \frac{1}{\Gamma(s/2)} \int_0^{r_B^\alpha} t^{s/2} \left\| |\cdot|^{-\alpha s} e^{-t\mathcal{L}_a} \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f \right\|_{L^{q_0}(S_j(B))} \frac{dt}{t} \\
&\quad + \frac{1}{\Gamma(s/2)} \int_{r_B^\alpha}^\infty t^{s/2} \left\| |\cdot|^{-\alpha s} e^{-t\mathcal{L}_a} \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f \right\|_{L^{q_0}(S_j(B))} \frac{dt}{t}
\end{aligned}$$

$$=: \mathrm{E}_1 + \mathrm{E}_2. \tag{3.6}$$

For the term E_1, we have that

$$\begin{aligned}
\mathrm{E}_1 &\lesssim \int_0^{r_B^\alpha} t^{s/2} \left\| |\cdot|^{-\alpha s} e^{-t\mathcal{L}_a} f \right\|_{L^{q_0}(S_j(B))} \frac{dt}{t} \\
&\quad + \sum_{k=1}^m \int_0^{r_B^\alpha} t^{s/2} \left\| |\cdot|^{-\alpha s} e^{-(t + k r_B^\alpha)\mathcal{L}_a} f \right\|_{L^{q_0}(S_j(B))} \frac{dt}{t}.
\end{aligned} \tag{3.7}$$

Since the associated kernel of the linear operator

$$f(x) \mapsto |x|^{-\alpha s/2}\, e^{-(t + k r_B^\alpha)\mathcal{L}_a} f(x)$$

is $|x|^{-\alpha s/2}\, p_{t + k r_B^\alpha}(x, y)$ for each $k \in \{0, 1, \ldots, m\}$, it follows, from Lemma 2.10, that the kernel $|x|^{-\alpha s/2} p_{t + k r_B^\alpha}(x, y)$ is dominated by

$$\begin{aligned}
&|x|^{-\alpha s/2}\, D_\sigma(x, t + k r_B^\alpha) D_\sigma(y, t + k r_B^\alpha)\, (t + k r_B^\alpha)^{-d/\alpha} \left[\frac{(t + k r_B^\alpha)^{1/\alpha} + |x - y|}{(t + k r_B^\alpha)^{1/\alpha}}\right]^{-d-\alpha} \\
&\lesssim (t + k r_B^\alpha)^{-\alpha s/2}\, D_{\sigma + \alpha s/2}(x, t + k r_B^\alpha) D_\sigma(y, t + k r_B^\alpha) \\
&\quad \times (t + k r_B^\alpha)^{-d/\alpha} \left[\frac{(t + k r_B^\alpha)^{1/\alpha} + |x - y|}{(t + k r_B^\alpha)^{1/\alpha}}\right]^{-d-\alpha}.
\end{aligned}$$

Therefore, by this, Lemma 2.9, and the fact that, for any $t \in (0, r_B^\alpha)$, $t + kr_B^\alpha \sim r_B^\alpha$, we find that

$$\left\| |\cdot|^{-\alpha s/2} e^{-t\mathcal{L}_a} f \right\|_{L^{q_0}(S_j(B))} \lesssim \left|2^j B\right|^{1/q_0} t^{-s/2} \left(\frac{r_B}{t^{1/\alpha}}\right)^d \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_B |f|^{q_0}\, dx\right)^{1/q_0}$$

and, for any $k \in \{1, 2, \ldots, m\}$,

$$\left\| |\cdot|^{-\alpha s/2} e^{-(t+kr_B^\alpha)\mathcal{L}_a} f \right\|_{L^{q_0}(S_j(B))} \lesssim \left|2^j B\right|^{1/q_0} r_B^{-s\alpha/2} 2^{-j(d+\alpha)} \left(\fint_B |f|^{q_0}\, dx\right)^{1/q_0}.$$

From this and (3.7), we deduce that

$$\begin{aligned} \mathrm{E}_1 &\lesssim \left|2^j B\right|^{1/q_0} \left(\fint_B |f|^{q_0}\, dx\right)^{1/q_0} \\ &\quad \times \left[\int_0^{r_B^\alpha} \left(\frac{r_B}{t^{1/\alpha}}\right)^d \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \frac{dt}{t} + \int_0^{r_B^\alpha} t^{s/2} r_B^{-s\alpha/2} 2^{-j(d+\alpha)} \frac{dt}{t}\right] \\ &\lesssim 2^{-j(d+\alpha)} \left|2^j B\right|^{1/q_0} \left(\fint_B |f|^{q_0}\, dx\right)^{1/q_0}. \end{aligned} \tag{3.8}$$

Now, we estimate the term E_2. By the facts that

$$\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m = \int_0^{r_B^\alpha} \cdots \int_0^{r_B^\alpha} \mathcal{L}_a^m e^{-(s_1+\cdots+s_m)\mathcal{L}_a}\, d\vec{s},$$

where $d\vec{s} := ds_1 \cdots ds_m$, and the associated kernel to the linear operator

$$f(x) \mapsto |x|^{-\alpha s/2} e^{-t\mathcal{L}_a} \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f(x)$$

is

$$\int_0^{r_B^\alpha} \cdots \int_0^{r_B^\alpha} |x|^{-\alpha s/2} p_{t+s_1+\cdots+s_m,m}(x,y)\, d\vec{s},$$

Proposition 2.12, and the fact that $t + s_1 + \cdots + s_m \sim t$ when $t \ge r_B^\alpha$ and $s_i \in (0, r_B^\alpha]$ for $i \in \{1, \ldots, m\}$, we conclude that $|x|^{-\alpha s/2} p_{t+s_1+\cdots+s_m,m}(x,y)$ is dominated by

$$\begin{aligned} &|x|^{-\alpha s/2} t^{-m} D_\sigma(x,t) D_\sigma(y,t) t^{-d/\alpha} \left[\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}}\right]^{-d-\alpha} \\ &\lesssim t^{-(s/2+m)} D_{\sigma+\alpha s/2}(x,t) D_\sigma(y,t) \left(t + kr_B^\alpha\right)^{-d/\alpha} \left[\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}}\right]^{-d-\alpha}. \end{aligned}$$

From this and Lemma 2.9, it follows that

$$\begin{aligned} &\left\| |\cdot|^{-\alpha s/2} e^{-(t+kr_B^\alpha)\mathcal{L}_a} \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f \right\|_{L^{q_0}(S_j(B))} \\ &\lesssim \left|2^j B\right|^{1/q_0} t^{-(s/2+m)} \left(\frac{r_B}{t^{1/\alpha}}\right)^{d/q_0} \left(1 + \frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q_0} \left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_B |f|^{q_0}\, dx\right)^{1/q_0}, \end{aligned}$$

which further implies that

$$
\begin{aligned}
\mathrm{E}_2 &\lesssim \left|2^j B\right|^{1/q_0} \left(\fint_B |f|^{q_0}\, dx \right)^{1/q_0} \\
&\quad \times \int_{r_B^\alpha}^{\infty} \left(\frac{r_B^\alpha}{t}\right)^m \left(\frac{r_B}{t^{1/\alpha}}\right)^{d/q_0} \left(1+\frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q_0} \left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \frac{dt}{t} \\
&\lesssim \left|2^j B\right|^{1/q_0} \left(\fint_B |f|^{q_0}\, dx \right)^{1/q_0} \\
&\quad \times \int_{r_B^\alpha}^{2^{\alpha j} {r_B}^{\alpha}} \left(\frac{r_B^\alpha}{t}\right)^m \left(\frac{r_B}{t^{1/\alpha}}\right)^{d/q_0} \left(1+\frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q_0} \left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \frac{dt}{t} \\
&\quad + \left|2^j B\right|^{1/q_0} \left(\fint_B |f|^{q_0}\, dx \right)^{1/q_0} \\
&\quad \times \int_{2^{\alpha j} r_B^\alpha}^{\infty} \left(\frac{r_B^\alpha}{t}\right)^m \left(\frac{r_B}{t^{1/\alpha}}\right)^{d/q_0} \left(1+\frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q_0} \left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \frac{dt}{t} \\
&\lesssim \left|2^j B\right|^{1/q_0} \left(\fint_B |f|^{q_0}\, dx \right)^{1/q_0} \int_{r_B^\alpha}^{2^{\alpha j} r_B^\alpha} \left(\frac{r_B^\alpha}{t}\right)^m \left(\frac{r_B}{t^{1/\alpha}}\right)^{d/q_0} \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \frac{dt}{t} \\
&\quad + \left|2^j B\right|^{1/q_0} \left(\fint_B |f|^{q_0}\, dx \right)^{1/q_0} \int_{2^{\alpha j} r_B^\alpha}^{\infty} \left(\frac{r_B^\alpha}{t}\right)^m \left(\frac{r_B}{t^{1/\alpha}}\right)^{d/q_0} \left(\frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q_0} \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \frac{dt}{t} \\
&\lesssim 2^{-j(d+\alpha)} \left|2^j B\right|^{1/q_0} \left(\fint_B |f|^{q_0}\, dx \right)^{1/q_0}.
\end{aligned}
$$

By this, (3.8), and (3.6), we find that (3.3) holds true.

Next, we show (3.4). Since

$$
\mathcal{A}_{r_B} = \sum_{k=1}^{m} (-1)^k \binom{m}{k} e^{-k r_B^\alpha \mathcal{L}_a},
$$

where $\binom{m}{k} := \frac{m!}{k!(m-k)!}$, it follows, from Lemma 2.10, that the kernel of $\mathcal{A}_{r_B}$ is dominated by

$$
D_\sigma(x, r_B) D_\sigma(y, r_B) {r_B}^{-d} \left(\frac{r_B + |x-y|}{r_B}\right)^{-d-\alpha}.
$$

Therefore, applying (2.3) in Lemma 2.9 and similar to the proof of (3.3), we prove (3.4). The details are omitted here. This finishes the proof of Lemma 3.1. □

Let $\gamma \in (0,\infty)$. Now, we consider the following square function that, for any $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$
S_{\mathcal{L}_a,\gamma} f(x) := \left(\int_0^\infty \left| (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}. \tag{3.9}
$$

We remark that, by the functional calculus theory in [27], we find that the square function $S_{\mathcal{L}_a,\gamma}$ is bounded on $L^2(\mathbb{R}^d)$. Moreover, the weighted $L^p(\mathbb{R}^d)$ boundedness of $S_{\mathcal{L}_a,\gamma}$ was given by Bui and Bui in [5].

Lemma 3.2. *Let $d\in\mathbb{N}$, $\gamma\in(0,\infty)$, $\alpha\in(0,2\wedge d)$, $a\in[a^*,\infty)$ with a^* being as in (1.2), and σ be as in (1.3). Assume further that $p\in(d'_\sigma,d_\sigma)$ and $\omega\in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)\cap RH_{(\frac{d_\sigma}{p})'}(\mathbb{R}^d)$, where d_σ is as in (1.4). Then there exists a positive constant C, depending on p, d, $[\omega]_{A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)}$, and $[\omega]_{RH_{(\frac{d_\sigma}{p})'}(\mathbb{R}^d)}$, such that, for any $f\in L^p_\omega(\mathbb{R}^d)$,*

$$C^{-1}\|f\|_{L^p_\omega(\mathbb{R}^d)}\le\|S_{\mathcal{L}_a,\gamma}f\|_{L^p_\omega(\mathbb{R}^d)}\le C\|f\|_{L^p_\omega(\mathbb{R}^d)}.$$

As a consequence, for any given $s\in(0,2]$, $p\in(d'_\sigma,d_\sigma)$, $\omega\in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)\cap RH_{(\frac{d_\sigma}{p})'}(\mathbb{R}^d)$, and any $f\in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$,

$$\left\|\left(\int_0^\infty t^{-s}\left|t\mathcal{L}_ae^{-t\mathcal{L}_a}f\right|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p_\omega(\mathbb{R}^d)}\le C\left\|\mathcal{L}_a^{s/2}f\right\|_{L^p_\omega(\mathbb{R}^d)},$$

where C is a positive constant independent of f.

Theorem 3.3. *Let $T_{\mathcal{L}_a,s}$ be as in (3.2) with $s\in(0,\infty)$ being as in Lemma 3.1, $d\in\mathbb{N}$, $\alpha\in(0,2\wedge d)$, $a\in[a^*,\infty)$ with a^* being as in (1.2), σ be as in (1.3), and $q\in(d'_\sigma,d_{\sigma+s\alpha/2})$, where d_σ and $d_{\sigma+s\alpha/2}$ are as in (1.4). Assume further that $d'_\sigma<p_0<q<q_0<d_{\sigma+\alpha s/2}$, the weights ω and v satisfy that $\omega\in RH_t(\mathbb{R}^d)$ with some $t\in((\frac{q_0}{q})',\infty]$, and*

$$\left[\omega,v^{1-(\frac{q}{p_0})'}\right]_{A_{\frac{q}{p_0}}(\mathbb{R}^d)}:=\sup_{B\subseteq\mathbb{R}^d}\left[\fint_B\omega\,dx\right]\left[\fint_B v^{1-(\frac{q}{p_0})'}\,dx\right]^{\frac{q}{p_0}-1}<\infty, \tag{3.10}$$

where the surpremum is taken over all balls B of $\mathbb{R}^d$. Then $T_{\mathcal{L}_a,s}$ is bounded from $L^q_v(\mathbb{R}^d)$ to $L^q_\omega(\mathbb{R}^d)$, and there exists a positive constant C, depending on q, d, s, and $[\omega]_{RH_t(\mathbb{R}^d)}$, such that, for any $f\in L^q_v(\mathbb{R}^d)$,

$$\|T_{\mathcal{L}_a,s}f\|_{L^q_\omega(\mathbb{R}^d)}\le C\|f\|_{L^q_v(\mathbb{R}^d)}.$$

Before proving Theorem 3.3, we need some conclusions which was proved in Yang and Yang [31].

Lemma 3.4. *Let $\gamma\in[0,1)$, $p_1,p_2,p_3\in(0,\infty]$ satisfy $p_3>p_1\vee p_2$, and $F,f\in L^1_{loc}(\mathbb{R}^d)$. Assume that, for any ball B of $\mathbb{R}^d$, there exist two measurable functions F_B and R_B on B such that $|F|\le|F_B|+|R_B|$ on B,*

$$\left(\fint_B|R_B|^{p_3}\,dx\right)^{1/p_3}\le C_1\left\{\left[\mathcal{M}\left(|f|^{p_1}\right)(x_1)\right]^{\frac{1}{p_1}}+\left[\mathcal{M}_\gamma\left(|f|^{p_2}\right)(x_2)\right]^{\frac{1}{p_2}}\right\} \tag{3.11}$$

with the usual modification made when $p_3=\infty$, and

$$\left(\fint_B|F_B|^{p_1}\,dx\right)^{1/p_1}\le\epsilon\left[\mathcal{M}\left(|f|^{p_1}\right)(x_1)\right]^{\frac{1}{p_1}}+C_2\left[\mathcal{M}_\gamma\left(|f|^{p_2}\right)(x_2)\right]^{\frac{1}{p_2}} \tag{3.12}$$

for any $x_1, x_2 \in B$, *where* C_1, C_2, *and* ϵ *are positive constants independent of the functions* F, f, R_B, F_B, *and* B. *Assume further that* $\omega \in RH_s(\mathbb{R}^d)$ *with some* $s \in (1, \infty]$, *and* $a \in (1, \frac{p_3}{p_1})$. *Then there exists a positive constant* $\beta_0 \in [1, \infty)$, *depending only on* C_1, C_2, n, p_1, p_2, p_3, a, *and* $[\omega]_{RH_s}(\mathbb{R}^d)$, *such that, for any given* $\beta \in [\beta_0, \infty)$, *there exist an* $\varepsilon_0 \in (0, \infty)$ *and a* $\kappa_0 \in (0, 1)$, *depending only on* C_1, C_2, n, p_1, p_2, p_3, a, $[\omega]_{RH_s(\mathbb{R}^n)}$, *and* β, *such that, if* $\epsilon \in [0, \epsilon_0)$ *and* $\kappa \in (0, \kappa_0)$, *then, for any* $\lambda \in (0, \infty)$,

$$\omega\left(E\left(\beta\lambda\right)\right) \leq \beta^{-\frac{(s-1)a}{s}} \omega\left(E\left(\lambda\right)\right) + \omega\left(\left\{x \in \mathbb{R}^d : \mathcal{M}_\gamma\left(|f|^{p_2}\right)(x) > (\kappa\lambda)^{\frac{p_2}{p_1}}\right\}\right), \tag{3.13}$$

where, for any given $\lambda \in (0, \infty)$,

$$E(\lambda) := \left\{x \in \mathbb{R}^d : \mathcal{M}\left(|F|^{p_1}\right)(x) > \lambda\right\}.$$

Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called a *Young function* if Φ is continuous, convex, strictly increasing, $\Phi(0) = 0$, and $\frac{\Phi(t)}{t} \to \infty$ as $t \to \infty$ (see, for instance, [13]). Moreover, it is said that a Young function Φ is *doubling* if there exists a positive constant C such that, for any $t \in [0, \infty)$, $\Phi(2t) \leq C\Phi(t)$.

Let Φ be a Young function and B a ball in $\mathbb{R}^n$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the *normalized Luxembourg norm* $\|f\|_{\Phi,\, B}$ of f on B is defined by setting

$$\|f\|_{\Phi,\, B} := \inf\left\{\lambda \in (0, \infty) : \fint_B \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Let $p \in (1, \infty)$, B be a ball in $\mathbb{R}^n$, and $\Phi(t) := t^p$ for any $t \in [0, \infty)$. Then Φ is a Young function and, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|f\|_{\Phi,\, B} = \left(\fint_B |f|^p \, dx\right)^{\frac{1}{p}} =: \|f\|_{p,\, B}.$$

Lemma 3.5. *Let* $p_1, p_2, p_3 \in (0, \infty]$ *satisfy* $p_3 \in (p_1 \vee p_2, \infty)$, $q \in (p_1 \vee p_2, p_3)$, Φ *be a doubling Young function satisfying*

$$\int_c^\infty \left[\frac{t^{(\frac{q}{p_2})'}}{\Phi(t)}\right]^{\frac{q}{p_2}-1} \frac{dt}{t} < \infty \tag{3.14}$$

for some constant $c \in (0, \infty)$, *and* $F, f \in L^1_{loc}(\mathbb{R}^d)$. *Assume that the weights* ω *and* v *satisfy that* $\omega \in RH_s(\mathbb{R}^d)$ *with some* $s \in ((\frac{p_3}{q})', \infty]$, *and*

$$\sup_{B \subseteq \mathbb{R}^d} \left(\fint_B \omega dx\right) \left\|v^{-\frac{p_2}{q}}\right\|_{\Phi, B}^{\frac{q}{p_2}} < \infty, \tag{3.15}$$

where the supremum is taken over all balls B *in* $\mathbb{R}^d$. *Assume further that* F *and* f *satisfy* (3.11) *and* (3.12) *with* $\gamma = 0$ *and some* $\epsilon \in (0, \infty)$ *such that* (3.13) *holds true. Then there exists a positive constant* C, *independent of* F *and* f, *such that*

$$\|F\|_{L^q_\omega(\mathbb{R}^d)} \leq C \|f\|_{L^q_v(\mathbb{R}^d)}.$$

Next, we show Theorem 3.3 by using Lemmas 3.4 and 3.5.

Proof of Theorem 3.3. Assume that $B := B(x_B, r_B)$, with $x_B \in \mathbb{R}^d$ and $r_B \in (0,\infty)$, is a ball of $\mathbb{R}^d$, $m > d/\alpha q_0' + 1$, and $f \in L_c^\infty(\mathbb{R}^d)$. Let $F := T_{\mathcal{L}_a,s}(f)$, $F_B := T_{\mathcal{L}_a,s}(I - e^{-r_B^\alpha \mathcal{L}_a})^m(f)$, and $R_B := T_{\mathcal{L}_a,s}[I - (I - e^{-r_B^\alpha \mathcal{L}_a})^m](f)$. Then $|F| \le |F_B| + |R_B|$ on B.

Now, let $d_\sigma' < \bar{p} < p_0 < q_0 < \bar{q} < d_{\sigma+s\alpha/2}$, where $\bar{p}$, $\bar{q} \in (1,\infty)$. By (3.10), we conclude that (3.14) and (3.15) hold true for $\Phi(t) := t^{(\frac{q_0}{q})'}$, $p_1 = p_2 := \bar{p}$, and $p_3 := \bar{q}$.

To show Theorem 3.3, by Lemma 3.5, it suffices to prove that, for any $f \in C_c^\infty(\mathbb{R}^d)$ and any $x_1 \in B$,

$$\left(\fint_B |F_B|^{\bar{p}}\,dx\right)^{1/\bar{p}} \lesssim \sum_{j=1}^\infty g_1(j)\left(\fint_{2^{j+1}B} |f|^{\bar{p}}\,dx\right)^{1/\bar{p}} \lesssim \left[\mathcal{M}\left(|f|^{\bar{p}}\right)(x_1)\right]^{\frac{1}{\bar{p}}}, \tag{3.16}$$

where $\sum_{j=1}^\infty g_1(j) \lesssim 1$, and, for any $x_2 \in B$,

$$\left(\fint_B |R_B|^{\bar{q}}\ dx\right)^{1/\bar{q}} \lesssim \sum_{j=1}^\infty g_2(j)\left(\fint_B |f|^{\bar{p}}\ dx\right)^{1/\bar{p}} \lesssim \left[\mathcal{M}\left(|f|^{\bar{p}}\right)(x_2)\right]^{\frac{1}{\bar{p}}}, \tag{3.17}$$

where $\sum_{j=1}^\infty g_2(j) \lesssim 1$.

To obtain (3.16), we first have

$$\begin{aligned}\left(\fint_B |F_B|^{\bar{p}}\,dx\right)^{1/\bar{p}} &= \left[\fint_B \left|T_{\mathcal{L}_a,s}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f)\right|^{\bar{p}}\ dx\right]^{1/\bar{p}}\\ &\lesssim \sum_{j=1}^\infty \left[\fint_B \left|T_{\mathcal{L}_a,s}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f_j)\right|^{\bar{p}}\ dx\right]^{1/\bar{p}},\end{aligned} \tag{3.18}$$

where, for any $j \in \mathbb{N}$, $f_j := f\chi_{S_j(B)}$. By the formula

$$\mathcal{L}_a^{-s/2} = \frac{1}{\Gamma(s/2)}\int_0^\infty t^{s/2} e^{-t\mathcal{L}_a}\frac{dt}{t},$$

and the fact that

$$I - \mathcal{A}_{r_B} = \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m,$$

we conclude that

$$T_{\mathcal{L}_a,s}\left(I - \mathcal{A}_{r_B}\right) f_j(x) = \frac{1}{\Gamma(s/2)}\int_0^\infty t^{s/2}|x|^{-\alpha s} e^{-t\mathcal{L}_a}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f_j(x)\,\frac{dt}{t}.$$

Applying the Minkowski inequality to (3.18), we obtain that

$$\begin{aligned}&\left\|T_{\mathcal{L}_a,s}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f_j\right\|_{L^{\bar{p}}(B)}\\ &\lesssim \frac{1}{\Gamma(s/2)}\int_0^\infty t^{s/2}\left\||\cdot|^{-\alpha s} e^{-t\mathcal{L}_a}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f_j\right\|_{L^{\bar{p}}(B)}\frac{dt}{t}\\ &\lesssim \frac{1}{\Gamma(s/2)}\int_0^{r_B^\alpha} t^{s/2}\left\||\cdot|^{-\alpha s} e^{-t\mathcal{L}_a}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m f_j\right\|_{L^{\bar{p}}(B)}\frac{dt}{t}\end{aligned}$$

$$+\frac{1}{\Gamma(s/2)}\int_{r_B^\alpha}^\infty t^{s/2}\left\||\cdot|^{-\alpha s}e^{-t\mathcal{L}_a}\left(I-e^{-r_B^\alpha\mathcal{L}_a}\right)^m f_j\right\|_{L^{\bar p}(B)}\frac{dt}{t}$$
$$=: \mathrm{F}_1+\mathrm{F}_2.$$

For the term F_1, we find that

$$\mathrm{F}_1 \lesssim \int_0^{r_B^\alpha} t^{s/2}\left\||\cdot|^{-\alpha s}e^{-t\mathcal{L}_a}f_j\right\|_{L^{\bar p}(B)}\frac{dt}{t}$$
$$+\sum_{k=1}^m\int_0^{r_B^\alpha} t^{s/2}\left\||\cdot|^{-\alpha s}e^{-(t+kr_B^\alpha)\mathcal{L}_a}f_j\right\|_{L^{\bar p}(B)}\frac{dt}{t}. \tag{3.19}$$

Since the associated kernel of the linear operator

$$f(x)\mapsto |x|^{-\alpha s/2}e^{-(t+kr_B^\alpha)\mathcal{L}_a}f(x)$$

is $|x|^{-\alpha s/2}p_{t+kr_B^\alpha}(x,y)$ for each $k\in\{0,1,\dots,m\}$, it follows, from Lemma 2.10, that the kernel $|x|^{-\alpha s/2}p_{t+kr_B^\alpha}(x,y)$ is dominated by

$$C|x|^{-\alpha s/2}D_\sigma(x,t+kr_B^\alpha)D_\sigma(y,t+kr_B^\alpha)\,(t+kr_B^\alpha)^{-d/\alpha}\left[\tfrac{(t+kr_B^\alpha)^{1/\alpha}+|x-y|}{(t+kr_B^\alpha)^{1/\alpha}}\right]^{-d-\alpha}$$
$$\lesssim (t+kr_B^\alpha)^{-\alpha s/2}\,D_{\sigma+\alpha s/2}(x,t+kr_B^\alpha)D_\sigma(y,t+kr_B^\alpha)\,(t+kr_B^\alpha)^{-d/\alpha}$$
$$\times\left[\tfrac{(t+kr_B^\alpha)^{1/\alpha}+|x-y|}{(t+kr_B^\alpha)^{1/\alpha}}\right]^{-d-\alpha}.$$

Therefore, applying Lemma 2.9, we conclude that

$$\left\||\cdot|^{-\alpha s/2}e^{-t\mathcal{L}_a}f_j\right\|_{L^{\bar p}(B)}$$
$$\lesssim |B|^{1/\bar p}\,t^{-s/2}\left(\frac{2^jr_B}{t^{1/\alpha}}\right)^d\left(\frac{2^jr_B}{t^{1/\alpha}}\right)^{-d-\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}$$
$$\lesssim |B|^{1/\bar p}t^{-s/2}\left(\frac{2^jr_B}{t^{1/\alpha}}\right)^{-\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}$$

and, for any $k\in\{1,2,\dots,m\}$ and $t\in(0,r_B^\alpha)$,

$$\left\||\cdot|^{-\alpha s/2}e^{-(t+kr_B^\alpha)\mathcal{L}_a}f_j\right\|_{L^{\bar p}(B)}\lesssim |B|^{1/\bar p}r_B^{-s\alpha/2}2^{jd}2^{-j(d+\alpha)}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}$$
$$\sim |B|^{1/\bar p}r_B^{-s\alpha/2}2^{-j\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p},$$

which, combined with (3.19), further implies that

$$\mathrm{F}_1\lesssim |B|^{1/\bar p}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}\left[\int_0^{r_B^\alpha}\left(\frac{2^jr_B}{t^{1/\alpha}}\right)^{-\alpha}\frac{dt}{t}+\int_0^{r_B^\alpha}t^{s/2}r_B^{-s\alpha/2}2^{-j\alpha}\frac{dt}{t}\right]$$

$$\lesssim 2^{-j\alpha} |B|^{1/\bar{p}} \left(\fint_{S_j(B)} |f_j|^{\bar{p}} dx \right)^{1/\bar{p}}.$$

Next, we estimate F_2. Using the facts that

$$\left(I - e^{-r_B^\alpha \mathcal{L}_a} \right)^m = \int_0^{r_B^\alpha} \cdots \int_0^{r_B^\alpha} \mathcal{L}_a^m e^{-(s_1+\cdots+s_m)\mathcal{L}_a} d\vec{s},$$

where $d\vec{s} := ds_1 \cdots ds_m$, and the associated kernel to the linear operator

$$f(x) \mapsto |x|^{-\alpha s/2} e^{-t\mathcal{L}_a} \left(I - e^{-r_B^\alpha \mathcal{L}_a} \right)^m f(x)$$

is

$$\int_0^{r_B^\alpha} \cdots \int_0^{r_B^\alpha} |x|^{-\alpha s/2} p_{t+s_1+\cdots+s_m,m}(x,y)\, d\vec{s},$$

from Proposition 2.12 and the fact that $t + s_1 + \cdots + s_m \sim t$ for $t \geq r_B^\alpha$ and $s_i \in (0, r_B^\alpha]$ for any $i \in \{1, \ldots, m\}$, we deduce that $|x|^{-\alpha s/2} p_{t+s_1+\cdots+s_m,m}(x,y)$ is dominated by

$$\begin{aligned} &C\, |x|^{-\alpha s/2}\, t^{-m} D_\sigma(x,t) D_\sigma(y,t) t^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha} \\ &\lesssim t^{-(s/2+m)} D_{\sigma+\alpha s/2}(x,t) D_\sigma(y,t)\, (t + k r_B^\alpha)^{-d/\alpha} \left(\frac{t^{1/\alpha} + |x-y|}{t^{1/\alpha}} \right)^{-d-\alpha}. \end{aligned}$$

Then, applying Lemma 2.9, we find that

$$\begin{aligned} &\left\| |\cdot|^{-\alpha s/2} e^{-(t+kr_B^\alpha)\mathcal{L}_a} \left(I - e^{-r_B^\alpha \mathcal{L}_a} \right)^m f_j \right\|_{L^{\bar{p}}(B)} \\ &\lesssim\ |B|^{1/\bar{p}}\, t^{-(s/2+m)} \left(\frac{2^j r_B}{t^{1/\alpha}} \right)^{d/\bar{p}} \left(1 + \frac{t^{1/\alpha}}{r_B} \right)^{d/\bar{p}} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \left(\fint_{S_j(B)} |f_j|^{\bar{p}}\, dx \right)^{1/\bar{p}}. \end{aligned}$$

From this, we deduce that

$$\begin{aligned} \mathrm{F}_2 &\lesssim |B|^{1/\bar{p}} \left(\fint_{S_j(B)} |f_j|^{\bar{p}}\, dx \right)^{1/\bar{p}} \\ &\quad \times \int_{r_B^\alpha}^\infty \left(\frac{r_B^\alpha}{t} \right)^m \left(\frac{2^j r_B}{t^{1/\alpha}} \right)^{d/\bar{p}} \left(1 + \frac{t^{1/\alpha}}{r_B} \right)^{d/\bar{p}} \left(1 + \frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \frac{dt}{t} \\ &\lesssim |B|^{1/\bar{p}} \left(\fint_{S_j(B)} |f|^{\bar{p}}\ dx \right)^{1/\bar{p}} \int_{r_B^\alpha}^\infty \left(\frac{r_B^\alpha}{t} \right)^m \left(\frac{2^j r_B}{t^{1/\alpha}} \right)^{d/\bar{p}} \left(\frac{t^{1/\alpha}}{r_B} \right)^{d/\bar{p}} \left(\frac{2^j r_B}{t^{1/\alpha}} \right)^{-d-\alpha} \frac{dt}{t} \\ &\lesssim 2^{-j\alpha} |B|^{1/\bar{p}} \left(\fint_{S_j(B)} |f_j|^{\bar{p}}\, dx \right)^{1/\bar{p}}. \end{aligned}$$

In combination with the estimates of F_1 and F_2, we complete the proof of (3.16).

Next, we prove inequality (3.17). By Lemma 2.13, we find that

$$\begin{aligned}
\left(\fint_B |R_B|^{\bar{q}}\, dx\right)^{1/\bar{q}} &= \left(\fint_B |T_{\mathcal{L}_a,s}\mathcal{A}_{r_B}(f)|^{\bar{q}}\, dx\right)^{1/\bar{q}}\\
&\lesssim \left(\int_{\mathbb{R}^d} \frac{1}{|B|}\,|\mathcal{A}_{r_B}(f)|^{\bar{q}}\, dx\right)^{1/\bar{q}}\\
&\lesssim \sum_{j=1}^{\infty}\left(\int_{S_j(B)} \frac{1}{|B|}\,|\mathcal{A}_{r_B}(f)|^{\bar{q}}\, dx\right)^{1/\bar{q}}\\
&\lesssim \sum_{j=1}^{\infty}\left(\fint_{S_j(B)} 2^{jd}\,|\mathcal{A}_{r_B}(f)|^{\bar{q}}\, dx\right)^{1/\bar{q}}. \qquad (3.20)
\end{aligned}$$

Applying (3.4) to (3.20), we conclude that

$$\begin{aligned}
\left(\fint_B |R_B|^{\bar{q}}\, dx\right)^{1/\bar{q}} &\lesssim \sum_{j=1}^{\infty}\left(\fint_{S_j(B)} 2^{jd}\,|\mathcal{A}_{r_B}(f)|^{\bar{q}}\, dx\right)^{1/\bar{q}}\\
&\lesssim \sum_{j=1}^{\infty} 2^{jd/\bar{q}}2^{-(d+\alpha)j}\left(\fint_B |f|^{\bar{p}}\, dx\right)^{1/\bar{p}} \lesssim \left[\mathcal{M}\left(|f|^{\bar{p}}\right)(x_2)\right]^{\frac{1}{\bar{p}}},
\end{aligned}$$

which completes the proof of (3.17).

From (3.16) and (3.17), it follows that (3.11) and (3.12) hold true for $p_1 = p_2 := \bar{p}$, $p_3 := \bar{q}$, and $\epsilon := 0$. Thus, applying Lemma 3.5, we finish the proof of Theorem 3.3. ◻

We now prove the two-weight boundedness of $S_{\mathcal{L}_a,\gamma}$ in the scale of weighted Lebesgue spaces.

Theorem 3.6. *Let $\gamma \in (0,\infty)$, $S_{\mathcal{L}_a,\gamma}$ be as in* (3.9)*, $d \in \mathbb{N}$, $\alpha \in (0, 2\wedge d)$, $a \in [a^*,\infty)$ with a^* being as in* (1.2)*, σ be as in* (1.3)*, and $q \in (d'_\sigma, d_\sigma)$ with d_σ being as in* (1.4)*. Assume further that $d'_\sigma < p_0 < q < q_0 < d_\sigma$ and the weights ω and v satisfy that $\omega \in RH_s(\mathbb{R}^d)$ with some $s \in ((\frac{q_0}{q})',\infty]$ and,*

$$\left[\omega, v^{1-(\frac{q}{p_0})'}\right]_{A_{\frac{q}{p_0}}(\mathbb{R}^d)} := \sup_{B\subseteq\mathbb{R}^d}\left[\fint_B \omega\, dx\right]\left[\fint_B v^{1-(\frac{q}{p_0})'}\, dx\right]^{\frac{q}{p_0}-1} < \infty, \qquad (3.21)$$

where the surpremum is taken over all balls B of $\mathbb{R}^d$. Then $S_{\mathcal{L}_a,\gamma}$ is bounded from $L^q_v(\mathbb{R}^d)$ to $L^q_\omega(\mathbb{R}^d)$, and there exists a positive constant C such that, for any $f \in L^q_v(\mathbb{R}^d)$,

$$\|S_{\mathcal{L}_a,\gamma}f\|_{L^q_\omega(\mathbb{R}^d)} \le C\|f\|_{L^q_v(\mathbb{R}^d)}.$$

As a consequence, for any $s \in (0,2]$ and $p \in (d'_\sigma, d_\sigma)$,

$$\left\|\left(\int_0^\infty t^{-s}\left|t\mathcal{L}_a e^{-t\mathcal{L}_a}f\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^q_\omega(\mathbb{R}^d)} \le C\left\|\mathcal{L}_a^{s/2}f\right\|_{L^q_v(\mathbb{R}^d)}.$$

Proof. Assume that $B := B(x_B, r_B)$, with $x_B \in \mathbb{R}^d$ and $r_B \in (0,\infty)$, is a ball of $\mathbb{R}^d$, $m > d/\alpha + 1$, and $f \in C_c^\infty(\mathbb{R}^d)$.

Let $F := S_{\mathcal{L}_a,\gamma}(f)$,

$$F_B := S_{\mathcal{L}_a,\gamma}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f),$$

and

$$R_B := S_{\mathcal{L}_a,\gamma}\left[I - \left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m\right](f).$$

Then $|F| \le |F_B| + |R_B|$ on B. Let $d'_\sigma < \bar{p} < p_0 < q_0 < \bar{q} < d_\sigma$, where $\bar{p}$, $\bar{q} \in (1,\infty)$. By (3.21), we conclude that (3.14) and (3.15) hold true for $\Phi(t) := t^{(\frac{q_0}{q})'}$, $p_1 = p_2 := \bar{p}$, and $p_3 := \bar{q}$.

To show Theorem 3.6, by Lemma 3.5, it suffices to prove that, for any $f \in C_c^\infty(\mathbb{R}^d)$ and any $x_1 \in B$,

$$\left(\fint_B |F_B|^{\bar{p}}\, dx\right)^{1/\bar{p}} \lesssim \sum_{j=1}^\infty g_1(j)\left(\fint_{2^{j+1}B} |f|^{\bar{p}}\, dx\right)^{1/\bar{p}} \lesssim \left[\mathcal{M}\left(|f|^{\bar{p}}\right)(x_1)\right]^{\frac{1}{\bar{p}}}, \tag{3.22}$$

where $\sum_{j=1}^\infty g_1(j) \lesssim 1$, and for any $x_2 \in B$,

$$\left(\fint_B |R_B|^{\bar{q}}\, dx\right)^{1/\bar{q}} \lesssim \sum_{j=1}^\infty g_2(j)\left(\fint_B |f|^{\bar{p}}\, dx\right)^{1/\bar{p}} \lesssim \left[\mathcal{M}\left(|f|^{\bar{p}}\right)(x_2)\right]^{\frac{1}{\bar{p}}}, \tag{3.23}$$

where $\sum_{j=1}^\infty g_2(j) \lesssim 1$.

We prove (3.22) by considering the following two cases.

Case 1. $\bar{p} \le 2$. In this case, by the Hölder inequality, we have

$$\begin{aligned}\left(\fint_B |F_B|^{\bar{p}}\, dx\right)^{1/\bar{p}} &= \left(\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f)\right|^{\bar{p}} dx\right)^{1/\bar{p}}\\ &\lesssim \left(\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f)\right|^{2} dx\right)^{1/2}.\end{aligned}$$

Thus, it remains to show that, for any $x_1 \in B$,

$$\left[\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f)\right|^{2} dx\right]^{1/2} \lesssim \left[\mathcal{M}\left(|f|^{\bar{p}}\right)(x_1)\right]^{1/\bar{p}}.$$

We observe that

$$\begin{aligned}&\left[\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f)\right|^{2} dx\right]^{1/2}\\ &\quad\lesssim \sum_{j=1}^\infty \left[\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f_j)\right|^{2} dx\right]^{1/2},\end{aligned}$$

where, for any $j \in \mathbb{N}$, $f_j := f\chi_{S_j(B)}$. Meanwhile, note that, for every $g \in L^2(\mathbb{R}^d)$ and $s \in (0,\infty)$,

$$\int_t^\infty \mathcal{L}_a^{s+1} e^{-\tau\mathcal{L}_a} g\, d\tau = \mathcal{L}_a^s e^{-t\mathcal{L}_a} g,$$

which, combined with Proposition 2.11, implies that

$$\left\|\mathcal{L}_a^s e^{-t\mathcal{L}_a} g\right\|_{L^2(B)} \le \int_t^\infty \left\|\mathcal{L}_a^{s+1} e^{-\tau\mathcal{L}_a} g\right\|_{L^2(B)} \, d\tau.$$

By this, we conclude that

$$\begin{aligned}
&\left\{\int_0^\infty \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a} g\right\|_{L^2(B)}^2 \frac{dt}{t}\right\}^{1/2} \\
&\le \left[\int_0^\infty t^{2\gamma} \left(\int_t^\infty \left\|\mathcal{L}_a^{s+1} e^{-\tau\mathcal{L}_a} g\right\|_{L^2(B)}^2 \, d\tau\right)^2 \frac{dt}{t}\right]^{1/2} \\
&\le \left[\int_0^\infty \left\|(\tau\mathcal{L}_a)^{s+1} e^{-\tau\mathcal{L}_a} g\right\|_{L^2(B)}^2 \frac{d\tau}{\tau}\right]^{1/2}.
\end{aligned}$$

From now on, let $\gamma > 1 + 2\,[0 \vee (\sigma/\alpha)]$. For $j = 0$, from the $L^2(\mathbb{R}^d)$-boundedness of $S_{\mathcal{L}_a,\gamma}$ and $\mathcal{A}_{r_B}$, we deduce that, for any $x_1 \in B$,

$$\left(\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f_0)\right|^2 dx\right)^{1/2} \lesssim \left[\mathcal{M}\left(|f|^{\bar{p}}\right)(x_1)\right]^{1/\bar{p}}.$$

For $j \ge 1$, applying the Minkowski inequality, we find that

$$\begin{aligned}
&\left[\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I - e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f_j)\right|^2 dy\right]^{1/2} \\
&\lesssim |B|^{-1/2}\left(\int_0^\infty \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I - \mathcal{A}_{r_B}\right) f_j\right\|_{L^2(B)}^2 \frac{dt}{t}\right)^{1/2} \\
&\lesssim |B|^{-1/2}\left(\int_0^{r_B^\alpha} \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I - \mathcal{A}_{r_B}\right) f_j\right\|_{L^2(B)}^2 \frac{dt}{t}\right)^{1/2} \\
&\quad + |B|^{-1/2}\left(\int_{r_B^\alpha}^\infty \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I - \mathcal{A}_{r_B}\right) f_j\right\|_{L^2(B)}^2 \frac{dt}{t}\right)^{1/2} \\
&=: \mathrm{E}_1 + \mathrm{E}_2.
\end{aligned}$$

Since

$$\begin{aligned}
(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I - \mathcal{A}_{r_B}\right) &= (t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I - e^{-t\mathcal{L}_a}\right)^m \\
&= \sum_{k=0}^m (-1)^k \binom{m}{k} (t\mathcal{L}_a)^\gamma e^{-(t+kr_B^\alpha)\mathcal{L}_a} \\
&= (t\mathcal{L}_a)^\gamma e^{-t_B\mathcal{L}_a} + \sum_{k=1}^m (-1)^k \binom{m}{k} (t\mathcal{L}_a)^\gamma e^{-(t+kr_B^\alpha)\mathcal{L}_a},
\end{aligned}$$

it follows, from Proposition 2.12 with $\gamma > 1 + 2[0 \vee (\sigma/\alpha)]$ and the fact that $t + kr_B^\alpha \sim r_B^\alpha$ for any $t \in (0, r_B^\alpha)$ and $k \ge 1$, that the kernel of $(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}(I -$

$\mathcal{A}_{r_B}$) is dominated by

$$Ct^\gamma D_\sigma(x,t)D_\sigma(y,t)t^{-(\gamma+d/\alpha)}\left(\frac{t^{1/\alpha}+|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha}$$
$$+\,Ct^\gamma D_\sigma(x,r_B^\alpha)D_\sigma(y,r_B^\alpha)\,(r_B^\alpha)^{-(\gamma+d/\alpha)}\left(\frac{r_B+|x-y|}{r_B}\right)^{-d-\alpha}.$$

Consequently, applying (2.4) in Lemma 2.9, we infer that

$$\begin{aligned}
&|B|^{-1/2}\left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I-\mathcal{A}_{r_B}\right)f_j\right\|_{L^2(B)}\\
&\lesssim \left(\frac{2^jr_B}{t^{1/\alpha}}\right)^d\left(1+\frac{t^{1/\alpha}}{r_B}\right)^{d/2}\left(1+\frac{2^jr_B}{t^{1/\alpha}}\right)^{-d-\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}\\
&\quad+\left(\frac{t}{r_B^\alpha}\right)^\gamma(2^j)^d2^{d/2}\left(1+2^j\right)^{-d-\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}\\
&\lesssim \left(\frac{2^jr_B}{t^{1/\alpha}}\right)^d\left(\frac{2^jr_B}{t^{1/\alpha}}\right)^{-d-\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}\\
&\quad+\left(\frac{t}{r_B^\alpha}\right)^\gamma 2^{-j\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p},
\end{aligned}$$

which further implies that

$$\mathrm{E}_1\lesssim 2^{-j\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}.$$

Next, we estimate E_2. Note first that

$$\left(I-e^{-r_B^\alpha\mathcal{L}_a}\right)^m=\int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha}\mathcal{L}_a^m e^{-(s_1+\cdots+s_m)\mathcal{L}_a}\,d\vec{s},$$

where $d\vec{s}:=ds_1\cdots ds_m$. Therefore,

$$\begin{aligned}
&\left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I-\mathcal{A}_{r_B}\right)f_j\right\|_{L^2(B)}\\
&\le\int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha}\left\|t^\gamma{\mathcal{L}_a}^{\gamma+m}e^{-(t+s_1+\cdots+s_m)\mathcal{L}_a}f_j\right\|_{L^2(B)}\,d\vec{s}.
\end{aligned}$$

From this, Proposition 2.12, and the fact that $t+s_1+\cdots+s_m\sim t$ for any $t\in[r_B^\alpha,\infty)$ and $s_i\in(0,r_B^\alpha]$ with $i\in\{1,\dots,m\}$, we deduce that

$$\begin{aligned}
&|B|^{-1/2}\left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I-\mathcal{A}_{r_B}\right)f_j\right\|_{L^2(B)}\\
&\lesssim\int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha}t^{-m}\left(\frac{2^jr_B}{t^{1/\alpha}}\right)^{d/2}\left(1+\frac{t^{1/\alpha}}{r_B}\right)^{d/2}\left(1+\frac{2^jr_B}{t^{1/\alpha}}\right)^{-d-\alpha}\left(\fint_{S_j(B)}|f_j|^{\bar p}\,dx\right)^{1/\bar p}d\vec{s}
\end{aligned}$$

$$\lesssim \left(\frac{r_B^\alpha}{t}\right)^m \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d/2} \left(1+\frac{t^{1/\alpha}}{r_B}\right)^{d/2} \left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_{S_j(B)} |f_j|^{\bar{p}}\,dx\right)^{1/\bar{p}},$$

which further implies that

$$\begin{aligned}
\mathrm{E}_2 &\lesssim \int_{r_B^\alpha}^\infty \left(\frac{r_B^\alpha}{t}\right)^m \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d/2} \left(1+\frac{t^{1/\alpha}}{r_B}\right)^{d/2} \left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_{S_j(B)} |f_j|^{\bar{p}}\,dx\right)^{1/\bar{p}} \frac{dt}{t}\\
&\lesssim \int_{r_B^\alpha}^\infty \left(\frac{r_B^\alpha}{t}\right)^m \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{d/2} \left(\frac{t^{1/\alpha}}{r_B}\right)^{d/2} \left(\frac{2^j r_B}{t^{1/\alpha}}\right)^{-d-\alpha} \left(\fint_{S_j(B)} |f_j|^{\bar{p}}\,dx\right)^{1/\bar{p}} \frac{dt}{t}\\
&\lesssim 2^{-j(d/2+\alpha)} \left(\fint_{S_j(B)} |f_j|^{\bar{p}}\,dx\right)^{1/\bar{p}}
\end{aligned}$$

as $m > d/\alpha + 1$. By the estimates of E_1 and E_2, we conclude that, for any $x_1 \in B$,

$$\left[\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I-e^{-r_B^\alpha \mathcal{L}_{a,s}}\right)^m (f)\right|^2 dx\right]^{1/2} \lesssim \left[\mathcal{M}\left(|f|^{p_0}\right)(x_1)\right]^{1/p_0}.$$

Case 2. $\bar{p} > 2$. In this case, applying the Minkowski inequality, we obtain that

$$\begin{aligned}
&\left[\fint_B \left|S_{\mathcal{L}_a,\gamma}\left(I-e^{-r_B^\alpha \mathcal{L}_a}\right)^m (f)\right|^{\bar{p}} dx\right]^{1/\bar{p}}\\
&\lesssim |B|^{-1/2}\left[\int_0^\infty \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I-e^{-r_B^\alpha \mathcal{L}_a}\right)^m f_j\right\|_{L^{\bar{p}}(B)}^2 \frac{dt}{t}\right]^{1/2}\\
&\lesssim |B|^{-1/2}\left[\int_0^{r_B^\alpha} \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I-e^{-r_B^\alpha \mathcal{L}_a}\right)^m f_j\right\|_{L^{\bar{p}}(B)}^2 \frac{dt}{t}\right]^{1/2}\\
&\quad + |B|^{-1/2}\left[\int_{r_B^\alpha}^\infty \left\|(t\mathcal{L}_a)^\gamma e^{-t\mathcal{L}_a}\left(I-e^{-r_B^\alpha \mathcal{L}_a}\right)^m f_j\right\|_{L^{\bar{p}}(B)}^2 \frac{dt}{t}\right]^{1/2}\\
&\lesssim 2^{-j\alpha}\left(\fint_{S_j(B)} |f_j|^{\bar{p}}\,dx\right)^{1/\bar{p}} + 2^{-j(d/2+\alpha)}\left(\fint_{S_j(B)} |f_j|^{\bar{p}}\,dx\right)^{1/\bar{p}}.
\end{aligned}$$

The remainder proof in the case of $\bar{p} > 2$ is similar to that in the case of $\bar{p} \le 2$, and we omit the details. In combination with the estimates in both Cases 1 and 2, we then finish the proof of (3.22).

Next, we prove that (3.23). Using Lemma 3.2 and (3.4), by suitable modification for the proof of (3.17), we can show that, for any $x_2 \in B$,

$$\left(\fint_B |R_B|^{\bar{q}}\,dx\right)^{1/\bar{q}} = \left[\fint_B \left|S_{\mathcal{L}_a,\gamma}\mathcal{A}_{r_B}(f)\right|^{\bar{q}} dx\right]^{1/\bar{q}}$$

$$
\begin{aligned}
&\lesssim \left[\int_{\mathbb{R}^d} \frac{1}{|B|} \left|\mathcal{A}_{r_B}(f)\right|^{\bar{q}} dx\right]^{1/\bar{q}}\\
&\lesssim \sum_{j=1}^{\infty} \left[\int_{S_j(B)} \frac{1}{|B|} \left|\mathcal{A}_{r_B}(f)\right|^{\bar{q}} dx\right]^{1/\bar{q}}\\
&\lesssim \sum_{j=1}^{\infty} \left[\fint_{S_j(B)} 2^{jd} \left|\mathcal{A}_{r_B}(f)\right|^{\bar{q}} dx\right]^{1/\bar{q}}\\
&\lesssim \sum_{j=1}^{\infty} 2^{jd/\bar{q}} 2^{-(d+\alpha)j} \left(\fint_B |f|^{\bar{p}}\, dx\right)^{1/\bar{p}} \lesssim \left[\mathcal{M}\left(|f|^{\bar{p}}\right)(x_2)\right]^{\frac{1}{\bar{p}}},
\end{aligned}
$$

which completes the proof of (3.23).

Then, from (3.22) and (3.23), it follows that (3.11) and (3.12) hold true for $p_1 = p_2 := \bar{p}$, $p_3 := \bar{q}$, and $\epsilon := 0$. Thus, applying Lemma 3.5, we finish the proof of Theorem 3.6. □

Finally, we are turning to the proof of Theorem 1.1.

Proof of Theorem 1.1. By the functional calculus associated with $\mathcal{L}_a$, we have that, for any $g \in L^{q'}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$
\int_{\mathbb{R}^d} f(x)g(x)\, dx = c_{(\alpha)} \int_{\mathbb{R}^d} \int_0^{\infty} (t\mathcal{L}_a)^{2\gamma} e^{-2t\mathcal{L}_a} f(x)g(x)\, \frac{dt}{t}\, dx,
$$

where $c_{(\alpha)} := \int_0^{\infty} t^{2\gamma} e^{-2t}\, \frac{dt}{t}$. Then applying the Hölder inequality, we find that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d} f(x)g(x)\, dx\right| &\lesssim \left|\int_{\mathbb{R}^d} \int_0^{\infty} (t\mathcal{L}_a)^{2\gamma} e^{-2t\mathcal{L}_a} f(x)g(x)\, \frac{dt}{t}\, dx\right|\\
&\lesssim \int_{\mathbb{R}^d} \int_0^{\infty} \left|(t\mathcal{L}_a)^{\gamma} e^{-t\mathcal{L}_a} f(x)\, (t\mathcal{L}_a)^{\gamma} e^{-t\mathcal{L}_a} g(x)\right| \frac{dt}{t}\, dx\\
&\lesssim \int_{\mathbb{R}^d} S_{\mathcal{L}_a,\gamma} f(x) S_{\mathcal{L}_a,\gamma} g(x)\, dx.
\end{aligned}
$$

Furthermore, for any $q \in (p_0, q_0)$, we have $q' \in (q_0', p_0')$. Assume that

$$
\left[\omega^{1-q'}, v^{(1-q')(1-(\frac{q'}{q_0'})')}\right]_{A_{\frac{p}{q_0'}}(\mathbb{R}^d)} < \infty,
$$

and $\omega^{1-q'} \in RH_{s_0}(\mathbb{R}^d)$ with $s_0 \in ((\frac{p_0'}{q'})', \infty]$. Then, from Theorem 3.6, we deduce that

$$
\|S_{\mathcal{L}_a,\gamma} g\|_{L^{q'}_{\omega^{1-q'}}(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'}_{v^{1-q'}}(\mathbb{R}^d)},
$$

which further implies that

$$
\int_{\mathbb{R}^d} S_{\mathcal{L}_a,\gamma} f(x) S_{\mathcal{L}_a,\gamma} g(x)\, dx \lesssim \|S_{\mathcal{L}_a,\gamma} f\|_{L^q_\omega(\mathbb{R}^d)} \|S_{\mathcal{L}_a,\gamma} g\|_{L^{q'}_{\omega^{1-q'}}(\mathbb{R}^d)}
$$

$$\lesssim \|S_{\mathcal{L}_a,\gamma} f\|_{L^q_\omega(\mathbb{R}^d)} \|g\|_{L^{q'}_{v^{1-q'}}(\mathbb{R}^d)}.$$

As a consequence, we obtain that

$$\|f\|_{L^q_v(\mathbb{R}^d)} \lesssim \|S_{\mathcal{L}_a,\gamma} f\|_{L^q_\omega(\mathbb{R}^d)}.$$

Fix $0 < s \le 2$ and $d'_\sigma < p < d_{s\alpha/2+\sigma}$. Then, by Theorems 3.3 and 3.6 and Lemma 2.14, we conclude that

$$\begin{aligned}
&\left\|(-\Delta)^{\alpha s/4} f\right\|_{L^p_\omega(\mathbb{R}^d)} \\
&\lesssim \left\|\left\{\int_0^\infty t^{-s}\left|t(-\Delta)^{\alpha/2} e^{-t(-\Delta)^{\alpha/2}} f\right|^2 \frac{dt}{t}\right\}^{1/2}\right\|_{L^p_\omega(\mathbb{R}^d)} \\
&\lesssim \left\|\left\{\int_0^\infty t^{-s}\left|\left(t\mathcal{L}_a e^{-t\mathcal{L}_a} - t\,(-\Delta)^{\alpha/2}\, e^{-t(-\Delta)^{\alpha/2}}\right) f\right|^2 \frac{dt}{t}\right\}^{1/2}\right\|_{L^p_\omega(\mathbb{R}^d)} \\
&\quad + \left\|\left(\int_0^\infty t^{-s}\left|t\mathcal{L}_a e^{-t\mathcal{L}_a} f\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^p_\omega(\mathbb{R}^d)} \\
&\lesssim \left\|\frac{f}{|x|^{s\alpha/2}}\right\|_{L^p_\omega(\mathbb{R}^d)} + \left\|\mathcal{L}_a^{s/2} f\right\|_{L^p_v(\mathbb{R}^d)} \lesssim \left\|\mathcal{L}_a^{s/2} f\right\|_{L^p_v(\mathbb{R}^d)}.
\end{aligned}$$

Conversely, for any $d'_\sigma < p < \frac{d}{(s\alpha/2)\vee\sigma}$, we find that

$$\begin{aligned}
\left\|\mathcal{L}_a^{s/2} f\right\|_{L^p_v(\mathbb{R}^d)} &\lesssim \left\|\left(\int_{r_B^\alpha}^\infty t^{-s}\left|t\mathcal{L}_a e^{-t\mathcal{L}_a} f\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^p_\omega(\mathbb{R}^d)} \\
&\lesssim \left\|(-\Delta)^{\alpha s/4} f\right\|_{L^p_\omega(\mathbb{R}^d)}.
\end{aligned}$$

This finishes the proof of Theorem 1.1. □

4. Vertical square functions associated with $\mathcal{L}_a$

In this section, we first recall the extrapolation theorem and the change of angle formula. Then we establish the weighted norm estimates related to vertical square functions for the operator $\mathcal{L}_a$. Using these estimates of vertical square functions, we then obtain an application to new Hardy spaces associated with $\mathcal{L}_a$.

4.1. Extrapolation and change of angle

In this subsection, we recall the extrapolation theorem and the change of angle formulas, which were proved by Chen, Martell, and Prisuelos Arribas in [11].

Lemma 4.1. *Let $\mathcal{F}$ be a given family of pairs (f, g) of non-negative and not identically zero measurable functions.*

(i) *Suppose that, for some fixed exponent* $p_0 \in [1,\infty)$, *any weight* $v \in A_{p_0}(\mathbb{R}^d)$, *and any* $(f,g) \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} f(x)^{p_0} v(x)\,dx \le C_{(v,p_0)} \int_{\mathbb{R}^d} g(x)^{p_0} v(x)\,dx.$$

Then, for any $p \in (1,\infty)$, *any* $v \in A_p(\mathbb{R}^d)$, *and any* $(f,g) \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} f(x)^{p} v(x)\,dx \le C_{(v,p)} \int_{\mathbb{R}^d} g(x)^{p} v(x)\,dx,$$

where $C_{(v,p)}$ *is a positive constant independent of* f *and* g.

(ii) *Suppose that, for some fixed exponent* $q_0 \in [1,\infty)$, *any weight* $v \in RH_{q_0'}(\mathbb{R}^d)$, *and any* $(f,g) \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} f(x)^{\frac{1}{q_0}} v(x)\,dx \le C_{(v,q_0)} \int_{\mathbb{R}^d} g(x)^{\frac{1}{q_0}} v(x)\,dx.$$

Then, for any $q \in (1,\infty)$, *any* $v \in RH_q(\mathbb{R}^d)$, *and any* $(f,g) \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} f(x)^{\frac{1}{q}} v(x)\,dx \le C_{(v,q)} \int_{\mathbb{R}^d} g(x)^{\frac{1}{q}} v(x)\,dx,$$

where $C_{(v,q)}$ *is a positive constant independent of* f *and* g.

(iii) *Suppose that, for some fixed exponent* $r_0 \in (0,\infty)$, *any* $v \in A_\infty(\mathbb{R}^d)$, *and any* $(f,g) \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} f(x)^{r_0} v(x)\,dx \le C_{(v,r_0)} \int_{\mathbb{R}^d} g(x)^{r_0} v(x)\,dx.$$

Then, for any $r \in (0,\infty)$, *any* $v \in A_\infty(\mathbb{R}^d)$, *and any* $(f,g) \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} f(x)^{r} v(x)\,dx \le C_{(v,r)} \int_{\mathbb{R}^d} g(x)^{r} v(x)\,dx,$$

where $C_{(v,r)}$ *is a positive constant independent of* f *and* g.

(iv) *Suppose that, for some fixed* $0 < p_0 < p < q_0 < \infty$, *any* $v \in A_{\frac{p}{p_0}}(\mathbb{R}^d) \cap RH_{(\frac{q_0}{p})'}(\mathbb{R}^d)$, *and any* $(f,g) \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} f(x)^{p} v(x)\,dx \le C_{(v,p)} \int_{\mathbb{R}^d} g(x)^{p} v(x)\,dx.$$

Then, for any $q \in (p_0,q_0)$, *any* $v \in A_{\frac{q}{p_0}}(\mathbb{R}^d) \bigcap RH_{(\frac{q_0}{q})'}(\mathbb{R}^d)$, *and any* $(f,g) \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} f(x)^{q} v(x)\,dx \le C_{(v,q)} \int_{\mathbb{R}^d} g(x)^{q} v(x)\,dx,$$

where $C_{(v,q)}$ *is a positive constant independent of* f *and* g.

Lemma 4.2. *Let $r \in (1,\infty)$ and $v \in RH_{r'}(\mathbb{R}^d)$. Then, for any $q \in (1,r)$, $\beta \in (0,1)$, and $t \in (0,\infty)$, there exists a positive constant C such that, for any measurable function h on $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,\infty)$,*

$$\int_{\mathbb{R}^d} \left(\int_{B(x,\beta t)} |h(y,t)| \frac{dy}{|B(y,\beta t)|} \right)^{\frac{1}{q}} v(x)\,dx$$
$$\leq C\beta^{d\left(\frac{1}{r}-\frac{1}{q}\right)} \int_{\mathbb{R}^d} \left(\int_{B(x,t)} |h(y,t)| \frac{dy}{|B(y,t)|} \right)^{\frac{1}{q}} v(x)\,dx.$$

4.2. Weighted norm estimates related to vertical square functions

In this subsection, we prove weighted norm estimates related to several vertical square functions associated with $\mathcal{L}_a$, which further implies the two-weight boundedness for vertical square functions.

By Lemma 4.2, we are now to show the following conclusions.

Theorem 4.3. *Let $d \in \mathbb{N}$, $B := B(x,t)$ with $(x,t) \in \mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,\infty)$, and σ be as in (1.3). Then, for any given $p \in (d'_\sigma, \infty)$ and any $v \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)$ with d_σ being as in (1.4), there exists a positive constant C such that, for any $f \in L^p_v(\mathbb{R}^d)$,*

$$\|\mathbf{S}_{m,H} f\|_{L^p_v(\mathbb{R}^d)} \leq C \|\mathbf{s}_{m,H} f\|_{L^p_v(\mathbb{R}^d)},$$

where $\mathbf{s}_{m,H}$ and $\mathbf{S}_{m,H}$ are as in (1.7) and (1.9), respectively.

Proof. Let $p \in (d'_\sigma, \infty)$ and $v \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)$. Then there exists a constant p_0 such that $d'_\sigma < p_0 < \min\{2,p\}$ and $v \in A_{\frac{p}{p_0}}(\mathbb{R}^d)$. To show Theorem 4.3, by Lemma 4.1(i), it suffices to prove that, for any $v_0 \in A_{\frac{2}{p_0}}(\mathbb{R}^d)$ and $f \in L^2_{v_0}(\mathbb{R}^d)$,

$$\|\mathbf{S}_{m,H} f\|_{L^2_{v_0}(\mathbb{R}^d)} \lesssim \|\mathbf{s}_{m,H} f\|_{L^2_{v_0}(\mathbb{R}^d)}. \tag{4.1}$$

Let $v_0 \in A_{\frac{2}{p_0}}(\mathbb{R}^d)$, $f \in L^2_{v_0}(\mathbb{R}^d)$, and $F(y,t) := (t\sqrt{\mathcal{L}_a})^m f(y)$ for any $(y,t) \in \mathbb{R}^{d+1}_+$. Applying (2.1), (2.4), and Remark 2.3, we find that, for any $x \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbf{S}_{m,H} f(x) &\lesssim \left[\int_0^\infty \fint_{B(x,t^{\frac{2}{\alpha}})} \left| e^{-t^2\mathcal{L}_a} F(y,t) \right|^2 \frac{dydt}{t} \right]^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^\infty 2^{-j\alpha} \left\{ \int_0^\infty \left[\fint_{B(x,2^{j+1}t^{\frac{2}{\alpha}})} \left| e^{-\frac{t^2}{2}\mathcal{L}_a} F(y,t) \right|^{p_0} dy \right]^{\frac{2}{p_0}} \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^\infty 2^{-j\alpha} \left[\int_0^\infty \fint_{B(x,2^{j+1}t^{\frac{2}{\alpha}})} \left| e^{-\frac{t^2}{2}\mathcal{L}_a} F(y,t) \right|^2 d\nu_0(y) \frac{dt}{t} \right]^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^\infty 2^{-j\alpha} \left[\int_0^\infty \int_{B(x,2^{j+1}t^{\frac{2}{\alpha}})} \left| e^{-\frac{t^2}{2}\mathcal{L}_a} F(y,t) \right|^2 \frac{d\nu_0(y)}{v_0(B(y,2^{j+1}t^{\frac{2}{\alpha}}))} \frac{dt}{t} \right]^{\frac{1}{2}},
\end{aligned}$$

which further implies that

$$
\begin{aligned}
&\|\mathbf{S}_{m,H}f\|_{L^2_{v_0}(\mathbb{R}^d)}\\
&\lesssim \sum_{j=1}^{\infty} 2^{-j\alpha}\left[\int_0^\infty \int_{\mathbb{R}^d} \left|e^{-\frac{t^2}{2}\mathcal{L}_a}F(y,t)\right|^2 \int_{B(y,2^{j+1}t^{\frac{2}{\alpha}})} \frac{d\nu_0(x)}{v_0(B(y,2^{j+1}t^{\frac{2}{\alpha}}))} d\nu_0(y)\frac{dt}{t}\right]^{\frac12}\\
&\lesssim \sum_{j=1}^{\infty} 2^{-j\alpha}\left[\int_0^\infty \int_{\mathbb{R}^d} \left|e^{-t^2\mathcal{L}_a}F\left(y,\sqrt{2}t\right)\right|^2 d\nu_0(y)\frac{dt}{t}\right]^{\frac12}\\
&\lesssim \left[\int_{\mathbb{R}^d}\int_0^\infty \left|e^{-t^2\mathcal{L}_a}F(y,t)\right|^2 \frac{dt}{t} d\nu_0(y)\right]^{\frac12} \sim \|\mathbf{s}_{m,H}f\|_{L^2_{v_0}(\mathbb{R}^d)}.
\end{aligned}
$$

This finishes the proof of (4.1) and hence of Theorem 4.3. □

Lemma 4.4. *Let $d \in \mathbb{N}$, $0 < p_0 < 2 < q_0 < \infty$, $r \in [q_0/2, \infty)$, $v_0 \in A_{\frac{2}{p_0}}(\mathbb{R}^d) \cap RH_{r'}(\mathbb{R}^d)$, $a \in [1,\infty)$, and $u \in (0,1/4)$. Then there exist positive constants C_1 and C_2 such that, for any measurable function F on $\mathbb{R}^{d+1}_+$,*

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^d}\int_0^\infty \int_{B(x,(at)^{\frac{2}{\alpha}})} \left|e^{-\frac{t^2}{4u}\mathcal{L}_a}F(y,t)\right|^2 \frac{dydt}{t(at)^{\frac{2d}{\alpha}}} d\nu_0(x)\right)^{\frac12}\\
&\le C_1\left[\int_0^\infty \int_{\mathbb{R}^d}\left(\int_{B(x,(at)^{\frac{2}{\alpha}})} \left|e^{-\frac{t^2}{4u}\mathcal{L}_a}F(y,t)\right|^{q_0} \frac{dy}{(at)^{\frac{2d}{\alpha}}}\right)^{\frac{2}{q_0}} d\nu_0(x)\frac{dt}{t}\right]^{\frac12}\\
&\le C_2 u^{\frac{2}{\alpha}d(\frac{1}{4r}-\frac{1}{2q_0})}\left(\int_{\mathbb{R}^d}\int_0^\infty |F(y,t)|^2\frac{dt}{t}v_0(y)dy\right)^{\frac12}.
\end{aligned}
$$

Proof. We fix v_0, p_0, q_0, r, a, and u as in Lemma 4.4, and let

$$
\mathrm{I} := \left(\int_{\mathbb{R}^d}\int_0^\infty \int_{B(x,(at)^{\frac{2}{\alpha}})} \left|e^{-\frac{t^2}{4u}\mathcal{L}_a}F(y,t)\right|^2 \frac{dydt}{t(at)^{\frac{2d}{\alpha}}} d\nu_0(x)\right)^{\frac12}.
$$

Then, by the Jenssen inequality and the Fubini theorem, we conclude that

$$
\mathrm{I} \lesssim \left[\int_0^\infty \int_{\mathbb{R}^d}\left(\int_{B(x,(at)^{\frac{2}{\alpha}})} \left|e^{-\frac{t^2}{4u}\mathcal{L}_a}F(y,t)\right|^{q_0} \frac{dy}{(at)^{\frac{2d}{\alpha}}}\right)^{\frac{2}{q_0}} d\nu_0(x)\frac{dt}{t}\right]^{\frac12} =: \mathrm{II}. \tag{4.2}
$$

Then, from (2.1), (2.4), Lemma 4.2 with $\beta := (2\sqrt{u})^{\frac{2}{\alpha}} < 1$, and $q := \frac{q_0}{2}$, we deduce that

$$
\mathrm{II} = \left\{\int_0^\infty \int_{\mathbb{R}^d}\left[\int_{B(x,(\frac{at}{2\sqrt{u}})^{\frac{2}{\alpha}}(2\sqrt{u})^{\frac{2}{\alpha}})} \left|e^{-\frac{t^2}{4u}\mathcal{L}_a}F(y,t)\right|^{q_0} \frac{dy}{(at)^{\frac{2d}{\alpha}}}\right]^{\frac{2}{q_0}} d\nu_0(x)\frac{dt}{t}\right\}^{\frac12}
$$

$$\lesssim u^{\frac{2}{\alpha}d(\frac{1}{4r}-\frac{1}{2q_0})}\left\{\int_0^\infty\int_{\mathbb{R}^d}\left[\fint_{B(x,(\frac{at}{2\sqrt{u}})^{\frac{2}{\alpha}})}\left|e^{-\frac{t^2}{4u}\mathcal{L}_a}F(y,t)\right|^{q_0}dy\right]^{\frac{2}{q_0}}d\nu_0(x)\frac{dt}{t}\right\}^{\frac{1}{2}}$$

$$\lesssim u^{\frac{2}{\alpha}d(\frac{1}{4r}-\frac{1}{2q_0})}$$

$$\times\sum_j 2^{-j\alpha}\left\{\int_0^\infty\int_{\mathbb{R}^d}\left[\fint_{B(x,(2^{j+1}\frac{at}{2\sqrt{u}})^{\frac{2}{\alpha}})}\left|e^{-\frac{t^2}{4u}\mathcal{L}_a}F(y,t)\right|^{p_0}dy\right]^{\frac{2}{p_0}}d\nu_0(x)\frac{dt}{t}\right\}^{\frac{1}{2}}. \tag{4.3}$$

Moreover, notice that, for any $x\in\mathbb{R}^d$,

$$\left[\fint_{B(x,2^{j+1}(\frac{at}{2\sqrt{u}})^{\frac{2}{\alpha}})}\left|e^{-\frac{t^2}{4u}\mathcal{L}_a}F(y,t)\right|^{p_0}dy\right]^{\frac{1}{p_0}}\lesssim\mathcal{M}_{p_0}\left(F(\cdot,t)\right)(x).$$

Then, from the boundedness of $\mathcal{M}_{p_0}$ on $L^2(v_0)$ [recall that $v_0\in A_{\frac{2}{p_0}}(\mathbb{R}^d)$, see Lemma 2.4], the Fubini theorem, and (4.3), it follows that

$$\mathrm{II}\lesssim u^{\frac{2}{\alpha}d(\frac{1}{4r}-\frac{1}{2q_0})}\left[\int_{\mathbb{R}^d}\int_0^\infty|F(y,t)|^2\frac{dt}{t}v_0(y)dy\right]^{\frac{1}{2}},$$

which, combined with (4.2), implies that the conclusion of Lemma 4.4 holds true. □

Theorem 4.5. *Let $d\in\mathbb{N}$, $B:=B(x,t)$ with $(x,t)\in\mathbb{R}^{d+1}_+$ and σ be as in* (1.3). *Then, for any given $p\in(d'_\sigma,d_\sigma)$ and any $v\in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d)\cap RH_{(\frac{d_\sigma}{p})'}(\mathbb{R}^d)$ with d_σ being as in* (1.4), *there exist a positive constant C such that, for any $f\in L^p_v(\mathbb{R}^d)$,*

$$\|\mathbf{S}_{m,P}f\|_{L^p_v(\mathbb{R}^d)}\le C\,\|\mathbf{s}_{m,P}f\|_{L^p_v(\mathbb{R}^d)},$$

where $\mathbf{s}_{m,P}$ and $\mathbf{S}_{m,P}$ are as in (1.6) *and* (1.8), *respectively.*

Proof. To show Theorem 4.5, by Lemma 4.1(iv), it suffices to show that, for any $v_0\in A_{\frac{2}{d'_\sigma}}(\mathbb{R}^d)\cap RH_{(\frac{d_\sigma}{2})'}(\mathbb{R}^d)$ and $f\in L^2_{v_0}(\mathbb{R}^d)$,

$$\|\mathbf{S}_{m,P}f\|_{L^2_{v_0}(\mathbb{R}^d)}\lesssim\|\mathbf{s}_{m,P}f\|_{L^2_{v_0}(\mathbb{R}^d)}. \tag{4.4}$$

Now, we prove (4.4). Let $v_0\in A_{\frac{2}{d'_\sigma}}(\mathbb{R}^d)\cap RH_{(\frac{d_\sigma}{2})'}(\mathbb{R}^d)$ and $f\in L^2_{v_0}(\mathbb{R}^d)$. Since $v_0\in A_{\frac{2}{d'_\sigma}}(\mathbb{R}^d)\cap RH_{(\frac{d_\sigma}{2})'}(\mathbb{R}^d)$, it follows that there exist $p_0,q_0\in(1,\infty)$ such that ${d_\sigma}'<p_0<p<q_0<d_\sigma$ with $q_0>2$, $p_0<2$, and $v_0\in A_{\frac{2}{p_0}}(\mathbb{R}^d)\cap RH_{(\frac{q_0}{2})'}(\mathbb{R}^d)$.

Changing the variable t into $2t$, applying the subordination formula (1.5) and the Minkowski integral inequality, we find that

$$\|\mathbf{S}_{m,P}f\|_{L^2_{v_0}(\mathbb{R}^d)}\lesssim\int_0^{\frac{1}{4}}u^{\frac{1}{2}}\mathrm{I}(u)\frac{du}{u}+\int_{\frac{1}{4}}^\infty u^{\frac{1}{2}}e^{-u}\mathrm{I}(u)\frac{du}{u}=:\mathrm{E}_1+\mathrm{E}_2, \tag{4.5}$$

where, for any $u\in(0,\infty)$,

$$\mathrm{I}(u):=\left[\int_{\mathbb{R}^d}\int_0^\infty\int_{B(x,(2t)^{\frac{2}{\alpha}})}\left|e^{-\frac{t^2}{4u}\mathcal{L}_a}\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t\sqrt{\mathcal{L}_a}}f(y)\right|^2\frac{dydt}{t(2t)^{\frac{2}{\alpha}}}d\nu_0(x)\right]^{\frac12}.$$

Since $v_0\in A_{\frac{2}{p_0}}(\mathbb{R}^d)\cap RH_{(\frac{q_0}{2})'}(\mathbb{R}^d)$, then, it follows, from Lemma 4.4 with

$$F(y,t):=\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t\sqrt{\mathcal{L}_a}}f(y),$$

$a:=2$, and $r:=\frac{q_0}{2}$, we deduce that, for any $u\in(0,1/4)$,

$$\mathrm{I(u)}\lesssim\left[\int_{\mathbb{R}^d}\int_0^\infty\left|\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t\sqrt{\mathcal{L}_a}}f(y)\right|^2\frac{dt}{t}d\nu_0(y)\right]^{\frac12}\sim\|\mathbf{s}_{m,P}f\|_{L^2_{v_0}(\mathbb{R}^d)}.$$

Therefore,

$$\mathrm{E}_1\lesssim\int_0^{\frac14}u^{\frac12}\frac{du}{u}\|\mathbf{s}_{m,P}f\|_{L^2_{v_0}(\mathbb{R}^d)}\lesssim\|\mathbf{s}_{m,P}f\|_{L^2_{v_0}(\mathbb{R}^d)}. \tag{4.6}$$

To estimate E_2, applying Remark 2.3, (2.4) [recall that $v_0\in A_{\frac{2}{p_0}}(\mathbb{R}^d)$], and the Fubini theorem, we obtain that

$$\begin{aligned}\mathrm{E}_2&\lesssim\sum_j 2^{-j\alpha}\int_{\frac14}^\infty u^\theta e^{-u}\frac{du}{u}\\&\quad\times\left[\int_{\mathbb{R}^d}\int_0^\infty\left(\fint_{B(x,2^{j+1}(2t)^{\frac{2}{\alpha}})}\left|\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t\sqrt{\mathcal{L}_a}}f(y)\right|^{p_0}dy\right)^{\frac{2}{p_0}}\frac{dt}{t}d\nu_0(x)\right]^{\frac12}\\&\lesssim\sum_j 2^{-j\alpha}\left[\int_{\mathbb{R}^d}\int_0^\infty\fint_{B(x,2^{j+1}(2t)^{\frac{2}{\alpha}})}\left|\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t\sqrt{\mathcal{L}_a}}f(y)\right|^2 d\nu_0(y)\frac{dt}{t}d\nu_0(x)\right]^{\frac12}\\&\lesssim\sum_j 2^{-j\alpha}\left[\int_{\mathbb{R}^d}\int_0^\infty\left|\left(t\sqrt{\mathcal{L}_a}\right)^m e^{-t\sqrt{\mathcal{L}_a}}f(y)\right|^2\fint_{B(x,2^{j+1}(2t)^{\frac{2}{\alpha}})}d\nu_0(x)\frac{dt}{t}d\nu_0(y)\right]^{\frac12}\\&\lesssim\|\mathbf{s}_{m,P}f\|_{L^2_{v_0}(\mathbb{R}^d)},\end{aligned}$$

which, combined with (4.5) and (4.6), implies that (4.4) holds true. This finishes the proof of Theorem 4.5. ◻

In addition, applying the subordination formula (1.5) and changing the variable t into $2\sqrt{u}t$, we find that

$$\mathbf{s}_{m,P}f(x)\lesssim\int_0^\infty u^{\frac{1+m}{2}}e^{-u}\frac{du}{u}\mathbf{s}_{m,H}f(x)\lesssim\mathbf{s}_{m,H}f(x). \tag{4.7}$$

By the boundedness of the $(-\Delta)^{\frac{\alpha}{2}}\mathcal{L}_a^{-1}$, we have following conclusions.

Theorem 4.6. *Let $d\in\mathbb{N}$, $\alpha\in(0,2\wedge d)$, $B:=B(x,t)$ with $(x,t)\in\mathbb{R}^{d+1}_+$, and σ be as in* (1.3). *Then, for any $p\in(d'_\sigma,d_{\sigma+\alpha})$ with $d_{\sigma+\alpha}\in(2,\infty)$, $m\in[2,\infty)$,* *and any* $v \in A_{\frac{p}{d_\sigma'}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+\alpha}}{p})'}(\mathbb{R}^d)$*, where* d_σ *and* $d_{\sigma+\alpha}$ *are as in* (1.4)*, there exist positive constants* C_1 *and* C_2 *such that, for any* $f \in L^p_v(\mathbb{R}^d)$*,*

$$C_1^{-1} \|\mathbf{s}_{m,P} f\|_{L^p_v(\mathbb{R}^d)} \le \|\mathbf{g}_{m-2,P} f\|_{L^p_v(\mathbb{R}^d)} \le C_1 \|\mathbf{s}_{m,P} f\|_{L^p_v(\mathbb{R}^d)}$$

and

$$C_2^{-1} \|\mathbf{s}_{m,H} f\|_{L^p_v(\mathbb{R}^d)} \le \|\mathbf{g}_{m-2,H} f\|_{L^p_v(\mathbb{R}^d)} \le C_2 \|\mathbf{s}_{m,H} f\|_{L^p_v(\mathbb{R}^d)},$$

where $\mathbf{s}_{m,P}$*,* $\mathbf{s}_{m,H}$ *,* $\mathbf{g}_{m,P}$*, and* $\mathbf{g}_{m,H}$ *are as in* (1.6)*,* (1.7)*,* (1.10)*, and* (1.11)*, respectively.*

Proof. To show Theorem 4.6, by Lemma 4.1(iv), it suffices to prove that, for any $f \in L^2(\mathbb{R}^d)$ and any $v_0 \in A_{\frac{2}{d_\sigma'}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+\alpha}}{2})'}(\mathbb{R}^d)$,

$$\|\mathbf{s}_{m,H} f\|_{L^p_{v_0}(\mathbb{R}^d)} \lesssim \|\mathbf{g}_{m-2,H} f\|_{L^2_{v_0}(\mathbb{R}^d)} \lesssim \|\mathbf{s}_{m,H} f\|_{L^2_{v_0}(\mathbb{R}^d)} \tag{4.8}$$

and

$$\|\mathbf{s}_{m,P} f\|_{L^p_{v_0}(\mathbb{R}^d)} \lesssim \|\mathbf{g}_{m-2,P} f\|_{L^2_{v_0}(\mathbb{R}^d)} \lesssim \|\mathbf{s}_{m,P} f\|_{L^2_{v_0}(\mathbb{R}^d)}. \tag{4.9}$$

Next, we prove (4.8) and (4.9). Let $f \in L^2(\mathbb{R}^d)$ and $F(y,t) := e^{-t^2\mathcal{L}_a} f(y)$ or $e^{-t\sqrt{\mathcal{L}_a}} f(y)$ for any $(y,t) \in \mathbb{R}^{d+1}_+$.

Applying the Fubini theorem and Lemma 2.16, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_0^\infty \left| t^2 (-\Delta)^{\frac{\alpha}{2}} \left(t\sqrt{\mathcal{L}_a}\right)^{m-2} F(y,t) \right|^2 \frac{\mathrm{d}t}{t} d\nu_0(y) \\
&= \int_0^\infty \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{\alpha}{2}} \mathcal{L}_a^{-1} \left(t\sqrt{\mathcal{L}_a}\right)^m F(y,t) \right|^2 d\nu_0(y) \frac{\mathrm{d}t}{t} \\
&\sim \int_0^\infty \int_{\mathbb{R}^d} \left| \left(t\sqrt{\mathcal{L}_a}\right)^m F(y,t) \right|^2 d\nu_0(y) \frac{\mathrm{d}t}{t},
\end{aligned}$$

which, further implies that (4.8) and (4.9) hold true. This finishes the proof of Theorem 4.6. □

Since $\mathcal{L}_a = (-\Delta)^{\alpha/2} + a|x|^{-\alpha}$, it follows that

$$t^2(-\Delta)^{\alpha/2} e^{-t^2\mathcal{L}_a} = t^2 \mathcal{L}_a e^{-t^2\mathcal{L}_a} - a|x|^{-\alpha} t^2 e^{-t^2\mathcal{L}_a}.$$

By Proposition 2.12 and suitable modification to the proof of Theorem 3.1, we conclude that the kernel of $t^2(-\Delta)^{\alpha/2} e^{-t^2\mathcal{L}_a}$ is dominated by

$$Ct^{-2d/\alpha} \left(\frac{t^{2/\alpha} + |x-y|}{t^{2/\alpha}} \right)^{-d-\alpha} D_{\sigma+\alpha}(x,t^2) D_\sigma(y,t^2).$$

Hence, if $d_\sigma' < p \le q < d_{\sigma+\alpha}$, then, the kernel of $t^2(-\Delta)^{\alpha/2} e^{-t^2\mathcal{L}_a}$ satisfies Lemma 2.9.

Thus, by using the same ideas as that used in the proof of Theorem 4.3, we have following conclusion.

Theorem 4.7. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2\wedge d)$, and σ be as in (1.3). Then, for any given $p \in (d'_\sigma, d_{\sigma+\alpha})$ with $d_{\sigma+\alpha} \in (2,\infty)$, and any $v \in A_{\frac{p}{d_\sigma{}'}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+\alpha}}{p})'}(\mathbb{R}^d)$, where d_σ and $d_{\sigma+\alpha}$ are as in (1.4), there exists a positive constant C such that, for any $f \in L^p_v(\mathbb{R}^d)$,*

$$\|\mathbf{G}_{K,H} f\|_{L^p_v(\mathbb{R}^d)} \le C \|\mathbf{g}_{K,H} f\|_{L^p_v(\mathbb{R}^d)},$$

where $\mathbf{g}_{K,H}$ and $\mathbf{G}_{K,H}$ are as in (1.11) and (1.13), respectively.

Theorem 4.8. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2\wedge d)$, $B := B(x,t)$ with $(x,t) \in \mathbb{R}^{d+1}_+$ and σ be as in (1.3). Then, for any given $p \in (d'_\sigma, d_{\sigma+\alpha})$ with $d_{\sigma+\alpha} \in (2,\infty)$ and any $v \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+\alpha}}{p})'}(\mathbb{R}^d)$, where d_σ and $d_{\sigma+\alpha}$ are as in (1.4), there exists a positive constant C such that, for any $f \in L^p_v(\mathbb{R}^d)$,*

$$\|\mathbf{G}_{K,P} f\|_{L^p_v(\mathbb{R}^d)} \le C \|\mathbf{s}_{K,P} f\|_{L^p_v(\mathbb{R}^d)},$$

where $\mathbf{s}_{K,P}$ and $\mathbf{G}_{K,P}$ are as in (1.6) and (1.12), respectively.

Proof. To prove Theorem 4.8, by Lemma 4.1(iv), it suffices to show that, for any given $v_0 \in A_{\frac{2}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+\alpha}}{2})'}(\mathbb{R}^d)$ and any $f \in L^2_{v_0}(\mathbb{R}^d)$,

$$\|\mathbf{G}_{K,P} f\|_{L^2_{v_0}(\mathbb{R}^d)} \lesssim \|\mathbf{s}_{K,P} f\|_{L^2_{v_0}(\mathbb{R}^d)}. \tag{4.10}$$

Now, we prove (4.10). Since $v_0 \in A_{\frac{2}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+\alpha}}{2})'}(\mathbb{R}^d)$, it follows that there exist p_0, $q_0 \in (1,\infty)$ such that, $d'_\sigma < p_0 < p < q_0 < d_{\sigma+\alpha}$, with $q_0 > 2$ and $p_0 < 2$, and $v_0 \in A_{\frac{2}{p_0}}(\mathbb{R}^d) \cap RH_{(\frac{q_0}{2})'}(\mathbb{R}^d)$. Changing the variable t into $2t$ applying the subordination formula (1.5), and the Minkowski integral inequality, we find that

$$\|\mathbf{G}_{K,P} f\|_{L^2_{v_0}(\mathbb{R}^d)} \lesssim \int_0^{\frac14} u^{\frac32} \mathrm{II}(u) \frac{du}{u} + \int_{\frac14}^\infty u^{\frac32} e^{-u} \mathrm{II}(u) \frac{du}{u} =: \mathrm{III} + \mathrm{IV}, \tag{4.11}$$

where, for any $u \in (0,\infty)$,

$$\mathrm{II}(u) := \left[\int_{\mathbb{R}^d} \int_0^\infty \int_{B(x,(2t)^{\frac{2}{\alpha}})} \left| \frac{t^2}{4u} (-\Delta)^{\frac{\alpha}{2}} e^{-\frac{t^2}{4u}\mathcal{L}_a} \left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t\sqrt{\mathcal{L}_a}} f(y) \right|^2 \frac{dydt}{t(2t)^{\frac{2}{\alpha}}} d\nu_0(x)\right]^{\frac12}.$$

Since $v_0 \in A_{\frac{2}{p_0}}(\mathbb{R}^d) \cap RH_{(\frac{q_0}{2})'}(\mathbb{R}^d)$, then, it follows, from by Lemma 4.4 with

$$F(y,t) := (t\sqrt{\mathcal{L}_a})^K e^{-t\sqrt{\mathcal{L}_a}} f(y),$$

$a := 2$, and $r := \frac{q_0}{2}$, we deduce that, for any $u \in (0, 1/4)$,

$$\mathrm{II}(u) \lesssim \left[\int_{\mathbb{R}^d} \int_0^\infty \left| \left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t\sqrt{\mathcal{L}_a}} f(y) \right|^2 \frac{dt}{t} d\nu_0(y)\right]^{\frac12} = \|\mathbf{s}_{K,P} f\|_{L^2_{v_0}(\mathbb{R}^d)}.$$

Therefore,

$$\mathrm{III} \lesssim \int_0^{\frac{1}{4}} u^{\frac{3}{2}} \frac{du}{u} \left\|\mathbf{s}_{K,P}\right\|_{L^2_{v_0}(\mathbb{R}^d)} \lesssim \left\|\mathbf{s}_{K,P} f\right\|_{L^2_{v_0}(\mathbb{R}^d)} . \tag{4.12}$$

To estimate IV, applying Remark 2.3 [recall that $v_0 \in A_{\frac{2}{p_0}}(\mathbb{R}^d)$] and the Fubini theorem, we conclude that

$$\begin{aligned}
\mathrm{IV} &\lesssim \sum_j 2^{-j\alpha} \int_{\frac{1}{4}}^{\infty} u^{\theta} e^{-u} \frac{du}{u} \\
&\quad \times \left\{ \int_{\mathbb{R}^d} \int_0^{\infty} \left[\fint_{B(x, 2^{j+1}(2t)^{\frac{2}{\alpha}})} \left| \left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t\sqrt{\mathcal{L}_a}} f(y) \right|^{p_0} dy \right]^{\frac{2}{p_0}} \frac{dt}{t} d\nu_0(x) \right\}^{\frac{1}{2}} \\
&\lesssim \sum_j 2^{-j\alpha} \left[\int_{\mathbb{R}^d} \int_0^{\infty} \fint_{B(x, 2^{j+1}(2t)^{\frac{2}{\alpha}})} \left| \left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t\sqrt{\mathcal{L}_a}} f(y) \right|^2 d\nu_0(y) \frac{dt}{t} d\nu_0(x) \right]^{\frac{1}{2}} \\
&\lesssim \sum_j 2^{-j\alpha} \left[\int_{\mathbb{R}^d} \int_0^{\infty} \left| \left(t\sqrt{\mathcal{L}_a}\right)^K e^{-t\sqrt{\mathcal{L}_a}} f(y) \right|^2 \fint_{B(x, 2^{j+1}(2t)^{\frac{2}{\alpha}})} d\nu_0(x) \frac{dt}{t} d\nu_0(y) \right]^{\frac{1}{2}} \\
&\lesssim \left\|\mathbf{s}_{K,P} f\right\|_{L^2_{v_0}(\mathbb{R}^d)} ,
\end{aligned}$$

which, combined with (4.11) and (4.12), implies that (4.10) holds true. This finishes the proof of Theorem 4.8. □

Proofs of Theorems 1.2 and 1.3. By (4.7), and Theorems 4.3, 4.6, 4.7, and 4.8, we find that the conclusions of Theorems 1.2 and 1.3 hold true. □

Remark 4.9. By Theorems 1.2 and 1.3, and Lemma 3.2, we obtain that $\mathbf{S}_{m,H}$, $\mathbf{S}_{m,P}$ and $\mathbf{s}_{m,P}$ are bounded on $L^p_v(\mathbb{R}^d)$ for any $p \in (d'_\sigma, d_\sigma)$ and any $v \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_\sigma}{p})'}(\mathbb{R}^d)$. Moreover, we obtain that $\mathbf{g}_{m,H}$, $\mathbf{g}_{m,P}$, $\mathbf{G}_{K,H}$ and $\mathbf{G}_{K,P}$ are bounded on $L^p_v(\mathbb{R}^d)$ for any $p \in (d'_\sigma, d_{\sigma+\alpha})$ and any $v \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{d_{\sigma+\alpha}}{p})'}(\mathbb{R}^d)$ with $d_{\sigma+\alpha} \in (2, \infty)$.

Next, we consider the reverse conclusion. For a locally square integrable function f on $\mathbb{R}^{d+1}_+$, let

$$Sf(x) := \left(\iint_{|x-y|<t^{\frac{2}{\alpha}}} |f(y,t)|^2 \frac{dydt}{t^{1+\frac{2d}{\alpha}}} \right)^{\frac{1}{2}},$$

$$Vf(y) := \left(\int_0^{\infty} |f(y,t)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

From [1, Proposition 2.3], we deduce that the following conclusion holds true.

Lemma 4.10. *For any given $p \in (0,2)$ and any $\omega \in RH_{(\frac{2}{p})'}(\mathbb{R}^d)$, there exists a positive constant C, depending on p and ω, such that, for any locally square integrable function f on $\mathbb{R}^{d+1}_+$,*

$$\|Vf\|_{L^p_\omega(\mathbb{R}^d)} \le C\,\|Sf\|_{L^p_\omega(\mathbb{R}^d)}.$$

Theorem 4.11. *For any given $p \in (0,2)$ and any $v \in RH_{(\frac{2}{p})'}(\mathbb{R}^d)$, there exists a positive constant C, depending on p and v, such that, for any $f \in L^p_v(\mathbb{R}^d)$,*

$$\|\mathbf{s}_{m,H}f\|_{L^p_v(\mathbb{R}^d)} \le C\,\|\mathbf{S}_{m,H}f\|_{L^p_v(\mathbb{R}^d)}, \tag{4.13}$$

$$\|\mathbf{s}_{m,P}f\|_{L^p_v(\mathbb{R}^d)} \le C\,\|\mathbf{S}_{m,P}f\|_{L^p_v(\mathbb{R}^d)},$$

$$\|\mathbf{g}_{K,H}f\|_{L^p_v(\mathbb{R}^d)} \le C\,\|\mathbf{G}_{K,H}f\|_{L^p_v(\mathbb{R}^d)}, \tag{4.14}$$

and

$$\|\mathbf{g}_{K,P}f\|_{L^p_v(\mathbb{R}^d)} \le C\,\|\mathbf{G}_{K,P}f\|_{L^p_v(\mathbb{R}^d)}.$$

Remark 4.12. In summary, from Lemma 3.2, Theorems 1.2, 1.3, and 4.11, it follows that

$$\|\mathbf{s}_{m,H}f\|_{L^p_v(\mathbb{R}^d)} \sim \|\mathbf{S}_{m,H}f\|_{L^p_v(\mathbb{R}^d)} \sim \|f\|_{L^p_v(\mathbb{R}^d)}, \tag{4.15}$$

$$\|\mathbf{s}_{m,P}f\|_{L^p_v(\mathbb{R}^d)} \sim \|\mathbf{S}_{m,P}f\|_{L^p_v(\mathbb{R}^d)} \lesssim \|f\|_{L^p_v(\mathbb{R}^d)},$$

and

$$\|\mathbf{g}_{K,H}f\|_{L^p(v)} \sim \|\mathbf{G}_{K,H}f\|_{L^p_v(\mathbb{R}^d)} \sim \|f\|_{L^p_v(\mathbb{R}^d)} \tag{4.16}$$

for any $p \in (d'_\sigma, 2)$ and any $v \in A_{\frac{p}{d'_\sigma}}(\mathbb{R}^d) \cap RH_{(\frac{2}{p})'}(\mathbb{R}^d)$.

For $m \in [0,\infty)$, let the operator $T := \mathbf{S}_{m,H}$, $\mathbf{s}_{m,P}$, $\mathbf{S}_{m,P}$, $\mathbf{g}_{m,H}$, $\mathbf{g}_{m,P}$, $\mathbf{G}_{m,H}$, or $\mathbf{G}_{m,P}$. Then, by Theorems 3.6, 1.2, and 1.3, we conclude that the following conclusion holds true.

Theorem 4.13. *Let $d \in \mathbb{N}$, $\alpha \in (0, 2\wedge d)$, $a \in [a^*,\infty)$ with a^* being as in (1.2), σ be as in (1.3), and $q \in (d'_\sigma, d_{\sigma+\alpha})$ with d_σ and $d_{\sigma+\alpha}$ being as in (1.4). Assume further that $d'_\sigma < p_0 < q < q_0 < d_{\sigma+\alpha}$, and the weights ω and v satisfy that $\omega \in RH_s(\mathbb{R}^d) \cap A_{\frac{q}{d'_\sigma}}(\mathbb{R}^d)$ with some $s \in ((\frac{q_0}{q})', \infty]$, and*

$$\left[\omega, v^{1-(\frac{q}{p_0})'}\right]_{A_{\frac{q}{p_0}}(\mathbb{R}^d)} := \sup_{B\subseteq\mathbb{R}^d}\left[\fint_B \omega\,dx\right]\left[\fint_B v^{1-(\frac{q}{p_0})'}\,dx\right]^{\frac{q}{p_0}-1} < \infty,$$

where the surpremum is taken over all balls B of $\mathbb{R}^d$. For any $m \in [0,\infty)$, let $T := \mathbf{S}_{m,H}$, $\mathbf{s}_{m,P}$, $\mathbf{S}_{m,P}$, $\mathbf{g}_{m,H}$, $\mathbf{g}_{m,P}$, $\mathbf{G}_{m,H}$, or $\mathbf{G}_{m,P}$. Then T is bounded *from* $L_v^q(\mathbb{R}^d)$ *to* $L_\omega^q(\mathbb{R}^d)$*, and there exists a positive constant* C *such that, for any* $f \in L_v^q(\mathbb{R}^d)$*,* $\|Tf\|_{L_\omega^q(\mathbb{R}^d)} \le C\|f\|_{L_v^q(\mathbb{R}^d)}$.

4.3. Applications

In this subsection, we define the new Hardy space associate with the operator $\mathcal{L}_a$. Using the weighted norm estimates for square functions obtained in Section 4.2, we further obtain that the operators $\mathbf{s}_{M,H}$ and $\mathbf{g}_{K,H}$ are bounded from the new Hardy space to $L^p(\mathbb{R}^d)$ and the equivalence between the new Hardy space and the Lebesgue space $L^p(\mathbb{R}^d)$.

For each M, $K \in [0,\infty)$ and $p \in (0,\infty)$, define

$$D_{S_M,p} := \left\{f \in L^2(\mathbb{R}^d) : \mathbf{S}_{M,H} f \in L^p(\mathbb{R}^d)\right\}$$

and

$$D_{G_K,p} := \left\{f \in L^2(\mathbb{R}^d) : \mathbf{G}_{K,H} f \in L^p(\mathbb{R}^d)\right\},$$

where $\mathbf{S}_{K,H}$ and $\mathbf{G}_{K,H}$ are as in (1.9) and (1.13), respectively.

Definition 4.14. Assume that M, $K \in [0,\infty)$, and $p \in (0,\infty)$. Then the Hardy spaces $H^p_{\mathcal{L}_a, S_M}(\mathbb{R}^d)$ and $H^p_{\mathcal{L}_a, G_K}(\mathbb{R}^d)$ associated to $\mathcal{L}_a$ are, respectively, defined as the completion of the space $D_{S_M,p}$ and $D_{G_K,p}$ with respect to the *quasi-norms*

$$\|f\|_{H^p_{\mathcal{L}_a, S_M}(\mathbb{R}^d)} := \|\mathbf{S}_{M,H} f\|_{L^p(\mathbb{R}^d)} \text{ and } \|f\|_{H^p_{\mathcal{L}_a, G_K}(\mathbb{R}^d)} := \|\mathbf{G}_{K,H} f\|_{L^p(\mathbb{R}^d)}.$$

We recall that the molecular characterization of the Hardy space $H^p_{\mathcal{L}_a, S_M}(\mathbb{R}^d)$ was established by Bui and Nader [10]. In recent years, the study on the real-variable theory of Hardy spaces associated with different differential operators has aroused great interests (see, for instance, [8, 9, 15, 16, 21, 22, 25, 26]).

By (4.13), (4.14), (4.15), (4.16), and Definition 4.14, we have the following conclusion.

Theorem 4.15. *Let* M*,* $K \in [0,\infty)$*,* $d \in \mathbb{N}$*,* $a \in [a^*,\infty)$ *with* a^* *being as in* (1.2)*, and* σ *be as in* (1.3)*.*

(i) *For any given* $p \in (d'_\sigma, 2)$ *with* d_σ *being as in* (1.4) *and any* $f \in L^2(\mathbb{R}^d)$*,* $\|f\|_{H^p_{\mathcal{L}_a, S_M}(\mathbb{R}^d)} \sim \|f\|_{L^p(\mathbb{R}^d)}$ *and* $\|f\|_{H^p_{\mathcal{L}_a, G_K}(\mathbb{R}^d)} \sim \|f\|_{L^p(\mathbb{R}^d)}$*, where the positive equivalence constants are independent of* f*. Thus, for any given* $p \in (d'_\sigma, 2)$*, the spaces* $H^p_{\mathcal{L}_a, S_M}(\mathbb{R}^d)$ *and* $H^p_{\mathcal{L}_a, G_K}(\mathbb{R}^d)$ *are equivalent with* $L^p(\mathbb{R}^d)$*.*

(ii) *For any given* $p \in (0, d'_\sigma]$ *with* d_σ *being as in* (1.4) *and any* $f \in L^2(\mathbb{R}^d)$*, there exists a positive constant* C *such that, for any* $f \in H^p_{\mathcal{L}_a, S_M}(\mathbb{R}^d)$*,* $\|\mathbf{s}_{M,H} f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{H^p_{\mathcal{L}_a, S_M}(\mathbb{R}^d)}$*, and for any* $f \in H^p_{\mathcal{L}_a, G_K}(\mathbb{R}^d)$*,* $\|\mathbf{g}_{K,H} f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{H^p_{\mathcal{L}_a, G_K}(\mathbb{R}^d)}$*, where* $\mathbf{s}_{M,H}$ *and* $\mathbf{g}_{K,H}$ *are as in* (1.6) *and* (1.11)*, respectively.*

Acknowledgements. The authors would like to thank the referee for her/his carefully reading and many valuable comments which definitely improve the presentation of this article.

References

[1] P. Auscher, S. C. Hofmann, and J. Martell, *Vertical versus conical square functions*, Trans. Amer. Math. Soc. **364** (2012), no. 10, 5469–5489. https://doi.org/10.1090/S0002-9947-2012-05668-6

[2] F. Bernicot and J. Zhao, *New abstract Hardy spaces*, J. Funct. Anal. **255** (2008), no. 7, 1761–1796. https://doi.org/10.1016/j.jfa.2008.06.018

[3] R. M. Blumenthal and R. K. Getoor, *Some theorems on stable processes*, Trans. Amer. Math. Soc. **95** (1960), 263–273. https://doi.org/10.2307/1993291

[4] K. Bogdan, T. Grzywny, T. Jakubowski, and D. Pilarczyk, *Fractional Laplacian with Hardy potential*, Comm. Partial Differential Equations **44** (2019), no. 1, 20–50. https://doi.org/10.1080/03605302.2018.1539102

[5] T. A. Bui and T. Q. Bui, *Maximal regularity of parabolic equations associated to generalized Hardy operators in weighted mixed-norm spaces*, J. Differential Equations **303** (2021), 547–574. https://doi.org/10.1016/j.jde.2021.09.029

[6] T. A. Bui and P. D'Ancona, *Generalized Hardy operators*, Nonlinearity **36** (2023), no. 1, 171–198.

[7] T. A. Bui, P. D'Ancona, X. T. Duong, J. Li, and F. K. Ly, *Weighted estimates for powers and smoothing estimates of Schrödinger operators with inverse-square potentials*, J. Differential Equations **262** (2017), no. 3, 2771–2807. https://doi.org/10.1016/j.jde.2016.11.008

[8] T. A. Bui, T. D. Do, and N. N. Trong, *Heat kernels of generalized degenerate Schrödinger operators and Hardy spaces*, J. Funct. Anal. **280** (2021), no. 1, Paper No. 108785, 49 pp. https://doi.org/10.1016/j.jfa.2020.108785

[9] T. A. Bui, X. T. Duong, and F. K. Ly, *Maximal function characterizations for new local Hardy-type spaces on spaces of homogeneous type*, Trans. Amer. Math. Soc. **370** (2018), no. 10, 7229–7292. https://doi.org/10.1090/tran/7289

[10] T. A. Bui and G. Nader, *Hardy spaces associated to generalized Hardy operators and applications*, NoDEA Nonlinear Differential Equations Appl. **29** (2022), no. 4, Paper No. 40, 40 pp. https://doi.org/10.1007/s00030-022-00765-4

[11] L. Chen, J. Martell, and C. Prisuelos-Arribas, *Conical square functions for degenerate elliptic operators*, Adv. Calc. Var. **13** (2020), no. 1, 75–113. https://doi.org/10.1515/acv-2016-0062

[12] S. Cho, P. Kim, R. Song, and Z. Vondraček, *Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings*, J. Math. Pures Appl. (9) **143** (2020), 208–256. https://doi.org/10.1016/j.matpur.2020.09.006

[13] D. V. Cruz-Uribe, J. Martell, and C. Pérez Moreno, *Weights, extrapolation and the theory of Rubio de Francia*, Operator Theory: Advances and Applications, 215, Birkhäuser/Springer Basel AG, Basel, 2011. https://doi.org/10.1007/978-3-0348-0072-3

[14] J. Duoandikoetxea, *Fourier analysis*, translated and revised from the 1995 Spanish original by David Cruz-Uribe, Graduate Studies in Mathematics, 29, Amer. Math. Soc., Providence, RI, 2001. https://doi.org/10.1090/gsm/029

[15] X. T. Duong and J. Li, *Hardy spaces associated to operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus*, J. Funct. Anal. **264** (2013), no. 6, 1409–1437. https://doi.org/10.1016/j.jfa.2013.01.006

[16] X. T. Duong and L. Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. **18** (2005), no. 4, 943–973. https://doi.org/10.1090/S0894-0347-05-00496-0

[17] R. L. Frank, E. H. Lieb, and R. Seiringer, *Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators*, J. Amer. Math. Soc. **21** (2008), no. 4, 925–950. https://doi.org/10.1090/S0894-0347-07-00582-6

[18] R. L. Frank, K. Merz, and H. K. H. Siedentop, *Equivalence of Sobolev norms involving generalized Hardy operators*, Int. Math. Res. Not. IMRN **2021** (2021), no. 3, 2284–2303. https://doi.org/10.1093/imrn/rnz135

[19] L. Grafakos, *Classical Fourier Analysis*, second edition, Graduate Texts in Mathematics, 249, Springer, New York, 2008.

[20] W. Hebisch, *Functional calculus for slowly decaying kernels*, Preprint, 1995.

[21] S. C. Hofmann, G. Lu, D. Mitrea, M. Mitrea, and L. Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*, Mem. Amer. Math. Soc. **214** (2011), no. 1007, vi+78 pp. https://doi.org/10.1090/S0065-9266-2011-00624-6

[22] S. C. Hofmann, S. Mayboroda, and A. G. R. McIntosh, *Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 5, 723–800. https://doi.org/10.24033/asens.2154

[23] T. Jakubowski and J. Wang, *Heat kernel estimates of fractional Schrödinger operators with negative Hardy potential*, Potential Anal. **53** (2020), no. 3, 997–1024. https://doi.org/10.1007/s11118-019-09795-7

[24] R. Killip, C. Miao, M. Visan, J. Zhang, and J. Zheng, *Sobolev spaces adapted to the Schrödinger operator with inverse-square potential*, Math. Z. **288** (2018), no. 3-4, 1273–1298. https://doi.org/10.1007/s00209-017-1934-8

[25] S. Liang and L. Yan, *Riesz transforms associated to Schrödinger operators on weighted Hardy spaces*, J. Funct. Anal. **259** (2010), no. 6, 1466–1490. https://doi.org/10.1016/j.jfa.2010.05.015

[26] S. Liang and L. Yan, *A maximal function characterization for Hardy spaces associated to nonnegative self-adjoint operators satisfying Gaussian estimates*, Adv. Math. **287** (2016), 463–484. https://doi.org/10.1016/j.aim.2015.09.026

[27] A. G. R. McIntosh, *Operators which have an H_∞ functional calculus*, in Miniconference on operator theory and partial differential equations (North Ryde, 1986), 210–231, Proc. Centre Math. Anal. Austral. Nat. Univ., 14, Austral. Nat. Univ., Canberra, 1986.

[28] K. Merz, *On scales of Sobolev spaces associated to generalized Hardy operators*, Math. Z. **299** (2021), no. 1-2, 101–121. https://doi.org/10.1007/s00209-020-02651-0

[29] C. Prisuelos-Arribas, *Vertical square functions and other operators associated with an elliptic operator*, J. Funct. Anal. **277** (2019), no. 12, 108296, 63 pp. https://doi.org/10.1016/j.jfa.2019.108296

[30] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton Univ. Press, Princeton, NJ, 1993.

[31] S. Yang and Z. Yang, *A Two-weight boundedness criterion and its applications*, arXiv: 2112.04252.

Tongxin Kang
School of Mathematics and Statistics
Gansu Key Laboratory of Applied Mathematics and Complex Systems
Lanzhou University
Lanzhou 730000, P. R. China
Email address: kangtx20@lzu.edu.cn

Yang Zou
School of Mathematics and Statistics
Gansu Key Laboratory of Applied Mathematics and Complex Systems
Lanzhou University
Lanzhou 730000, P. R. China
Email address: zouy21@lzu.edu.cn