Bull. Korean Math. Soc. **60** (2023), No. 6, pp. 1621–1639
https://doi.org/10.4134/BKMS.b220769
pISSN: 1015-8634 / eISSN: 2234-3016

SECOND MAIN THEOREM FOR MEROMORPHIC MAPPINGS ON p-PARABOLIC MANIFOLDS INTERSECTING HYPERSURFACES IN SUBGENERAL POSITION

Yuehuan Zhu

Abstract. In this paper, we give an improvement for the second main theorems of algebraically non-degenerate meromorphic maps from generalized p-parabolic manifolds into projective varieties intersecting hypersurfaces in subgeneral position with some index, which extends the results of Han [6] and Chen-Thin [3].

1. Introduction

In 1933, Cartan [2] established a second main theorem for linearly nondegenerate holomorphic curves into complex projective spaces intersecting hyperplanes in general position. Later, Ahlfors [1], using an innovative geometry method, extended Cartan's second main theorem to linearly nondegenerate meromorphic maps on $\mathbb{C}^m$. Stoll and Wong [17,18] generalized the above results to algebraically non-degenerate meromorphic maps defined on parabolic manifolds. In 2004, Ru [13], using the filtration of the vector space of homogeneous polynomials, established a defect relation for linearly nondegenerate meromorphic mappings from parabolic manifolds into the projective space intersecting hypersurfaces. Subsequently, Ru [11] obtained a second main theorem of algebraically nondegenerate holomorphic curves into projective varieties, solving the Shiffman's conjecture [15]. Han [6] generalized Ru's results to meromorphic maps from p-parabolic manifolds into smooth projective varieties intersecting hypersurfaces in general position. The result of Han [6] was generalized by Chen-Thin [3] to the case of intersecting hypersurfaces in subgeneral position.

Recently, Ji-Yan-Yu [7] introduced the concept of the index of subgeneral position, and gave interesting improvements of some previously known second main theorems. Motivated by this new notion, we will prove a second main

Received November 6, 2022; Revised May 8, 2023; Accepted May 16, 2023.
2020 *Mathematics Subject Classification.* Primary 32H30.
Key words and phrases. Second main theorem, p-parabolic manifold, index.

theorem for meromorphic maps from p-parabolic manifolds into projective varieties intersecting hypersurfaces in subgeneral position with index, which are improvements and extensions of the results in Han [6] and Chen-Thin [3].

To state our result, we give some basic definitions and notations of p-parabolic manifolds. For more details, we refer the reader to [18, 19].

Definition. For $1 \leq p \leq m$, a Kahler manifold (M, ω) of dimension m is said to be a generalized p-parabolic manifold if there exists a plurisubharmonic function ϕ such that

- $\{\phi = -\infty\}$ is a closed subset of M with strictly lower dimension,
- ϕ is smooth on the open dense set $M \backslash \{\phi = -\infty\}$ satisfying

$$(dd^c\phi)^{p-1} \wedge \omega^{m-p} \not\equiv 0 \quad \text{and} \quad (dd^c\phi)^p \wedge \omega^{m-p} \equiv 0.$$

Note that m-parabolic manifolds are just ordinary parabolic manifolds. Write $\tau := \mathrm{e}^{\phi}$ and $\sigma := d^c\phi \wedge (dd^c\phi)^{p-1} \wedge \omega^{m-p}$, where $\tau \geq 0$ is called a p-parabolic exhaustion on M. For any positive real number $r > 0$, define

$$M[r] := \left\{x \in M : \tau(x) \leq r^2\right\}, \quad M(r) := \left\{x \in M : \tau(x) < r^2\right\}.$$

Then the pseudo-spheres associated with τ are defined as

$$M\langle r\rangle := M[r] \backslash M(r) = \left\{x \in M : \tau(x) = r^2\right\}.$$

By [6], we have, for all $r > 0$,

$$\int_{M\langle r\rangle} \sigma = \varsigma,$$

where ς is a constant depending only on the structure of M.

We next introduce the notion of associated maps. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map defined on a complex manifold of dimension m, and let $\mathbf{f}_z : U_z \to \mathbb{C}^{n+1}$ be a reduced representation of f on some a chart (z, U_z). If a global meromorphic $(m-1, 0)$-form B is given on M, we define the first B-derivative f'_B of $\mathbf{f}_z$ on U_z, by

$$d\mathbf{f}_z \wedge B = f'_B dz_1 \wedge dz_2 \wedge \cdots \wedge dz_m.$$

This operation can be iterated such that the k-th B-derivative $f_B^{(k)}$ is defined as

$$df_B^{(k-1)} \wedge B = f_B^{(k)} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_m$$

for $k = 1, \ldots, n$. Then the k-th associated map $f_k : M \to \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right)$ is defined by $f_k \mid_{U_z} = \mathbb{P}(\mathbf{f}_k)$ on U_z, where $\mathbb{P}$ is the projection. We note that the associated maps are independent of the choice of local charts, and thus are globally well-defined.

With the notions as above, we give some general conditions on p-parabolic manifolds.

(1) (M, τ, ω) denotes a p-parabolic manifold which possesses a globally defined meromorphic form B of degree $(m-1, 0)$, such that, for any linearly non-degenerate meromorphic map $f : M \to \mathbb{P}^n(\mathbb{C})$, the k-th associated map f_k is well defined for $k = 0, 1, \ldots, n$, where we put $f_0 := f$ and where f_n is a constant.

(2) There exists a Hermitian holomorphic line bundle $(\mathfrak{L}, \hbar)$ which admits a holomorphic section μ such that, for some increasing function $Y(\tau)$, we have

$$(-1)^{(m-1)(m-2)/2} m! \left(\frac{\sqrt{-1}}{2\pi}\right)^{m-1} |\mu|_{\hbar}^2 B \wedge \bar{B} \leq Y(\tau)\, (dd^c\tau)^{p-1} \wedge \omega^{m-p}.$$

A p-parabolic manifold (M, τ, ω) with the above assumptions is called an admissible p-parabolic manifold.

For $1 \leq p \leq m$, A_p is the p-th symmetric polynomial of the matrix $(\tau_{a\bar{b}})$ with respect to the Kahler metric ω. Actually, A_1 is the trace of $\tau_{a\bar{b}}$, while A_m is the determinant $\det(\tau_{a\bar{b}})\,(> 0)$. We denote

$$m_0(\mathfrak{L}; r, s) = \frac{1}{2}\int_{M\langle r\rangle} \log \frac{1}{|\mu|_{\hbar}^2}\sigma - \frac{1}{2}\int_{M\langle s\rangle} \log \frac{1}{|\mu|_{\hbar}^2}\sigma.$$

Following [7,20], we give a definition for hypersurfaces being in N-subgeneral position with index κ.

Definition. Let $X \subseteq \mathbb{P}^n(\mathbb{C})$ be an algebraic subvariety, and let $\{D_1, \ldots, D_q\}$ be a family of hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. Let N and κ be two positive integers satisfying $N \geq \dim X \geq \kappa$.

(1) The hypersurfaces $\{D_1, \ldots, D_q\}$ are called in general position in X if for any subset $I \subseteq \{1, \ldots, q\}$ with $\sharp I \leq \dim X + 1$,

$$\operatorname{codim}\left(\bigcap_{i\in I} D_i \bigcap X\right) \geq \sharp I.$$

(2) The hypersurfaces $\{D_1, \ldots, D_q\}$ are called in N-subgeneral position in X if for any subset $I \subseteq \{1, \ldots, q\}$ with $\sharp I \leq N + 1$,

$$\dim\left(\bigcap_{i\in I} D_i \bigcap X\right) \leq N - \sharp I.$$

(3) The hypersurfaces $\{D_1, \ldots, D_q\}$ are called in N-subgeneral position with index κ in X if $\{D_1, \ldots, D_q\}$ are in N-subgeneral position and for any subset $I \subseteq \{1, \ldots, q\}$ with $\sharp I \leq \kappa$,

$$\operatorname{codim}\left(\bigcap_{i\in I} D_i \bigcap X\right) \geq \sharp I$$

(Here we set $\dim \emptyset = -1$).

Our main result is the following.

Theorem 1.1. *Let $f : M \to X \subseteq \mathbb{P}^n(\mathbb{C})$ be an algebraically nondegenerate meromorphic map defined on an admissible p-parabolic manifold M, where X is a smooth variety of dimension $\ell \geq 1$. Let $\{D_1, \ldots, D_q\}$ be a collection of hypersurfaces in N-subgeneral position with index κ in X, and $\deg Q_j = d_j$ ($j = 1, \ldots, q$). Then, for any $\varepsilon > 0$ and $r > s > 0$, we have*[1]

$$\| \left(q - \frac{N - \ell + \kappa}{\kappa} (\ell + 1) - \varepsilon \right) T_f(r, s)$$

$$\leq \sum_{j=1}^{q} d_j^{-1} N_f^{\mathfrak{m}}(r, s; D_j) + c \left(m_0(\mathfrak{L}; r, s) + \mathrm{Ric}_p(r, s) + \varsigma \log^+ Y\left(r^2\right) + \varsigma \log^+ r \right),$$

where $c \gg 1$ is a constant, $\mathfrak{m} \leq \deg X d^\ell e^\ell \left(1 + \frac{u}{\ell}\right)^\ell$ is a positive integer with u controlled by (16), and $\mathrm{Ric}_p(r, s)$, $N_{Ramf}(r, s)$ are the counting functions of $\operatorname{div} A_p$ and the ramification divisor $\tilde{\theta}$, respectively. Whenever s is fixed, take $\mathfrak{m}$ to be the largest integer less than

$$(\deg X)^{\ell+1} \left[\frac{e d^{\ell+1} (N - \ell + \kappa)(2\ell + 5) l}{\kappa \varepsilon} \right]^\ell,$$

where $l = \frac{q!(\ell - \kappa + 1)}{\kappa!(n - \ell + 1)!(q - N - 1)!} + q$.

Letting $\mathfrak{m} \to \infty$, we get the following second main theorem without truncation.

Corollary 1.2. *Under the assumptions of Theorem 1.1, we have, for any $\varepsilon > 0$ and $r > s > 0$,*

$$\| \sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq \left(\frac{N - \ell + \kappa}{\kappa} (\ell + 1) + \varepsilon \right) T_f(r, s)$$

$$+ c \left(m_0(\mathfrak{L}; r, s) + \mathrm{Ric}_p(r, s) + \varsigma \log^+ Y\left(r^2\right) + \varsigma \log^+ r \right),$$

where $c \gg 1$ is a constant.

In this paper, we use the Hilbert weights method to prove a second main theorem with truncated counting functions, which extends the main result in [14] to the case of meromorphic maps from generalized p-parabolic manifolds into projective varieties. We note that the main theorem in [3] is just a special case of our main result when $\kappa = 1$. Next, we introduce a filtration of the vector space corresponding to the coordinate ring of the variety. This filtration is a generalization of Corvaja-Zannier's filtration [4], given by Dethloff-Tan [5]. By utilizing the algebraic properties of the filtration and properties of Hilbert polynomials, we provide an alternative proof of Corollary 1.2.

[1]Here, by the notation $\|$, we mean that the inequality holds for all $r \in (s, +\infty)$ outside a possible set of finite Lebesgue measure.

2. Basic notations and auxiliary results

In this section, we briefly recall some notations and facts in Nevanlinna theory on generalized p-parabolic manifolds.

2.1. Nevanlinna theory

Green-Jensen formula (on p-parabolic manifolds) [19], which is the fundamental formula in the theory of value distribution, is defined as follows, for $r > s > 0$,

$$\int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]} dd^c\varphi \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p} = \frac{1}{2}\int_{M\langle r\rangle} \varphi\sigma - \frac{1}{2}\int_{M\langle s\rangle} \varphi\sigma, \tag{1}$$

where φ is a plurisubharmonic function, and $dd^c\varphi$ denotes differentiation in the sense of currents.

Let $D \subseteq \mathbb{P}^n(\mathbb{C})$ be a hypersurface, and let $Q \in \mathbb{C}[x_0, \ldots, x_n]$ be the homogeneous polynomial of degree d defining D. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map such that $f(M) \nsubseteq D$. We choose a reduced representation $\mathbf{f}_z = (f_0, \ldots, f_n) : U_z \to \mathbb{C}^{n+1}$ on a local chart $U_z \subseteq M$. Then the Weil function of f with respect to D (or Q) is locally denoted as, for $x \notin (Q(\mathbf{f}_z))^{-1}(0)$,

$$\lambda_D(\mathbf{f}_z) := \lambda_D(f)|_{U_z} = \log \frac{\|\mathbf{f}_z\|^d \|Q\|}{|Q(\mathbf{f}_z)|},$$

where $\|Q\|$ is the maximum norm of the coefficients appearing in Q. Note that $\lambda_D(f)$ is independent of the reduced representations and hence is global well-defined. Correspondingly, the proximity function $m_f(r, D)$ is defined as

$$m_f(r, D) = \int_{M\langle r\rangle} \lambda_D(f)\sigma.$$

Without loss of generality, we may assume $\|Q\| = 1$ in the definition of the Weil function and the proximity function.

Put $\theta_f^D|_{U_z} := \operatorname{div}(Q(\mathbf{f}_z))|_{U_z}$ on the local chart (z, U_z). Given two reduced representations $\mathbf{f}_\alpha, \mathbf{f}_\beta$ on the overlapping charts U_α, U_β correspondingly, we have $\mathbf{f}_\alpha = h_{\alpha\beta}\mathbf{f}_\beta$ on $U_\alpha \cap U_\beta$, for a non-vanishing holomorphic function $h_{\alpha\beta}$, and thus θ_f^D is a global well-defined divisor on M. Then the counting function of f with respect to D is defined by

$$N_f(r, s; D) = \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]} \theta_f^D \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p}$$

for $0 < s < r$. Writing θ_f^D as locally finite sums $\theta_f^D = \sum_{\lambda\in A} k_\lambda v_\lambda$ of irreducible analytic hypersurfaces, the $\mathfrak{m}$-th truncated divisor is locally defined as $\theta_f^{\mathfrak{m},D} := \min\{\mathfrak{m}, k_\lambda\} v_\lambda$ for some positive integer $\mathfrak{m}$. Then the counting function with truncated level $\mathfrak{m}$ is defined by

$$N_f^{\mathfrak{m}}(r, s; D) = \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]} \theta_f^{\mathfrak{m},D} \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p}.$$

Accordingly, for any $r > s > 0$, the characteristic function of f is defined as

$$T_f(r,s) := \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} f^*\Omega_{\mathrm{FS}} \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p},$$

where Ω_{FS} is Fubini-Study form on $\mathbb{P}^n(\mathbb{C})$.

Now, the Green-Jensen formula (1) implies:

Theorem 2.1 (First Main Theorem [6]). *Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a nonconstant meromorphic map defined on a p-parabolic manifold M, and let $D \subset \mathbb{P}^n(\mathbb{C})$ be a hypersurface of degree d such that $f(M) \not\subseteq D$. Then for any $r > s > 0$,*

$$dT_f(r,s) = N_f(r,s;D) + m_f(r,D) - m_f(s,D).$$

2.2. Some lemmas

Let $X \subseteq \mathbb{P}^n$ be a projective variety of dimension ℓ. Set $V_u = \mathbb{C}\left[x_0,\ldots,x_n\right]_u$ and $\widehat{V_u} = \frac{\mathbb{C}[x_0,\ldots,x_n]_u}{\mathcal{I}(X)_u}$, where $\mathcal{I}(X)$ is the ideal of $\mathbb{C}\left[x_0,\ldots,x_n\right]$ defining X and $\mathcal{I}(X)_u = \mathcal{I}(X) \cap \mathbb{C}\left[x_0,\ldots,x_n\right]_u$. The Hilbert polynomial $H_X(u)$ of X is defined by

$$H_X(u) := \dim\left(\mathbb{C}\left[x_0,\ldots,x_n\right]_u / I_X(u)\right).$$

Then for u big enough, we have

$$H_X(u) = \dim_{\mathbb{C}} \frac{\mathbb{C}\left[x_0,\ldots,x_n\right]_u}{\mathcal{I}(X)_u} = \dim_{\mathbb{C}} \widehat{V_u} = \deg V \cdot \frac{u^\ell}{\ell!} + O\left(u^{\ell-1}\right),$$

by the theory of Hilbert polynomials (see [16]). The Hilbert Weight $S_X(u,\mathbf{c})$ of X with respect to some tuple $\mathbf{c} = (c_0,\ldots,c_n) \in \mathbb{R}^{n+1}$ is defined by

$$S_X(u,\mathbf{c}) = \max\left(\sum_{j=1}^{H_X(u)} \mathbf{a}_j \cdot \mathbf{c}\right),$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1},\ldots,\mathbf{x}^{\mathbf{a}_{H_X(u)}}$ whose residue classes modulo $\mathcal{I}(X)$ form a basis of $\mathbb{C}\left[x_0,\ldots,x_n\right]_u / \left(\mathcal{I}(X)\right)_u$, where $\mathbf{a_j} = (a_{j0},\ldots,a_{jn}) \in \mathbb{Z}_{\geq 0}^{n+1}$ is an $(n+1)$-dimensional multi-index, and $\mathbf{x}^{\mathbf{a_j}} = x_0^{a_{j0}} \ldots x_n^{a_{jn}}$.

Lemma 2.2 (see [11,12]). *Let $X \subseteq \mathbb{P}^n$ be an algebraic subvariety of dimension ℓ and degree $\triangle$. Let $u > \triangle$ be an integer, $\mathbf{c} \in \mathbb{R}_{\geq 0}^{n+1}$, and let $\{i_0,\ldots,i_\ell\}$ be a subset of $\{0,\ldots,n\}$ satisfying $\{x = [x_0 : \cdots : x_n] \in \mathbb{P}^n : x_{i_0} = \cdots = x_{i_\ell} = 0\} \cap X = \emptyset$. Then*

$$\frac{1}{uH_X(u)} S_X(u,\mathbf{c}) \geq \frac{1}{(\ell+1)}\left(c_{i_0} + \cdots + c_{i_\ell}\right) - \frac{(2\ell+1)\triangle}{u}\left(\max_{0 \leq i \leq n} c_i\right).$$

Lemma 2.3 (see [17]). *Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic map defined on a generalized p-parabolic manifold M, and let $\{H_j\}_{j=1}^q$*

be a collection of hyperplanes of $\mathbb{P}^n(\mathbb{C})$ *in general position. We have*

$$\sum_{j=1}^{q}\left(\theta_f^{H_j}-\theta_f^{n,H_j}\right)\leq\tilde{\theta}.$$

Lemma 2.4 (see [6]). *Let* $f:M\to\mathbb{P}^n(\mathbb{C})$ *be a linearly nondegenerate meromorphic map defined on an admissible p-parabolic manifold* M*. Let* $\{H_j\}_{j=1}^q$ *be arbitrary hyperplanes in* $\mathbb{P}^n(\mathbb{C})$*. Then, for* $r>s>0$*, we have*

$$\begin{aligned}
&\| \int_{M\langle r\rangle}\max_{\mathcal{K}}\sum_{j\in\mathcal{K}}\lambda_{H_j}(f)\sigma\\
&\leq (n+1)T_f(r,s)-N_{\mathrm{Ram}\,f}(r,s)\\
&\quad+\frac{1}{2}n(n+1)m_0(\mathfrak{L};r,s)+\frac{1}{2}n(n+1)\operatorname{Ric}_p(r,s)+\frac{1}{2}\varsigma n(n+1)\log^+T_f(r,s)\\
&\quad+\frac{1}{2}\varsigma n(n+1)\left(\log^+m_0(\mathfrak{L};r,s)+\log^+Y\left(r^2\right)+\log^+\operatorname{Ric}_p(r,s)+\log^+r\right),
\end{aligned}$$

where $\max_{\mathcal{K}}$ *ranges over all subsets* $\mathcal{K}$ *of* $\{1,\ldots,q\}$ *such that the hyperplanes* $\{H_j\}_{j\in\mathcal{K}}$ *are linearly independent.*

3. Second main theorems

3.1. Proof of Theorem 1.1

Proof. Firstly, we prove the main theorem for the case where the hypersurfaces have the same degree d. Let $Q_i\in\mathbb{C}[x_0,\ldots,x_n]$ be the homogeneous polynomial defining D_i for $1\leq i\leq q$. We choose a reduced representation $\mathbf{f}$ of f on an arbitrary local chart $U\subseteq M$. For any $z\in U$ (excluding the zeros of all $Q_j(\mathbf{f})$ in U), there exists a permutation $I_i=(i_1,\ldots,i_q)$ of $\{1,\ldots,q\}$ such that

$$(2)\qquad |Q_{i_1}\circ\mathbf{f}(z)|\leq|Q_{i_2}\circ\mathbf{f}(z)|\leq\cdots\leq\left|Q_{i_q}\circ\mathbf{f}(z)\right|.$$

We consider the following positive function [9]

$$h(z)=\max_{1\leq t\leq N+1}\left\{\frac{|Q_{i_t}(z)|}{\|z\|^d}\right\},$$

where $z=[z_0:\cdots:z_n]\in\mathbb{P}^n(\mathbb{C})$ and $\|z\|=\left(\sum_{i=0}^n|z_i|^2\right)^{\frac{1}{2}}$. We see that h is a positive continuous function on X. By the compactness of X, there exist two positive constants c_1 and c_2, independence of the choice of I_i, such that $c_1=\min_{z\in X}h(z)$ and $c_2=\max_{z\in X}h(z)$. Then, we have

$$(3)\qquad c_1\|\mathbf{f}\|^d\leq\max_{1\leq t\leq N+1}|Q_{i_t}(\mathbf{f})|\leq c_2\|\mathbf{f}\|^d.$$

Therefore (2) and (3) imply that

$$(4)\qquad \prod_{j=1}^{q}\frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f})(z)|}\leq\frac{1}{c_1^{q-N}}\prod_{k=1}^{N}\frac{\|\mathbf{f}(z)\|^d}{|Q_{i_k}(\mathbf{f})(z)|}.$$

Since the hypersurfaces $\{D_1, \ldots, D_q\}$ are located in N-subgeneral position with index κ in X, we get

$$\operatorname{codim}\left(\bigcap_{t=1}^{\kappa} D_{i_t} \bigcap X\right) \geq \kappa.$$

With respect to the hypersurfaces $\{D_{i_1}, \ldots, D_{i_N}\}$, we can construct $(\ell-\kappa)$-homogeneous polynomials of the following forms:

$$P_j = \sum_{t=\kappa+1}^{N-\ell+j} b_{jt} Q_{i_t}, \quad b_{jt} \in \mathbb{C}, \quad j = \kappa+1, \ldots, \ell, \tag{5}$$

such that $\{D_{i_1}, \ldots, D_{i_\kappa}, \tilde{D}_{i_{\kappa+1}}, \ldots, \tilde{D}_{i_\ell}\}$ are located in general position on X, where $\{\tilde{D}_{i_{\kappa+1}}, \ldots, \tilde{D}_{i_\ell}\}$ are defined by the above P_j's, respectively. This method of construction is due to Quang [8].

Now, we construct $P_{\kappa+1}$ as follows. Let Γ be the set of irreducible components of $(\bigcap_{t=1}^{\kappa} D_{i_t} \bigcap X)$ with codimension κ. For any $\Delta \in \Gamma$, let

$$X_\Delta = \Bigg\{ \mathbf{b} = (b_{\kappa+1}, \ldots, b_{N-\ell+\kappa+1}) \in \mathbb{C}^{N-\ell+1} : \Delta \subseteq \widetilde{D}, \text{ where } \widetilde{D} \text{ is the hypersurface defined by } \widetilde{Q} = \sum_{t=\kappa+1}^{N-\ell+\kappa+1} b_t Q_{i_t} \Bigg\}.$$

Observe that $\widetilde{D} = \mathbb{P}^n(\mathbb{C})$ in the case where $\widetilde{Q}$ is the zero polynomial. By definition, X_Δ is a subspace of $\mathbb{C}^{N-\ell+1}$. Since

$$\operatorname{codim}\left(\bigcap_{t=1}^{N-\ell+\kappa+1} D_{i_t} \bigcap X\right) \geq \kappa+1,$$

there exists some $t \in \{\kappa+1, \ldots, N-\ell+\kappa+1\}$ such that $\Delta \not\subseteq \tilde{D}_{i_t}$. This implies that X_Δ is a proper subspace of $\mathbb{C}^{N-\ell+1}$. In view of the fact that Γ is at most countable, we have

$$\mathbb{C}^{N-\ell+1} \setminus \bigcup_{\Delta \in \Gamma} X_\Delta \neq \emptyset.$$

We denote by $\widetilde{D}_{i_{\kappa+1}}$ the hypersurface defined by $\widetilde{P}_{\kappa+1} = \sum_{t=\kappa+1}^{N-\ell+\kappa+1} b_t Q_{i_t}$, where $\mathbf{b} = (b_{\kappa+1}, \ldots, b_{N-\ell+\kappa+1}) \in \mathbb{C}^{N-\ell+1} \setminus \cup_{\Delta\in\Gamma} V_\Delta$. This clearly implies that

$$\operatorname{codim}\left(\bigcap_{t=1}^{\kappa} D_{i_t} \bigcap X \bigcap \widetilde{D}_{i_{\kappa+1}}\right) \geq \kappa+1.$$

Next, let Γ' be the set of irreducible components of $\left(\bigcap_{t=1}^{\kappa} D_{i_t} \bigcap X \bigcap \widetilde{D}_{i_{\kappa+1}}\right)$ with codimension $\kappa+1$. For any $\Delta' \in \Gamma'$, put

$$X_{\Delta'} = \Bigg\{ \mathbf{b} = (b_{\kappa+1}, \ldots, b_{N-\ell+\kappa+2}) \in \mathbb{C}^{N-\ell+2} : \Delta' \subseteq \widetilde{D}, \text{ where } \widetilde{D} \text{ is the hypersurface defined by } \widetilde{Q} = \sum_{t=\kappa+1}^{N-\ell+\kappa+2} b_t Q_{i_t} \Bigg\}.$$

Similarly, Δ' is a subspace of $\mathbb{C}^{N-\ell+2}$. Since

$$\operatorname{codim}\left(\bigcap_{t=1}^{N-\ell+\kappa+2} D_{i_t} \bigcap X\right) \geq \kappa+2,$$

there exists some $t \in \{\kappa+1, \ldots, N-\ell+\kappa+2\}$ such that $X_{\Delta'} \not\subseteq \tilde{D}_{i_t}$. This implies that X_Δ is a proper subspace of $\mathbb{C}^{N-\ell+2}$. Since Γ' is at most countable,

$$\mathbb{C}^{N-\ell+2} \setminus \bigcup_{\Delta' \in \Gamma'} X_{\Delta'} \neq \emptyset.$$

Denote by $\widetilde{D}_{i_{\kappa+2}}$ the hypersurface defined by $\widetilde{P}_{\kappa+2} = \sum_{t=\kappa+1}^{N-\ell+\kappa+2} b_t Q_{i_t}$, where $\mathbf{b} = (b_{\kappa+1}, \ldots, b_{N-\ell+\kappa+2}) \in \mathbb{C}^{N-\ell+2} \setminus \bigcup_{\Delta' \in \Gamma'} X_{\Delta'}$. Obviously,

$$\operatorname{codim}\left(\bigcap_{t=1}^{\kappa} D_{i_t} \bigcap X \bigcap \widetilde{D}_{i_{\kappa+1}} \bigcap \widetilde{D}_{i_{\kappa+2}}\right) \geq \kappa+2.$$

Repeating the above argument, the construction is complete. Putting $\tilde{D}_{i_t} := D_{i_t}$ for $1 \leq t \leq \kappa$, then $\{\tilde{D}_{i_1}, \ldots, \tilde{D}_{i_\ell}\}$ are in general position on X. For any permutation $(i_1, \ldots, i_q)$ of $\{1, \ldots, q\}$, we can always construct homogeneous polynomials $\{P_{\kappa+1}, \ldots, P_\ell\}$ satisfying (5), correspondingly.

Since there are only finitely choices of N-polynomials in $\{Q_1, \ldots, Q_q\}$, we can find a constant $C > 0$, independent of z, such that

$$|P_t(\mathbf{f})(z)| \leq C \max_{\kappa+1 \leq j \leq N-\ell+t} \left|Q_{i_j}(\mathbf{f})(z)\right| = C \left|Q_{i_{N-\ell+t}}(\mathbf{f})(z)\right|$$

for $\kappa+1 \leq t \leq \ell$, and thus by the definition, we get

$$\lambda_{D_{i_{N-\ell+t}}}(\mathbf{f}(z)) \leq \lambda_{\tilde{D}_{i_t}}(\mathbf{f}(z)) + O(1) \quad \text{for } \kappa+1 \leq t \leq \ell. \tag{6}$$

Combining the above inequality with (4), we get

$$\begin{aligned} &\sum_{j=1}^{q} \lambda_{D_j}(\mathbf{f}(z)) \\ &\leq \sum_{t=1}^{\kappa} \lambda_{D_{i_t}}(\mathbf{f}(z)) + \sum_{t=\kappa+1}^{N-\ell+\kappa} \lambda_{D_{i_t}}(\mathbf{f}(z)) + \sum_{t=N-\ell+\kappa+1}^{N} \lambda_{D_{i_t}}(\mathbf{f}(z)) + O(1) \end{aligned} \tag{7}$$

$$\begin{aligned}
&\le \sum_{t=1}^{\kappa} \lambda_{\tilde{D}_{i_t}}(\mathbf{f}(z)) + \sum_{t=\kappa+1}^{N-\ell+\kappa} \lambda_{D_{i_t}}(\mathbf{f}(z)) + \sum_{t=\kappa+1}^{\ell} \lambda_{\tilde{D}_{i_t}}(\mathbf{f}(z)) + O(1)\\
&= \sum_{t=1}^{\ell} \lambda_{\tilde{D}_{i_t}}(\mathbf{f}(z)) + \sum_{t=\kappa+1}^{N-\ell+\kappa} \lambda_{D_{i_t}}(\mathbf{f}(z)) + O(1)\\
&\le \frac{N-\ell+\kappa}{\kappa}\left(\sum_{t=1}^{\ell} \lambda_{\tilde{D}_{i_t}}(\mathbf{f}(z))\right) + O(1).
\end{aligned}$$

By (5), we can also construct a homogeneous polynomial

$$P_{\ell+1} = \sum_{t=\kappa+1}^{N+1} b_{jt} Q_{i_t},$$

which defines $\tilde{D}_{i_{\ell+1}}$ such that $\{\tilde{D}_{i_1}, \ldots, \tilde{D}_{i_{\ell+1}}\}$ are in general position on X. Let I denote the set of all permutations of $\{1, \ldots, q\}$, written as $I = \{I_1, \ldots, I_{\#I}\}$. For each $I_i := (i_1, \ldots, i_q) \in I$, we use $P_{i,\kappa+1}, \ldots, P_{i,\ell+1}$ to denote the polynomials obtained from the hypersurfaces $\{D_{i_1}, \ldots, D_{i_{N+1}}\}$. For each $t \in \{\kappa + 1, \ldots, \ell+1\}$, the polynomial $P_{i,t}$ is determined only by $Q_{i_{\kappa+1}}, \ldots, Q_{i_{N-\ell+t}}$, so we can take a subset $\hat{I} \subseteq I$ with cardinality $l = \frac{q!}{\kappa!(N-\ell+1)!(q-N-1)!}$ to construct all possible polynomials of the above form [14]. By renumbering, we may put $\hat{I} = \{I_1, I_2 \ldots, I_l\}$. Consider the map $\chi : X \to \mathbb{P}^{k-1}(\mathbb{C})$ defined by

$$\chi(z) =: [Q_1 : \cdots : Q_q : P_{1,\kappa+1}(z) : \cdots : P_{1,\ell+1}(z) : \cdots : P_{l,\kappa+1}(z) : \cdots : P_{l,\ell+1}(z)]$$

for $k = (\ell - \kappa + 1)l + q$. Set $Z = \chi(X)$. Then χ is a finite morphism, Z is an ℓ-dimensional algebraic subvariety of $\mathbb{P}^{k-1}(\mathbb{C})$, and $\triangle := \deg Z \le d^{\ell} \deg X$.

Now, let $\{\mathbf{f}_\lambda, U_\lambda, \lambda \in \Lambda\}$ be a system of local reduced representations of f. Given any $z \notin \cup_{j=1}^{q} (Q_j(\mathbf{f}_\lambda))^{-1}(0)$, set

$$\mathbf{c}(z) = (c_{0,1}(z), \ldots, c_{0,q}(z), c_{1,\kappa+1}(z), \ldots, c_{1,\ell+1}(z), \ldots, c_{l,\kappa+1}(z), \ldots, c_{l,\ell+1}(z)),$$

in which $c_{i,t}(z) = \lambda_{D_t}(\mathbf{f}_\lambda(z))$ for $i = 0$, $1 \le t \le q$, and $c_{i,t}(z) = \lambda_{\tilde{D}_{i_t}}(\mathbf{f}_\lambda(z))$ for $1 \le i \le l$, $\kappa + 1 \le t \le \ell + 1$. Let $\mathcal{I}(Z)$ be the ideal in $\mathbb{C}[x_1, \ldots, x_k]$ defining Z. Put $\mathcal{I}(Z)_u = \mathbb{C}[x_1, \ldots, x_k]_u \cap \mathcal{I}(Z)$ for some positive integer $u > \triangle$. Since $\{\tilde{D}_{i_1}, \ldots, \tilde{D}_{i_{\ell+1}}\}$ are in general position with respect to X, we have, by Lemma 2.2 and (7),

$$p \sum_{j=1}^{q} \lambda_{D_j}(\mathbf{f}_\lambda(z)) \le \frac{S_Z(u, \mathbf{c}(z))}{u H_Z(u)} + \frac{(2\ell+1)\triangle}{u} \max_{i,t} c_{i,t}(z) \tag{8}$$

for $p = \frac{\kappa}{(N-\ell+\kappa)(\ell+1)}$. Fix a basis $\phi_0, \ldots, \phi_{n_u}$ for $\widehat{V_u}$, where $\widehat{V_u} = \frac{\mathbb{C}[x_1,\ldots,x_k]_u}{\mathcal{I}(Z)_u}$, and $n_u = H_Z(u) - 1$. We consider the map

$$F = [\phi_0(\chi \circ f) : \cdots : \phi_{n_u}(\chi \circ f)] : M \to \mathbb{P}^{n_u}.$$

Set $\mathbf{F}_\lambda = (F_{0,\lambda}, \ldots, F_{n_u,\lambda})$, where $F_{j,\lambda} = \phi_j\big(Q_1(\mathbf{f}_\lambda), \ldots, Q_q(\mathbf{f}_\lambda), P_{1,\kappa+1}(\mathbf{f}_\lambda), \ldots, P_{1,\ell+1}(\mathbf{f}_\lambda), \ldots, P_{l,\kappa+1}(\mathbf{f}_\lambda), \ldots, P_{l,\ell+1}(\mathbf{f}_\lambda)\big)$ for $j = 0, 1, \ldots, n_u$. Note that $\mathbf{F}_\lambda$ is a reduced representation of F on U_λ, and F is linearly nondegenerate.

For $\mathbf{a_j} = (a_{j1}, \ldots, a_{jk}) \in \mathbb{Z}^k_{\geq 0}$, put $\mathbf{x}^{\mathbf{a_j}} = x_1^{a_{j1}} \cdots x_k^{a_{jk}}$. By the definition of Hilbert weight, there exist monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{H_Z(u)}}$ (depending on z) whose residue classes modulo $\mathcal{I}(Z)_u$ form a basis of $\widehat{V_u}$ such that $\sum_{j=1}^{H_Z(u)} \mathbf{a}_j \cdot \mathbf{c} = S_Z(u, \mathbf{c}(z))$. For each $1 \leq j \leq H_Z(u)$, write $\mathbf{x}^{\mathbf{a}_j} = L_{j,z}\left(\phi_0, \ldots \phi_{n_u}\right)$, where $L_{j,z}$ are linear forms that are linearly independent for every fixed z. Note that there are only finitely many choices of $L_{j,z}$ in total. We get

$$L_{j,z}\left(\mathbf{F}_\lambda(z)\right) = \Big(Q_1\left(\mathbf{f}_\lambda\right)(z)\Big)^{a_{j1}} \cdots Q_q\Big(\left(\mathbf{f}_\lambda\right)(z)\Big)^{a_{jq}} \cdot \Big(P_{1,\kappa+1}\left(\mathbf{f}_\lambda\right)(z)\Big)^{a_{j,q+1}} \cdots P_{l,\ell+1}\Big(\left(\mathbf{f}_\lambda\right)(z)\Big)^{a_{jk}}.$$

This gives that

$$-\log\left|L_{j,z}\left(\mathbf{F}_\lambda(z)\right)\right| = \mathbf{a}_j \cdot \mathbf{c}(z) - u\log\left\|\mathbf{f}_\lambda(z)\right\|^d,$$

and then

$$(9) \qquad -\sum_{j=1}^{H_Z(u)} \log\left|L_{j,z}\left(\mathbf{F}_\lambda(z)\right)\right| = S_Z(u, \mathbf{c}(z)) - uH_Z(u)\log\left\|\mathbf{f}_\lambda(z)\right\|^d.$$

By (8) and (9), we have

$$(10) \quad p\sum_{j=1}^{q} \lambda_{D_j}(\mathbf{f}_\lambda(z)) \leq \frac{1}{uH_Z(u)} \sum_{j=1}^{H_Z(u)} \lambda_{L_{j,z}}(\mathbf{F}_\lambda(z)) + \frac{1}{u}\log\frac{\|\mathbf{f}_\lambda(z)\|^{du}}{\|\mathbf{F}_\lambda(z)\|} + \frac{(2\ell+1)\triangle}{u}\max_{i,t} c_{i,t}(z) + O\left(\frac{1}{u}\right),$$

where $O\left(\frac{1}{u}\right)$ denotes a bounded term independent of z. By the definition of F, we have

$$c_1\left\|\mathbf{f}_\lambda(z)\right\|^{du} \leq \left\|\mathbf{F}_\lambda(z)\right\| \leq c_2\left\|\mathbf{f}_\lambda(z)\right\|^{du}$$

for positive constants c_1 and c_2 independent of λ. We derive that

$$p\sum_{j=1}^{q} \lambda_{D_j}(\mathbf{f}_\lambda(z)) \leq \frac{1}{uH_Z(u)} \sum_{j=1}^{H_Z(u)} \lambda_{L_{j,z}}(\mathbf{F}_\lambda(z)) + \frac{(2\ell+1)\triangle}{u}\max_{i,t} c_{i,t}(z) + O(1),$$

where the bounded term $O(1)$ does not depend on z. Taking integration on both sides of the above inequality, we obtain

$$(11) \qquad \| p\sum_{j=1}^{q} m_f\left(r, D_j\right) \leq \frac{1}{uH_Z(u)} \int_{M\langle r\rangle} \max_{\mathcal{K}} \sum_{j\in\mathcal{K}} \lambda_{L_j}(\mathbf{F}(z))\sigma + \frac{(2\ell+1)\triangle}{u}\int_{M\langle r\rangle} \sum_{i,t} c_{i,t}(z)\sigma + O(1),$$

where $\max_{\mathcal{K}}$ ranges over all subsets of all possible linear forms $\{L_{j,z}\}$. By Lemma 2.4 and by the fact $T_F(r,s) = du \cdot T_f(r,s) + O(1)$ [6], we have, for any $\varepsilon' > 0$,

$$\begin{aligned}(12)\quad & \| p \sum_{j=1}^{q} m_f(r, D_j) \\ & \le dT_f(r,s) - \frac{N_{\mathrm{RamF}}(r,s)}{uH_Z(u)} + \frac{d\varepsilon'}{H_Z(u)} T_f(r,s) \\ & + \left(\frac{H_Z(u)-1}{2u} + \frac{\varepsilon'}{uH_Z(u)}\right)\left(m_0(\mathfrak{L}; r,s) + Ric_p(r,s) + \kappa \log^+ Y\left(r^2\right) + \kappa \log^+(r)\right) \\ & + \frac{(2\ell+1)\triangle}{u}\left[\sum_{1\le j\le q} m_f(r,D_j) + \sum_{\substack{1\le i\le l \\ \kappa+1\le t\le \ell+1}} m_f(r,\tilde{D}_{i_t})\right] + O(1).\end{aligned}$$

We next verify that

$$\begin{aligned}(13)\quad & \frac{N_{\mathrm{RamF}}(r,s)}{uH_Z(u)} \\ & \ge p\sum_{j=1}^{q}\left[N_f(r,s;D_j) - N_f^{n_u}(r,s;D_j)\right] \\ & - \frac{(2\ell+1)\triangle}{u}\left[\sum_{1\le j\le q} N_f(r,s;D_j) + \sum_{\substack{1\le i\le l \\ \kappa+1\le t\le \ell+1}} N_f(r,s;\tilde{D}_{i_t})\right].\end{aligned}$$

From the assumption of subgeneral position, there are at most N-hypersurfaces among $\{D_1, \ldots, D_q\}$ passing through $f(z)$ for any $z \in \cup_{j=1}^{q}\left(Q_j(\mathbf{f}_\lambda)\right)^{-1}(0)$. Without loss of generality, for fixed z, we may assume that

$$\begin{aligned}\mathrm{ord}_{E,z}(Q_1(\mathbf{f}_\lambda)) \ge \cdots \ge \mathrm{ord}_{E,z}(Q_{\mathfrak{s}}(\mathbf{f}_\lambda)) > 0 &= \mathrm{ord}_{E,z}(Q_{\mathfrak{s}+1}(\mathbf{f}_\lambda)) \\ &= \cdots = \mathrm{ord}_{E,z}(Q_p(\mathbf{f}_\lambda)),\end{aligned}$$

where $\mathrm{ord}_{E,z}(Q_j(\mathbf{f}_\lambda))$ is the vanishing order of $Q_j(\mathbf{f}_\lambda)$ along E at z for some fixed irreducible hypersurface E, and $\mathfrak{s} \in \{0, 1, \ldots, N\}$. Denote $P_{\kappa+1}, \ldots, P_{\ell+1}$ the polynomials obtained from $\{Q_1, \ldots, Q_{N+1}\}$, and then we have

$$\mathrm{ord}_{E,z}(P_t(\mathbf{f}_\lambda)) \ge \mathrm{ord}_{E,z}(Q_{N-\ell+t}(\mathbf{f}_\lambda)), \quad t = \kappa+1, \ldots, \ell+1.$$

We define

$$\mathbf{c} = (c_{0,1}, \ldots, c_{0,q}, c_{1,\kappa+1}, \ldots, c_{1,\ell+1}, \ldots, c_{t,\kappa+1}, \ldots, c_{t,\ell+1}),$$

where $c_{i,t} = \max\{0, \mathrm{ord}_{E,z}\big(Q_t(\mathbf{f}_\lambda)\big) - n_u\}$ for $i = 0$, $1 \le t \le q$, and $c_{i,t} = \max\{0, \mathrm{ord}_{E,z}(P_{i,t}(\mathbf{f}_\lambda)) - n_u\}$ for $1 \le i \le l$, $\kappa+1 \le t \le \ell+1$. Likewise, take monomials $\mathbf{x}^{\hat{\mathbf{a}}_1}, \ldots, \mathbf{x}^{\hat{\mathbf{a}}_{H_Z(u)}}$ whose residue classes modulo $\mathcal{I}(Z)_u$ form a basis of $\widehat{V_u}$ such that

$$\sum_{j=1}^{H_Z(u)} \hat{\mathbf{a}}_j \cdot \mathbf{c} = S_Z(u, \mathbf{c}) \quad \text{for} \;\; \hat{\mathbf{a}}_j = (\hat{a}_{j1}, \ldots, \hat{a}_{jk}) \in \mathbb{Z}^k_{\geq 0}.$$

Furthermore, there are linear forms $\{L_j\}_{j=1}^{H_Z(u)}$ such that $\mathbf{x}^{\hat{\mathbf{a}}_j} = L_j(\phi_0, \ldots, \phi_{n_u})$ for every $1 \leq j \leq H_Z(u)$. We then have

$$S_Z(u, \mathbf{c}) \leq \sum_{j=1}^{H_Z(u)} \max\{0, \operatorname{ord}_{E,z}(L_j(\mathbf{F}_\lambda)) - n_u\}. \tag{14}$$

On the flip side, by Lemma 2.2 we get

$$\begin{aligned} \frac{S_Z(u, \mathbf{c})}{uH_Z(u)} &\geq \frac{1}{\ell+1}\Bigg(\sum_{j=1}^{\kappa} \max\{0, \operatorname{ord}_{E,z}(Q_j(\mathbf{f}_\lambda)) - n_u\} \\ &\quad + \sum_{t=\kappa+1}^{\ell} \max\{0, \operatorname{ord}_{E,z}(Q_{N-\ell+t}(\mathbf{f}_\lambda)) - n_u\}\Bigg) - \frac{(2\ell+1)\triangle}{u} \max_{i,t} c_{i,t} \\ &\geq p\left(\sum_{t=1}^{N} \max\{0, \operatorname{ord}_{E,z}(Q_t(\mathbf{f}_\lambda)) - n_u\}\right) - \frac{(2\ell+1)\triangle}{u} \max_{i,t} c_{i,t} \\ &= p\left(\sum_{j=1}^{q} \max\{0, \operatorname{ord}_{E,z}(Q_j(\mathbf{f}_\lambda)) - n_u\}\right) - \frac{(2\ell+1)\triangle}{u} \max_{i,t} c_{i,t}. \end{aligned}$$

Combining (14), Lemma 2.3 and the above inequality, we get

$$\frac{\tilde{\theta}}{uH_Z(u)} \geq p\sum_{j=1}^{q}\left[\theta_f^{D_j} - \theta_f^{n_u, D_j}\right] - \frac{(2\ell+1)\triangle}{u}\left[\sum_{j=1}^{q}\theta_f^{D_j} + \sum_{\substack{1\leq i\leq l \\ \kappa+1\leq t\leq \ell+1}} \theta_f^{D_{i_t}}\right]. \tag{15}$$

Integrating both sides of (15), we thus get (13). By (12) and (13) yields

$$\begin{aligned} &\| \, (pq-1)\, T_f(r,s) \\ &\leq \left(\frac{\varepsilon'}{H_Z(u)} + \frac{(2\ell+1)\triangle k}{u}\right) T_f(r,s) + \left(\frac{H_Z(u)-1}{2du} + \frac{\varepsilon'}{duH_Z(u)}\right) \\ &\quad \cdot \Big(m_0(\mathfrak{L}; r, s) + Ric_p(r,s) + \kappa \log^+ Y\left(r^2\right) + \kappa \log^+(r)\Big) + \frac{p}{d}\sum_{j=1}^{q} N_f^{n_u}(r, s; D_j) \\ &\quad + \frac{(2\ell+1)\triangle}{du}\left[\sum_{1\leq j\leq q} m_f(s, D_j) + \sum_{\substack{1\leq i\leq l \\ \kappa+1\leq t\leq \ell+1}} m_f(s, \tilde{D}_{i_t})\right] + O(1). \end{aligned}$$

For any $\varepsilon > 0$, we choose u as the smallest integer such that

$$u > \frac{(2\ell+1)\triangle k}{p\varepsilon}, \quad \frac{\varepsilon'}{H_Z(u)} + \frac{(2\ell+1)\triangle k}{u} < p\varepsilon, \tag{16}$$

$$\sum_{1\le j\le q} m_f(s,D_j)+\sum_{\substack{1\le i\le l\\ \kappa+1\le t\le \ell+1}} m_f(s,\tilde{D}_{i_t})<u.$$

Hence

$$(17)\qquad \|\left(q-\frac{N-\ell+\kappa}{\kappa}(\ell+1)-\varepsilon\right)T_f(r,s)$$
$$\le \sum_{j=1}^{q}\frac{1}{d}N_f^{n_u}(r,s;D_j)+c\left(m_0(\mathfrak{L};r,s)+\mathrm{Ric}_p(r,s)+\varsigma\log^+ Y\left(r^2\right)+\varsigma\log^+ r\right),$$

where $c\le \frac{H_Z(u)-1}{2dpu}+1$. For some fixed s, we can choose u as the smallest integer satisfying

$$u>\frac{(2\ell+1)\triangle k}{p\varepsilon},\quad \frac{\varepsilon'}{H_Z(u)}+\frac{(2\ell+1)\triangle k}{u}<p\varepsilon$$

such that (17) makes sense. Then we give an explicit estimate for n_u:

$$n_u=H_Z(u)-1\le \triangle\binom{u+\ell}{\ell}\le \deg X d^{\ell}e^{\ell}\left(1+\frac{u}{\ell}\right)^{\ell}$$
$$\le (\deg X)^{\ell+1}\left[\frac{ed^{\ell+1}(N-\ell+\kappa)(2\ell+5)l}{\kappa\varepsilon}\right]^{\ell}.$$

If $\{Q_1,\dots,Q_q\}$ are not of the same degree, then we set $d:=\mathrm{lcm}\,(d_1,\dots,d_q)$ and apply (17) for the hypersurfaces $\{D_1,\dots,D_q\}$ defined by $Q_1^{d/d_1},\dots,Q_q^{d/d_q}$, respectively, which yields our result. □

3.2. Another proof of Corollary 1.2

Proof. Similarly, we only need to give proofs for the case, where $\{Q_1,\dots,Q_q\}$ have the same degree d.

For a positive integer L, let $V_L=\mathbb{C}[x_0,\dots,x_n]_L$ and $\widehat{V_L}=\frac{\mathbb{C}[x_0,\dots,x_n]_L}{\mathcal{I}(X)_L}$, where $\mathcal{I}(X)$ is the ideal of $\mathbb{C}[x_0,\dots,x_n]$ defining X and $\mathcal{I}(X)_L=\mathcal{I}(X)\cap\mathbb{C}[x_0,\dots,x_n]_L$. Denote $[\phi]$ the projection of ϕ in $\widehat{V_L}$. In what follows, we introduce a filtration of $\widehat{V_L}$ with respect to $\{Q_{i_1},\dots,Q_{i_\kappa},P_{\kappa+1},\dots,P_\ell\}$. For brevity, we put $P_t:=Q_{i_t}$ for $1\le t\le\kappa$.

We arrange, in lexicographic order, the ℓ-tuples $\mathbf{i}=(i_1,\dots,i_\ell)$ of non-negative integers and put $\|\mathbf{i}\|:=\sum_j i_j$.

Definition (see [5, 7, 10])**.**

(i) For every $\mathbf{i}\in\mathbb{Z}_{\ge 0}^{\ell}$ and non-negative integer L with $L\ge d\|\mathbf{i}\|$, denote by $I_L^{\mathbf{i}}$ the subspace of $\mathbb{C}[x_0,\dots,x_n]_{L-d\|\mathbf{i}\|}$ consisting of all $r\in\mathbb{C}[x_0,\dots,x_n]_{L-d\|\mathbf{i}\|}$ such that

$$P_1^{i_1}\cdots P_\ell^{i_\ell}r-\sum_{\mathbf{j}=(j_1,\dots,j_\ell)>\mathbf{i}} P_1^{j_1}\cdots P_\ell^{j_\ell}r_{\mathbf{j}}\in\mathcal{I}(X)_L$$

$$\text{or } \left[P_1^{i_1}\cdots P_\ell^{i_\ell} r\right] = \left[\sum_{\mathbf{j}>\mathbf{i}} P_1^{j_1}\cdots P_\ell^{i_\ell} r_{\mathbf{j}}\right] \text{ on } \widehat{V_L}$$

for some $r_{\mathbf{j}} \in \mathbb{C}[x_0,\ldots,x_n]_{L-d\|\mathbf{j}\|}$;

(ii) Let $I^{\mathbf{i}}$ denote the homogeneous ideal in $\mathbb{C}[x_0,\ldots,x_n]$ generated by

$$\bigcup_{L\geq d\|\mathbf{i}\|} I_L^{\mathbf{i}}.$$

Remark 3.1 (see [5, 7, 10]). From the above definition, we have the following properties.

(i) $(\mathcal{I}(X),P_1,\ldots,P_\ell)_{L-d\|\mathbf{i}\|} \subseteq I_L^{\mathbf{i}} \subseteq \mathbb{C}[x_0,\ldots,x_n]_{L-d\|\mathbf{i}\|}$, where $(\mathcal{I}(X),P_1,\ldots,P_\ell)$ is the ideal in $\mathbb{C}[x_0,\ldots,x_n]$ generated by $\mathcal{I}(X)\cup\{P_1,\ldots,P_\ell\}$;

(ii) $I^{\mathbf{i}} \cap \mathbb{C}[x_0,\ldots,x_n]_{L-d\|\mathbf{i}\|} = I_L^{\mathbf{i}}$;

(iii) If $\mathbf{i}_1 - \mathbf{i}_2 := (i_{1,1}-i_{2,1},\ldots,i_{1,\ell}-i_{2,\ell}) \in \mathbb{Z}_{\geq 0}^\ell$, then $I_L^{\mathbf{i}_2} \subseteq I_{L+d\|\mathbf{i}_1\|-d\|\mathbf{i}_2\|}^{\mathbf{i}_1}$. Hence $I^{\mathbf{i}_2} \subseteq I^{\mathbf{i}_1}$.

Here, we set

$$\Delta_L^{\mathbf{i}} := \dim \frac{\mathbb{C}[x_0,\ldots,x_n]_{L-d\|\mathbf{i}\|}}{I_L^{\mathbf{i}}}. \tag{18}$$

Lemma 3.2 (see [5, 7, 10])**.**

(i) $\left\{I^{\mathbf{i}} \mid \mathbf{i}\in\mathbb{Z}_{\geq 0}^\ell\right\}$ *is a finite set.*

(ii) *There exists a positive integer L_0 such that, for every $\mathbf{i}\in\mathbb{Z}_{\geq 0}^\ell$, $\Delta_L^{\mathbf{i}}$ is independent of L for all L satisfying $L-d\|\mathbf{i}\|>L_0$.*

(iii) *For all L and $\mathbf{i}$ with $L-d\|\mathbf{i}\|\geq 0$, $\Delta_L^{\mathbf{i}}$ is bounded.*

Subsequently, we construct the filtration of V_L and $\widehat{V_L}$ with respect to $\{P_1,\ldots,P_\ell\}$ for a fixed large enough integer L.

Let τ_L denote the set of all $\mathbf{i}\in\mathbb{Z}_{\geq 0}^\ell$ with $L-d\|\mathbf{i}\|\geq 0$, arranged by the lexicographic order. Define the spaces $W_{\mathbf{i}} = W_{L,\mathbf{i}}$ by

$$W_{\mathbf{i}} = \sum_{\mathbf{j}\geq\mathbf{i}} P_1^{j_1}\cdots P_\ell^{j_\ell} V_{L-d\|\mathbf{j}\|}.$$

Clearly, $W_{(0,\ldots,0)} = V_L$ and $W_{\mathbf{i}} \supset W_{\mathbf{i}'}$ if $\mathbf{i}'>\mathbf{i}$, thus $\{W_{\mathbf{i}}\}$ is a filtration of V_L. Set $\widehat{W_{\mathbf{i}}} = \{[g] \mid g\in W_{\mathbf{i}}\}$. Hence, $\{\widehat{W_{\mathbf{i}}}\}$ is a filtration of $\widehat{V_L}$.

Lemma 3.3 (see [5, 7, 10])**.** *If $\mathbf{i}'$ follows $\mathbf{i}$ in lexicographic ordering, then*

$$\frac{\widehat{W_{\mathbf{i}}}}{\widehat{W_{\mathbf{i}'}}} \cong \frac{\mathbb{C}[x_0,\ldots,x_n]_{L-d\|\mathbf{i}\|}}{I_L^{\mathbf{i}}} = \Delta_L^{\mathbf{i}}.$$

By Lemma 3.2, for every $\mathbf{i}\in\mathbb{Z}_{\geq 0}^\ell$, there is an integer L_0, such that$\Delta_L^{\mathbf{i}}$ is a constant for all L satisfying $L-d\|\mathbf{i}\|>L_0$. Here, we let $\Delta^{\mathbf{i}}$ be this constant.

Take $\Delta_0 := \min_{\mathbf{i}\in\mathbb{Z}^{\ell}_{\geq 0}} \Delta^{\mathbf{i}}$. Then $\Delta_0 = \Delta^{\mathbf{i}_0}$ for some $\mathbf{i}_0 \in \mathbb{Z}^{\ell}_{\geq 0}$. By (iii) of the remark below Section 3.2, if $\mathbf{i} - \mathbf{i}_0 \in \mathbb{Z}^{\ell}_{\geq 0}$, then $\Delta^{\mathbf{i}} \leq \Delta^{\mathbf{i}_0}$. Set

$$\tau_L^0 := \left\{\mathbf{i} \in \tau_L : L - d\|\mathbf{i}\| > L_0 \text{ and } \mathbf{i} - \mathbf{i}_0 \in \mathbb{Z}^{\ell}_{\geq 0}\right\}.$$

Then we have the following lemma.

Lemma 3.4 (see [5, 7, 10])**.**

(i) $\Delta_0 = \Delta^{\mathbf{i}}$ *for all* $\mathbf{i} \in \tau_L^0$;

(ii) $\sharp\tau_L^0 = \frac{1}{d^\ell}\frac{L^\ell}{\ell!} + O\left(L^{\ell-1}\right)$;

(iii) $\Delta_L^{\mathbf{i}} = \Delta d^\ell$ *for all* $\mathbf{i} \in \tau_L^0$, *where* $\Delta = \deg X$.

Now, for L big enough, divisible by d, and for every $1 \leq j \leq \ell$,

$$\sum_{\mathbf{i}\in\tau_L} i_j = \frac{\Delta L^{\ell+1}}{d^{\ell+1}(\ell+1)!} + O\left(L^\ell\right). \tag{19}$$

(For a proof see (3.6) in [10].) Then combining (19) with Lemma 3.4, for every $1 \leq j \leq \ell$, we have

$$\sum_{\mathbf{i}\in\tau_L} \Delta_L^{\mathbf{i}} i_j = \frac{\Delta L^{\ell+1}}{d(\ell+1)!} + O\left(L^\ell\right). \tag{20}$$

Let $\{U_\lambda, \lambda \in \Lambda\}$ be an open covering of M, and denote by $\mathbf{f}_\lambda : U_\lambda \to \mathbb{C}^{n+1}$ a reduced representation of f on U_λ, correspondingly. Set $\mathfrak{u} := \dim \widehat{V_L}$ and choose a basis $\mathcal{B} = \{\psi_1, \ldots, \psi_{\mathfrak{u}}\}$ for $\widehat{V_L}$ with respect to the filtration. Let ψ_s be an element of $\mathcal{B}$, which lies inside $\widehat{W_{\mathbf{i}}}\backslash\widehat{W'_{\mathbf{i}}}$. We thus write $\psi_s = [P_1^{i_1}\cdots P_\ell^{i_\ell} r]$, where $r \in V_{L-d\|\mathbf{i}\|}$. By the definition of the Weil function and (20), we get

$$\sum_{s=1}^{\mathfrak{u}} \lambda_{\psi_s}(\mathbf{f}_\lambda(z)) \geq \left(\frac{\Delta L^{\ell+1}}{d(\ell+1)!} + O\left(L^\ell\right)\right)\cdot\sum_{t=1}^{\ell}\lambda_{\tilde{D}_{i_t}}(\mathbf{f}_\lambda(z)) + O(1), \tag{21}$$

where $O(1)$ denotes a bounded term which depends only on the ψ_s', but not on $\mathbf{f}_\lambda$ and z.

We fix a basis $\{\phi_1, \ldots, \phi_{\mathfrak{u}}\}$ for $\widehat{V_L}$, and let $\mathbf{F}_\lambda = (\phi_1(\mathbf{f}_\lambda), \ldots, \phi_{\mathfrak{u}}(\mathbf{f}_\lambda))$. Then $\mathbf{F} = \mathbb{P}(\mathbf{F}_\lambda)$ is independent of choices of λ. Therefore, we can define a meromorphic map $\mathbf{F} : M \to \mathbb{P}^{\mathfrak{u}-1}(\mathbb{C})$. Write the basis $\mathcal{B}$ as linear forms $L_1, \ldots, L_{\mathfrak{u}}$ in $\phi_1, \ldots, \phi_{\mathfrak{u}}$ satisfying $\psi_s(\mathbf{f}_\lambda) = L_s(\mathbf{F}_\lambda)$, $s = 1, \ldots, \mathfrak{u}$. By the definition of $\mathbf{F}$, there exist positive constants c_1 and c_2, independent of λ, such that

$$c_1\|\mathbf{f}_\lambda(z)\|^L \leq \|\mathbf{F}_\lambda(z)\| \leq c_2\|\mathbf{f}_\lambda(z)\|^L.$$

Combining the above inequality with (21), we obtain

$$\sum_{s=1}^{\mathfrak{u}} \lambda_{L_s}(\mathbf{F}_\lambda(z)) \geq \left(\frac{\Delta L^{\ell+1}}{d(\ell+1)!} + O\left(L^\ell\right)\right)\cdot\sum_{t=1}^{\ell}\lambda_{\tilde{D}_{i_t}}(\mathbf{f}_\lambda(z)) + O(1). \tag{22}$$

The linear forms $L_1, \ldots, L_{\mathfrak{u}}$ are linearly independent, and we have, by the assumption of algebraic non-degeneracy of $\mathbf{f}$, that $\mathbf{F} : M \to \mathbb{P}^{\mathfrak{u}-1}(\mathbb{C})$ is linearly nondegenerate. Since there are only finitely many choices of N-polynomials in $\{Q_1,\ldots,Q_q\}$, then the collection of all possible linear forms L_s $(1\le s\le \mathfrak{u})$ is a finite set. For simplicity, we denote it by $\mathcal{L}:=\{L_j\}_{j=1}^{\Lambda}$, $\Lambda<\infty$.

Hence, by (7) and (22), taking integration on the pseudo-sphere of radius r, we have

$$\left(\frac{\Delta L^{\ell+1}}{d(\ell+1)!}+O\left(L^{\ell}\right)\right)\cdot\sum_{j=1}^{q}m_f\left(r,D_j\right) \le \frac{N-\ell+\kappa}{\kappa}\int_{M\langle r\rangle}\max_{\mathcal{K}}\sum_{j\in\mathcal{K}}\lambda_{L_j}\left(\mathbf{F}\left(z\right)\right)\sigma+O(1), \tag{23}$$

where the maximum is taken over all subsets $\mathcal{K}\subseteq\{1,\ldots,\Lambda\}$ with $\#\mathcal{K}=\mathfrak{u}$ such that $\{L_j\}_{j\in\mathcal{K}}$ are linearly independent. Since $N_{\mathrm{RamF}}\left(r,s\right)\ge 0$, Lemma 2.4 yields that, for $r>s>0$, and any $\varepsilon'>0$ (which will be chosen later),

$$\begin{aligned}\| \int_{M\langle r\rangle}&\max_{\mathcal{K}}\sum_{j\in\mathcal{K}}\lambda_{L_j}\left(\mathbf{F}\left(z\right)\right)\sigma\\ \le{}& (u+\varepsilon')T_F(r,s)\\ &+\left(\frac{u(u-1)}{2}+\varepsilon'\right)\Big(m_0(\mathfrak{L};r,s)+\mathrm{Ric}_p(r,s)+\varsigma\log^+Y\big(r^2\big)+\varsigma\log^+(r)\Big)+O(1).\end{aligned}$$

Combining the above inequality with (23), we have

$$\begin{aligned}\| &\left(\frac{\Delta L^{\ell+1}}{d(\ell+1)!}+O\left(L^{\ell}\right)\right)\cdot\sum_{j=1}^{q}m_f\left(r,D_j\right)\\ &\le\frac{N-\ell+\kappa}{\kappa}\bigg\{(u+\varepsilon')T_F(r,s)\\ &\quad+\left(\frac{u^2-u}{2}+\varepsilon'\right)\Big(m_0(\mathfrak{L};r,s)+\mathrm{Ric}_p(r,s)+\varsigma\log^+Y\big(r^2\big)+\varsigma\log^+(r)\Big)\bigg\}+O(1).\end{aligned}\tag{24}$$

Now, we encounter a comparison between $T_F\left(r,s\right)$ and $T_f\left(r,s\right)$. By [6], we get

$$T_F(r,s)=L\cdot T_f(r,s)+O(1).$$

Since we have, for L big enough

$$u=H_X(L)=\Delta\frac{L^{\ell}}{\ell!}+O\left(L^{\ell-1}\right),$$

(24) gives that

$$\begin{aligned}\| &\sum_{j=1}^{q}m_f\left(r,D_j\right)\\ \le{}&\frac{N-\ell+\kappa}{\kappa}\bigg[\frac{d(\ell+1)!}{\Delta L^{\ell}\big(1+o(1)\big)}(u+\varepsilon')T_f(r,s)\end{aligned}$$

$$+ C_L\Big(m_0(\mathfrak{L}; r, s) + \mathrm{Ric}_p(r, s) + \varsigma \log^+ Y\big(r^2\big) + \varsigma \log^+(r)\Big)\Big],$$

where

$$C_L := \Delta \frac{\ell+1}{2\ell!} L^{\ell-1} + O\left(L^{\ell-2}\right)$$

is a constant dependent on L. For L large enough, we may suppose

$$\frac{d(\ell+1)!(u+\varepsilon')}{\Delta L^{\ell}\big(1+o(1)\big)} \leq d(\ell+1) + \varepsilon.$$

Hence, we have

$$\begin{aligned} (25) \qquad & \| \sum_{j=1}^{q} m_f\left(r, D_j\right) \\ \leq\ & \frac{N-\ell+\kappa}{\kappa}\big(d(\ell+1)+\varepsilon\big) T_f(r, s) \\ & + C_L\left(m_0(\mathfrak{L}; r, s) + \mathrm{Ric}_p(r, s) + \varsigma \log^+ Y\big(r^2\big) + \varsigma \log^+ r\right). \end{aligned}$$

Thus, this completes the proof. □

Acknowledgements. We would like to express our heartfelt appreciation to the reviewing experts for their valuable comments and suggestions on our paper. Additionally, we would like to acknowledge the financial support provided by the Natural Science Research Project for Colleges and Universities of Anhui Province (Nos. 2022AH050329, 2022AH050290).

References

[1] L. V. Ahlfors, *The theory of meromorphic curves*, Acta Soc. Sci. Fennicae. Nova Ser. A **3** (1941), no. 4, 31 pp.

[2] H. Cartan, *Sur les zeros des combinaisons lineaires de p fonctions holomorphers donnees*, Mathematica **7** (1985), 5–31.

[3] W. Chen and N. V. Thin, *A general form of the second main theorem for meromorphic mappings from a p-parabolic manifold to a projective algebraic variety*, Indian J. Pure Appl. Math. **52** (2021), no. 3, 847–860. https://doi.org/10.1007/s13226-021-00095-8

[4] P. Corvaja and U. M. Zannier, *On a general Thue's equation*, Amer. J. Math. **126** (2004), no. 5, 1033–1055.

[5] G.-E. Dethloff and T. V. Tan, *Holomorphic curves into algebraic varieties intersecting moving hypersurface targets*, Acta Math. Vietnam. **45** (2020), no. 1, 291–308. https://doi.org/10.1007/s40306-019-00336-3

[6] Q. Han, *A defect relation for meromorphic maps on generalized p-parabolic manifolds intersecting hypersurfaces in complex projective algebraic varieties*, Proc. Edinb. Math. Soc. (2) **56** (2013), no. 2, 551–574. https://doi.org/10.1017/S0013091512000284

[7] Q. Ji, Q. Yan, and G. Yu, *Holomorphic curves into algebraic varieties intersecting divisors in subgeneral position*, Math. Ann. **373** (2019), no. 3-4, 1457–1483. https://doi.org/10.1007/s00208-018-1661-4

[8] S. D. Quang, *Degeneracy second main theorems for meromorphic mappings into projective varieties with hypersurfaces*, Trans. Amer. Math. Soc. **371** (2019), no. 4, 2431–2453. https://doi.org/10.1090/tran/7433

[9] S. D. Quang and D. P. An, *Second main theorem and unicity of meromorphic mappings for hypersurfaces in projective varieties*, Acta Math. Vietnam. **42** (2017), no. 3, 455–470. https://doi.org/10.1007/s40306-016-0196-6

[10] M. Ru, *A defect relation for holomorphic curves intersecting hypersurfaces*, Amer. J. Math. **126** (2004), no. 1, 215–226.

[11] M. Ru, *Holomorphic curves into algebraic varieties*, Ann. of Math. (2) **169** (2009), no. 1, 255–267. https://doi.org/10.4007/annals.2009.169.255

[12] M. Ru, *Some generalizations of the second main theorem intersecting hypersurfaces*, Methods Appl. Anal. **21** (2014), no. 4, 503–526. https://doi.org/10.4310/MAA.2014.v21.n4.a6

[13] M. Ru and J. T.-Y. Wang, *A second main theorem on parabolic manifolds*, Asian J. Math. **9** (2005), no. 3, 349–371. https://doi.org/10.4310/AJM.2005.v9.n3.a4

[14] L. Shi, *Degenerated second main theorem for holomorphic curves into algebraic varieties*, Internat. J. Math. **31** (2020), no. 6, 2050042, 18 pp. https://doi.org/10.1142/S0129167X20500421

[15] B. Shiffman, *On holomorphic curves and meromorphic maps in projective space*, Indiana Univ. Math. J. **28** (1979), no. 4, 627–641. https://doi.org/10.1512/iumj.1979.28.28044

[16] M. Sombra, *Bounds for the Hilbert function of polynomial ideals and for the degrees in the Nullstellensatz*, J. Pure Appl. Algebra **117/118** (1997), 565–599. https://doi.org/10.1016/S0022-4049(97)00028-5

[17] W. Stoll, *The Ahlfors-Weyl theory of meromorphic maps on parabolic manifolds*, in Value distribution theory (Joensuu, 1981), 101–219, Lecture Notes in Math., 981, Springer, Berlin, 1983. https://doi.org/10.1007/BFb0066385

[18] P.-M. Wong and W. Stoll, *Second main theorem of Nevanlinna theory for nonequidimensional meromorphic maps*, Amer. J. Math. **116** (1994), no. 5, 1031–1071. https://doi.org/10.2307/2374940

[19] P.-M. Wong and P. P. W. Wong, *The second main theorem on generalized parabolic manifolds*, in Some topics on value distribution and differentiability in complex and p-adic analysis, 3–41, Math. Monogr. Ser., 11, Sci. Press Beijing, Beijing, 2008.

[20] L. Xie and T.-B. Cao, *Second main theorem for meromorphic maps into algebraic varieties intersecting moving hypersurfaces targets*, Chinese Ann. Math. Ser. B **42** (2021), no. 5, 753–776. https://doi.org/10.1007/s11401-021-0289-y

Yuehuan Zhu
School of Mathematics and Physics
Anhui University of Technology
Ma'anshan, 243032, P. R. China
Email address: zhuyh@ahut.edu.cn