TIME ANALYTICITY FOR THE HEAT EQUATION UNDER BAKRY-ÉMERY RICCI CURVATURE CONDITION

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Abstract. Inspired by Hongjie Dong and Qi S. Zhang's article [3], we find that the analyticity in time for a smooth solution of the heat equation with exponential quadratic growth in the space variable can be extended to any complete noncompact Riemannian manifolds with Bakry-Émery Ricci curvature bounded below and the potential function being of at most quadratic growth. Therefore, our result holds on all gradient Ricci solitons. As a corollary, we give a necessary and sufficient condition on the solvability of the backward heat equation in a class of functions with the similar growth condition. In addition, we also consider the solution in certain L^p spaces with $p \in [2, +\infty)$ and prove its analyticity with respect to time.

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold. The Bakry-Émery Ricci curvature tensor of M ([1]) is defined as

$$(1) \qquad \mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\, f,$$

where f is a smooth function on M (called the potential function), and Ric and Hess f denote the Ricci curvature tensor and the Hessian of f, respectively. It is clear that when f is a constant, Ric_f reduces to the Ricci curvature tensor. A gradient Ricci soliton is a Riemannian manifold (M^n, g) with constant Bakry-Émery Ricci curvature, namely,

$$(2) \qquad \mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$$

for some constant λ. It is called a shrinking, steady, or expanding Ricci soliton when $\lambda > 0, = 0$, or < 0, respectively. Also, manifolds with Bakry-Émery Ricci curvature bound are closely related to the singularity analysis of the Ricci flow and Ricci limit spaces (see e.g., [6, 8, 10, 14, 15]). Therefore, many efforts have been made to extend the results under the Ricci curvature condition to the Bakry-Émery Ricci curvature condition.

The study of the analyticity of the heat equation has a rich history. For generic solutions, as expected, the space analyticity is valid. However, the time analyticity is more delicate and is indeed invalid. Because in the Euclidean space, it is easy to construct a non-time-analytic solution of the heat equation in a finite space-time cylinder. Therefore, it is meaningful to study the time analyticity of the heat equation.

Recently, Qi S. Zhang [18] discovered on a complete noncompact Riemannian manifold whose Ricci curvature is bounded from below, any ancient solution of the heat equation with exponential growth in the space variable is analytic in time. This result was improved to any solution with exponential quadratic growth by Hongjie Dong and Qi S. Zhang [3]. In particular, they gave a necessary and sufficient condition on the solvability of the backward heat equation. In [17], Jiayong Wu obtained a similar result on the time analyticity of the heat equation for complete noncompact gradient shrinking Ricci solitons. For more results, see [4], [5], [7], [16] and references therein.

In [3], for Riemannian manifolds with Ricci curvature bounded below the key estimate for proving the time analyticity of the heat equation is the parabolic mean value inequality, which can also be found in [13] under the Bakry-Émery Ricci curvature condition. Here we emphasize that our result generalizes Hongjie Dong and Qi S. Zhang's result [3] and can be extended to all gradient Ricci solitons.

Theorem 1.1. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric}_f \geq -Kg$ for some constant $K \geq 0$. For a fixed point $o \in M$, assume that there exist non-negative constants a and b such that*

$$(3) \qquad\qquad |f(x)| \leq ad^2(x, o) + b \ \ \text{for all} \ \ x \in M,$$

where $d(x, o)$ is the distance function from x to o. Let $u(x, t)$ be a smooth solution of the heat equation $(\Delta - \partial_t)u = 0$ on $M \times [-2, 0]$ and satisfies exponential quadratic growth in the space variable, i.e.,

$$(4) \qquad\qquad |u(x, t)| \leq A_1 e^{A_2 d^2(x, o)} \ \ \text{for all} \ \ (x, t) \in M \times [-2, 0],$$

where A_1 and A_2 are some positive constants. Then $u(x, t)$ is analytic in time $t \in [-1, 0]$ with radius $\delta > 0$ depending only on n, K, a, b and A_2. Besides, we have

$$(5) \qquad\qquad u(x, t) = \sum_{j=0}^{\infty} a_j(x) \frac{t^j}{j!}$$

with $\Delta a_j(x) = a_{j+1}(x)$ and

(6) $$|a_j(x)| \leq A_1 A_3^{j+1}(j+1)^j e^{A_4 d^2(x,o)}, \; j = 0, 1, 2, \ldots,$$

where A_3 and A_4 are two positive constants depending on K, n, a, b, A_2 and n, a, A_2, respectively.

Remark 1.2. If the potential function is 0, i.e., $a = b = 0$ in (13), after careful calculation, then we get $A_4 = 2A_2$ in (6). Theorem 1.1 reduces Honejie Dong and Qi S. Zhang's result [3].

Remark 1.3. The growth condition (4) is sharp due to the Tychonov's solution of the heat equation in $\mathbb{R}^n \times (-\infty, +\infty)$ (see Remark 2.3 in [3]).

The conditions in the above theorem are especially satisfied on gradient Ricci solitons. For gradient Ricci solitons, it is well known that

(7) $$S + |\nabla f|^2 = 2\lambda f + C,$$

where S is the scalar curvature of M, ∇f is the gradient of f and C is a constant.

For gradient shrinking solitons, it is showed in [2] that $S \geq 0$, then setting $\tilde{f} = f + \frac{C}{2\lambda}$, (7) implies

$$|\nabla \tilde{f}|^2 \leq 2\lambda \tilde{f},$$

so

$$|\tilde{f}(x)| \leq \lambda d^2(x,o) + 2|\tilde{f}(o)|.$$

For gradient expanding solitons, it is showed in [11, 19] that $S \geq n\lambda$, then (7) implies that

$$\left| \nabla \sqrt{-f - \frac{C - n\lambda}{2\lambda}} \right| \leq \sqrt{-\frac{\lambda}{2}}.$$

Hence

$$\sqrt{-f(x) - \frac{C - n\lambda}{2\lambda}} \leq \sqrt{-\frac{\lambda}{2}} d(x,o) + \sqrt{-f(o) - \frac{C - n\lambda}{2\lambda}},$$

setting $\tilde{f} = f + \frac{C - n\lambda}{2\lambda}$, then

$$|\tilde{f}(x)| \leq -\lambda d^2(x,o) + 2|\tilde{f}(o)|.$$

For gradient steady solitons, we know $S \geq 0$ in [2].

If $C = 0$ in (7), then f is a constant.

If $C \neq 0$ in (7), by scaling the metric g, we can get

$$S + |\nabla f|^2 = 1,$$

which implies

$$|f(x)| \leq d(x,o) + |f(o)|.$$

To sum up, for gradient Ricci solitons, we can always adjust f or the metric such that

(8) $$|f(x)| \leq a d^2(x,o) + b,$$

where a and b are two positive constants depending on λ and $f(o)$, respectively.

Therefore, Theorem 1.1 implies the analyticity in time for smooth solutions of the heat equation on complete noncompact gradient Ricci solitons.

Theorem 1.4. *Let (M^n, g) be a complete noncompact gradient Ricci soliton satisfying* (2). *Let $u(x, t)$ be a smooth solution of the heat equation $(\Delta - \partial_t)u = 0$ on $M \times [-2, 0]$ and satisfies the growth condition*

$$(9) \qquad |u(x, t)| \leq A_1 e^{A_2 d^2(x, o)} \quad \text{for all} \quad (x, t) \in M \times [-2, 0],$$

where A_1 and A_2 are some positive constants, and $d(x, o)$ is the distance function from x to a fixed point o. Then $u(x, t)$ is analytic in time $t \in [-1, 0]$ with radius $\delta > 0$ depending only on $n, \lambda, f(o)$ and A_2. Besides, we have

$$(10) \qquad u(x, t) = \sum_{j=0}^{\infty} a_j(x) \frac{t^j}{j!}$$

with $\Delta a_j(x) = a_{j+1}(x)$ and

$$(11) \qquad |a_j(x)| \leq A_1 A_3^{j+1}(j+1)^j e^{A_4 d^2(x, o)}, \; j = 0, 1, 2, \dots,$$

where A_3 and A_4 are two positive constants depending on $n, \lambda, f(o), A_2$ and n, λ, A_2, respectively.

Remark 1.5. In [17], Jiayong Wu obtained a similar result on the time analyticity of the heat equation on the complete noncompact gradient shrinking Ricci solitons. More precisely, he showed that the bound of $a_j(x)$ in (10) is

$$(12) \qquad |a_j(x)| \leq A_1 e^{-\frac{\mu}{2}} e^{\frac{f(x)}{2}} (f(x) + 1)^{\frac{n}{4}} A_3^{j+1} j^j e^{2A_2 d^2(x, o)}, \; j = 0, 1, 2, \dots,$$

where A_3 is a constant depending on n and A_2 and $\mu = \mu(g, 1)$ denotes Perelman's entropy functional.

Comparing (11) with (12), it is not difficult to find that our result does not depend on μ.

As an application of Theorem 1.1, we give a solvable result for the backward heat equation.

Corollary 1.6. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric}_f \geq -Kg$ for some constant $K \geq 0$. For a fixed point $o \in M$, assume that there exist non-negative constants a and b such that*

$$(13) \qquad |f(x)| \leq ad^2(x, o) + b \quad \text{for all} \quad x \in M,$$

where $d(x, o)$ is the distance function from x to o. The Cauchy problem for the backward heat equation

$$(14) \qquad \begin{cases} (\Delta + \partial_t)u = 0, \\ u(x, 0) = a(x) \end{cases}$$

has a smooth solution with exponential quadratic growth of the space variable in $M \times (0, \delta)$ for some $\delta > 0$ if and only if

$$(15) \qquad |\Delta^j a(x)| \leq A_3^{j+1}(j+1)^j e^{A_4 d^2(x,o)}, \ j = 0, 1, 2, \ldots,$$

where A_3 and A_4 are some positive constants.

In addition, we also consider the solution of the heat equation in L^p spaces with $p \in [2, +\infty)$ and prove its analyticity with respect to the time variable.

Theorem 1.7. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric}_f \geq -Kg$ for some constant $K \geq 0$. For a fixed point $o \in M$, assume that there exist non-negative constants a and b such that*

$$(16) \qquad |f(x)| \leq ad^2(x, o) + b \ \text{ for all } \ x \in M,$$

where $d(x, o)$ is the distance function from x to o. Let $u(x, t)$ be a smooth solution of the heat equation $(\Delta - \partial_t)u = 0$ on $M \times [-2, 0]$. For any $p \geq 2$, assume that there exists a positive constant L such that

$$(17) \qquad \left(\int_M |u(x,t)|^p dv \right)^{\frac{1}{p}} \leq L \ \text{ for all } \ t \in [-2, 0].$$

Then $u(x, t)$ is analytic in time $t \in [-1, 0]$ with radius $\delta > 0$ depending only on n, K, a, b and p.

Moreover, we have

$$(18) \qquad u(x, t) = \sum_{j=0}^{\infty} a_j(x) \frac{t^j}{j!}$$

with $\Delta a_j(x) = a_{j+1}(x)$ and

$$(19) \qquad |a_j(x)| \leq A_6^{j+1}(j+1)^j e^{A_7 d^2(x,o)} \mathrm{Vol}(B_o(1))^{-\frac{1}{p}} L, \ j = 0, 1, 2, \ldots,$$

where A_6 and A_7 are two positive constants depending on n, K, a, b, p and n, a, K, p, respectively.

The rest of this paper is organized as follows. In Section 2, we recall a volume comparison theorem and a parabolic mean value inequality from [13] for complete Riemannian manifolds with Bakry-Émery Ricci curvature bounded below and the potential function locally bounded. In Section 3, applying Hongjie Dong and Qi S. Zhang's method of proof [3], we utilize the mean value inequality of Section 2 to prove Theorem 1.1, Corollary 1.6 and Theorem 1.7.

2. Preliminaries

For a fixed point $o \in M$ and $R > 0$, we define

$$L(R) = \sup_{B_o(3R)} |f|,$$

where $B_o(3R)$ is the geodesic ball centered at $o \in M$ with radius $3R$.

Theorem 2.1 ([13]). *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_f \geq -Kg$ for some constant $K \geq 0$. Then the following conclusions are true.*

(a)(Laplacian comparison) Let $r = d(y, p)$ be the distance from any point y to some fixed point $p \in B_o(R)$ with $0 < r < R$. Then for $0 < r_1 < r_2 < R$, we have

$$(20) \qquad \int_{r_1}^{r_2} (\Delta r - \frac{n-1}{r}) dr \leq \frac{K}{6}(r_2^2 - r_1^2) + 6L(R).$$

(b)(Volume element comparison) Take any point $p \in B_o(R)$ and denote the volume form in geodesic polar coordinates centered at p with $J(r, \theta, p) dr d\theta$, where $r > 0$ and $\theta \in S_p(M)$, a unit tangent vector at p. Then for $0 < r_1 < r_2 < R$, we have

$$(21) \qquad \frac{J(r_2, \theta, p)}{J(r_1, \theta, p)} \leq \left(\frac{r_2}{r_1}\right)^{n-1} e^{\frac{K}{6}(r_2^2 - r_1^2) + 6L(R)}.$$

(c)(Volume comparison) For any $p \in B_o(R)$, $0 < r_1 < r_2 < R$, we have

$$(22) \qquad \frac{\mathrm{Vol}(B_p(r_2))}{\mathrm{Vol}(B_p(r_1))} \leq \left(\frac{r_2}{r_1}\right)^n e^{\frac{K}{6}(r_2^2 - r_1^2) + 6L(R)},$$

where $\mathrm{Vol}(\cdot)$ denotes the volume of a region.

In [13], combining Theorem 2.1 and using a similar argument as in the proof of Lemma 3.2 in [9], the authors obtained a local Sobolev inequality.

Theorem 2.2 ([13]). *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_f \geq -Kg$ for some constant $K \geq 0$. Then there exist constants $\mu = 4n - 2 > 2$, c_3 and c_4, all depending only on n such that*

$$(23) \qquad \left(\int_{B_o(r)} |u|^{\frac{2\mu}{\mu-2}} dv\right)^{\frac{\mu-2}{\mu}} \leq \frac{c_3 e^{c_4(Kr^2 + L(R))}}{\mathrm{Vol}(B_o(r))^{\frac{2}{\mu}}} r^2 \int_{B_o(r)} (|\nabla u|^2 + r^{-2} u^2) dv$$

for all $0 < r < R$, where $u \in C^\infty(B_o(r))$.

Following the argument of Theorem 5.2.9 in [12], [13] showed a parabolic mean value inequality which is crucial to prove the analyticity of time. Its proof technique is the Moser iteration applied to the Sobolev inequality (23).

Proposition 2.3 (Mean value inequality [13]). *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_f \geq -Kg$ for some constant $K \geq 0$. For any real number s and any $0 < \tilde{\delta} < \delta' \leq 1$, let u be a smooth non-negative subsolution of the heat equation in the cylinder $Q = B_o(r) \times (s - r^2, s)$, $0 < r < R$.*

For $2 \leq p < \infty$, there exist constants $\tilde{c}_1(n)$ and $\tilde{c}_2(n)$ such that

$$(24) \qquad \sup_{Q_{\tilde{\delta}}} u^p \leq \frac{\tilde{c}_1(n) e^{\tilde{c}_2(n)(Kr^2 + L(R))}}{(\delta' - \tilde{\delta})^{4n} r^2 \mathrm{Vol}(B_o(r))} \cdot \int_{Q_{\delta'}} u^p dv dt.$$

For $0 < p < 2$, there exist constants $\tilde{c}_3(n,p)$ and $\tilde{c}_4(n)$ such that

$$(25) \qquad \sup_{Q_{\tilde{\delta}}} u^p \leq \frac{\tilde{c}_3(n,p)e^{\tilde{c}_4(n)(Kr^2+L(R))}}{(\delta'-\tilde{\delta})^{4n}r^2 \operatorname{Vol}(B_o(r))} \cdot \int_{Q_{\delta'}} u^p dvdt.$$

Here $Q_{\tilde{\delta}} = B_o(\tilde{\delta}r) \times (s - \tilde{\delta}r^2, s)$, $Q_{\delta'} = B_o(\delta'r) \times (s - \delta'r^2, s)$.

3. Proof of the main results

In this section, we apply the volume comparison theorem and the parabolic mean value inequality in Section 2 to prove the results of this article. We first prove Theorem 1.1.

Proof of Theorem 1.1. Since the heat equation is linear, we can assume that $A_1 = 1$. Indeed, we just need to prove the time analyticity result at $(x,0)$ for any $x \in M$.

Given $R \geq 1$. For any point $x \in B_o(R)$ and a positive integer j, since the solution $u(x,t)$ is smooth, we choose $t \in [-\delta, 0]$ for $0 < \delta < 1$, by Taylor's theorem,

$$(26) \qquad u(x,t) - \sum_{i=0}^{j-1} \partial_t^i u(x,0)\frac{t^i}{i!} = \frac{t^j}{j!}\partial_s^j u(x,s),$$

where $s = s(x,t,j) \in [t,0]$. It suffices to prove that the right-hand side of (26) tends to zero when j tends to infinity for any $x \in B_o(R)$ and $t \in [-\delta, 0]$ with $\delta > 0$ sufficiently small.

Since u^2 is a non-negative subsolution to the heat equation, we apply Proposition 2.3 with $p = 1$. Given a point $(x_0, t_0) \in M \times [-1,0]$ and a positive integer k, by letting $s = t_0$, $r = \frac{1}{\sqrt{k}}$, $\tilde{\delta} = \frac{1}{2}$, $\delta' = 1$ in (25), we have

$$u^2(x_0,t_0) \leq \frac{c_1(n)e^{c_2(n)(K\frac{1}{k}+\sup\limits_{B_{x_0}(\frac{3}{\sqrt{k}})}|f|)}}{(\frac{1}{2})^{4n}\frac{1}{k}\operatorname{Vol}(B_{x_0}(\frac{1}{\sqrt{k}}))}\int_{B_{x_0}(\frac{1}{\sqrt{k}})\times[t_0-\frac{1}{k},t_0]}u^2 dvdt.$$

We observe that

$$\sup_{B_{x_0}(\frac{3}{\sqrt{k}})}|f| \leq \sup_{B_o(d(x_0,o)+\frac{3}{\sqrt{k}})}|f| \leq a\left(d(x_0,o)+\frac{3}{\sqrt{k}}\right)^2+b \leq 2ad^2(x_0,o)+\frac{18a}{k}+b,$$

then we have

$$(27) \qquad u^2(x_0,t_0) \leq \frac{c_3(n)e^{c_4(n)(K+ad^2(x_0,o)+a+b)}k}{\operatorname{Vol}(B_{x_0}(\frac{1}{\sqrt{k}}))}\int_{B_{x_0}(\frac{1}{\sqrt{k}})\times[t_0-\frac{1}{k},t_0]}u^2 dvdt.$$

Since $(\partial_t - \Delta)\partial_t^{k-1}u = 0$, from (27), we obtain

$$(28) \qquad (\partial_t^{k-1}u)^2(x_0,t_0)$$

$$\leq \frac{c_3(n)e^{c_4(n)(K+ad^2(x_0,o)+a+b)}k}{\operatorname{Vol}(B_{x_0}(\frac{1}{\sqrt{k}}))}\int_{B_{x_0}(\frac{1}{\sqrt{k}})\times[t_0-\frac{1}{k},t_0]}(\partial_t^{k-1}u)^2 dvdt$$

for a positive integer k.

Next, we will bound the right-hand side of (28).

For positive integers $j = 1, 2, \ldots, k$, we define the following domains:

$$\Omega_j^1 = B_{x_0}\left(\frac{j}{\sqrt{k}}\right) \times \left[t_0 - \frac{j}{k}, t_0\right],$$

$$\Omega_j^2 = B_{x_0}\left(\frac{j + 0.5}{\sqrt{k}}\right) \times \left[t_0 - \frac{j + 0.5}{k}, t_0\right].$$

It is easy to see that

(29) $$\Omega_j^1 \subset \Omega_j^2 \subset \Omega_{j+1}^1.$$

Let $\psi_j^{(1)}$ be a standard Lipschitz cut-off function supported in

$$B_{x_0}\left(\frac{j + 0.5}{\sqrt{k}}\right) \times \left(t_0 - \frac{j + 0.5}{k}, t_0 + \frac{j + 0.5}{k}\right)$$

satisfying

(30) $$\psi_j^{(1)} = 1 \ \text{in} \ \Omega_j^1 \ \text{and} \ |\nabla \psi_j^{(1)}|^2 + |\partial_t \psi_j^{(1)}| \leq Ck,$$

where C is a universal constant that may be changed line by line.

For the above cut-off function $\psi = \psi_j^{(1)}$, since $(\Delta - \partial_t)u = 0$, using integration by parts, we compute that

$$\int_{\Omega_j^2} (u_t)^2 \psi^2 \, dvdt$$

$$= \int_{\Omega_j^2} u_t \Delta u \psi^2 \, dvdt$$

$$= -\int_{\Omega_j^2} \langle (\nabla u)_t, \nabla u \rangle \psi^2 \, dvdt - \int_{\Omega_j^2} u_t \langle \nabla u, \nabla \psi^2 \rangle \, dvdt$$

$$= -\frac{1}{2} \int_{\Omega_j^2} (|\nabla u|^2)_t \psi^2 \, dvdt - 2 \int_{\Omega_j^2} u_t \psi \langle \nabla u, \nabla \psi \rangle \, dvdt$$

$$= -\frac{1}{2} \int_{B_{x_0}\left(\frac{j+0.5}{\sqrt{k}}\right)} (|\nabla u|^2 \psi^2)(x, t_0) \, dv + \frac{1}{2} \int_{\Omega_j^2} |\nabla u|^2 (\psi^2)_t \, dvdt$$

$$\quad - 2 \int_{\Omega_j^2} u_t \psi \langle \nabla u, \nabla \psi \rangle \, dvdt$$

$$\leq \frac{1}{2} \int_{\Omega_j^2} |\nabla u|^2 (\psi^2)_t \, dvdt + \frac{1}{2} \int_{\Omega_j^2} (u_t)^2 \psi^2 \, dvdt + 2 \int_{\Omega_j^2} |\nabla u|^2 |\nabla \psi|^2 \, dvdt.$$

By (29) and (30), we have that

(31) $$\int_{\Omega_j^1} (u_t)^2 \, dvdt \leq Ck \int_{\Omega_j^2} |\nabla u|^2 \, dvdt.$$

Let $\psi_j^{(2)}$ also be a standard Lipschitz cut-off function supported in

$$B_{x_0}\left(\frac{j+1}{\sqrt{k}}\right) \times \left(t_0 - \frac{j+1}{k}, t_0 + \frac{j+1}{k}\right)$$

satisfying

(32) $$\psi_j^{(2)} = 1 \ \text{ in } \ \Omega_j^2 \ \text{ and } \ |\nabla\psi_j^{(2)}|^2 + |\partial_t\psi_j^{(2)}| \leq Ck.$$

Then we can obtain

(33) $$\int_{\Omega_j^2} |\nabla u|^2 dvdt \leq Ck \int_{\Omega_{j+1}^1} u^2 dvdt.$$

To achieve (33), considering the cut-off function $\varphi = \psi_j^{(2)}$, by $(\Delta - \partial_t)u = 0$, using integration by parts, we are continue to calculate that

$$\frac{1}{2}\int_{\Omega_{j+1}^1} \partial_t(u^2\varphi^2)dvdt - \int_{\Omega_{j+1}^1} \varphi\varphi_t u^2 dvdt$$

$$= \int_{\Omega_{j+1}^1} uu_t\varphi^2 dvdt$$

$$= \int_{\Omega_{j+1}^1} u\Delta u\varphi^2 dvdt$$

$$= -\int_{\Omega_{j+1}^1} |\nabla u|^2\varphi^2 dvdt - 2\int_{\Omega_{j+1}^1} u\varphi\langle\nabla u, \nabla\varphi\rangle dvdt.$$

Noticing that

$$\frac{1}{2}\int_{\Omega_{j+1}^1} \partial_t(u^2\varphi^2)dvdt = \frac{1}{2}\int_{B_{x_0}\left(\frac{j+1}{\sqrt{k}}\right)} (u^2\varphi^2)(x, t_0)dv \geq 0,$$

we have that

$$\int_{\Omega_{j+1}^1} |\nabla u|^2\varphi^2 dvdt$$

$$\leq \int_{\Omega_{j+1}^1} \varphi\varphi_t u^2 dvdt - 2\int_{\Omega_{j+1}^1} u\varphi\langle\nabla u, \nabla\varphi\rangle dvdt$$

$$\leq \int_{\Omega_{j+1}^1} \varphi\varphi_t u^2 dvdt + \frac{1}{2}\int_{\Omega_{j+1}^1} |\nabla u|^2\varphi^2 dvdt + 2\int_{\Omega_{j+1}^1} u^2|\nabla\varphi|^2 dvdt.$$

This implies that

$$\int_{\Omega_{j+1}^1} |\nabla u|^2\varphi^2 dvdt \leq 2\int_{\Omega_{j+1}^1} \varphi\varphi_t u^2 dvdt + 4\int_{\Omega_{j+1}^1} u^2|\nabla\varphi|^2 dvdt.$$

Then (33) follows by (29) and (32).

Combining (31) and (33), we achieve that

$$\int_{\Omega_j^1} (u_t)^2 dvdt \leq Ck^2 \int_{\Omega_{j+1}^1} u^2 dvdt.$$

Since above inequality holds for all solutions of the heat equation, we can replace u by $\partial_t^j u$. By induction, we conclude that

$$\int_{\Omega_1^1} (\partial_t^{k-1} u)^2 \, dv dt \leq C^{k-1} k^{2(k-1)} \int_{\Omega_k^1} u^2 \, dv dt.$$

By the selection of Ω_1^1 and Ω_k^1, we substitute the above inequality into (28) to get that

$$(34) \quad (\partial_t^{k-1} u)^2(x_0, t_0) \leq \frac{c_3(n) e^{c_4(n)(K + ad^2(x_0, o) + a + b)} k}{\mathrm{Vol}(B_{x_0}(\frac{1}{\sqrt{k}}))} C^{k-1} k^{2(k-1)} \int_{\Omega_k^1} u^2 \, dv dt.$$

Using exponential quadratic growth condition (4) and the triangle inequality, for some point $(x, t) \in \Omega_k^1$ we deduce

$$(35) \quad |u(x,t)|^2 \leq e^{2A_2(d(x, x_0) + d(x_0, o))^2} \leq e^{2A_2(\sqrt{k} + d(x_0, o))^2} \leq e^{4A_2 k + 4A_2 d^2(x_0, o)}.$$

By the volume comparison theorem (22), we have

$$(36) \quad \frac{\mathrm{Vol}(B_{x_0}(\sqrt{k}))}{\mathrm{Vol}(B_{x_0}(\frac{1}{\sqrt{k}}))} \leq k^n e^{\frac{K}{6}\left(k - \frac{1}{k}\right) + 6 \sup_{B_{x_0}(3\sqrt{k})} |f|} \leq k^n e^{\frac{K}{6} k + 6(2ad^2(x_0, o) + 18ak + b)}.$$

Substituting (35) and (36) into (34) gives

$$(37) \qquad\qquad |\partial_t^{k-1} u(x_0, t_0)| \leq A_3^k k^{k-1} e^{A_4 d^2(x_0, o)}$$

for all integers $k \geq 1$. Here A_3 and A_4 are two positive constants depending on K, n, a, b, A_2 and n, a, A_2, respectively.

Combining (37), for (26), we know that, for $\delta < \frac{1}{A_3 e}$, the right-hand side of (26) converges to 0 uniformly for $x \in B_o(R)$ as $j \to \infty$. Hence

$$u(x,t) = \sum_{j=0}^{\infty} \partial_t^j u(x,0) \frac{t^j}{j!},$$

that is, $u(x,t)$ is time analytic with radius δ. Write $a_j = a_j(x) = \partial_t^j u(x,0)$. We have that

$$\partial_t u(x,t) = \sum_{j=0}^{\infty} a_{j+1}(x) \frac{t^j}{j!} \quad \text{and} \quad \Delta u(x,t) = \sum_{j=0}^{\infty} \Delta a_j(x) \frac{t^j}{j!},$$

where both series converge uniformly for $(x,t) \in B_o(R) \times [-\delta, 0]$. Since $(\Delta - \partial_t)u = 0$, this gives that

$$\Delta a_j(x) = a_{j+1}(x)$$

and

$$|a_j(x)| \leq A_3^{j+1} (j+1)^j e^{A_4 d^2(x, o)}.$$

Here A_3 and A_4 are two positive constants depending on K, n, a, b, A_2 and n, a, A_2, respectively. $\qquad \square$

Next we apply Theorem 1.1 to prove Corollary 1.6.

Proof of Corollary 1.6. Assume that $u(x,t)$ is a smooth solution to (14) with exponential quadratic growth of the space variable in $M \times (0,\delta)$. Then

$$(\Delta - \partial_t)u(x,-t) = 0 \text{ and } |u(x,-t)| \leq A_1 e^{A_2 d^2(x,o)},$$

where A_1 and A_2 are some positive constants. By Theorem 1.1, we have

$$u(x,-t) = \sum_{j=0}^{\infty} a_j(x)\frac{(-t)^j}{j!}.$$

Combining the initial condition of (14) with Theorem 1.1, then (15) follows.

On the other hand, suppose (15) holds. Setting $u(x,t) = \sum_{j=0}^{\infty} \Delta^j a(x)\frac{t^j}{j!}$, by (15), it is easy to see that

$$\sum_{j=0}^{\infty} \Delta^{j+1}a(x)\frac{t^j}{j!} \text{ and } \sum_{j=0}^{\infty} \Delta^j a(x)\frac{\partial_t t^j}{j!}$$

all converge absolutely and uniformly in $B_o(R) \times [-\delta, 0]$ for any fixed $R > 0$ and $\delta > 0$ sufficiently small. Hence

$$(\Delta - \partial_t)u(x,t) = 0 \text{ for } (x,t) \in M \times [-\delta, 0].$$

By (15) again, we get the exponential quadratic growth for u,

$$|u(x,t)| \leq \sum_{j=0}^{\infty} |\Delta^j a(x)|\frac{|t|^j}{j!} \leq A_3 e^{A_4 d^2(x,o)} \sum_{j=0}^{\infty} \frac{(A_3(j+1)|t|)^j}{j!} \leq A_5 e^{A_4 d^2(x,o)}$$

provided that $t \in [-\delta, 0]$ with $\delta > 0$ sufficiently small.

Then a smooth solution with desired growth condition to (14) follows by letting $u = u(x,-t)$. $\square$

The proof of Theorem 1.7 is similar to Theorem 1.1. We only present the key steps.

Proof of Theorem 1.7. From (34), we know, for $(x_0, t_0) \in M \times [-1, 0]$ and any positive integer k,

$$(38) \quad (\partial_t^{k-1}u)^2(x_0, t_0)$$

$$\leq \frac{\bar{c}_1(n)e^{\bar{c}_2(n)(K+ad^2(x_0,o)+a+b)k}}{\text{Vol}(B_{x_0}(\frac{1}{\sqrt{k}}))}C^{k-1}k^{2(k-1)}\int_{B_{x_0}(\sqrt{k})\times[t_0-1,t_0]} u^2 dvdt.$$

By mean value theorem, there exists $\xi \in (t_0 - 1, t_0)$ such that

$$(39) \quad \int_{B_{x_0}(\sqrt{k})\times[t_0-1,t_0]} u^2 dvdt = \int_{B_{x_0}(\sqrt{k})} u^2(x,\xi)dv$$

$$\leq \left(\int_{B_{x_0}(\sqrt{k})} |u|^p(x,\xi)dv\right)^{\frac{2}{p}} \text{Vol}(B_{x_0}(\sqrt{k}))^{1-\frac{2}{p}}$$

$$\leq L^2 \text{Vol}(B_{x_0}(\sqrt{k}))^{1-\frac{2}{p}}$$

for $p \geq 2$, where we used Hölder inequality in the second line and in the last line we used the assumption (17).

By volume comparison theorem (22) and $k \geq 1$, we have

$$(40) \qquad \frac{\mathrm{Vol}(B_{x_0}(\sqrt{k}))^{1-\frac{2}{p}}}{\mathrm{Vol}(B_{x_0}(\frac{1}{\sqrt{k}}))} = \frac{\mathrm{Vol}(B_{x_0}(\sqrt{k}))\,\mathrm{Vol}(B_{x_0}(\sqrt{k}))^{-\frac{2}{p}}}{\mathrm{Vol}(B_{x_0}(\frac{1}{\sqrt{k}}))}$$

$$\leq (k)^n e^{\frac{K}{6}k+12ad^2(x_0,o)+108ak+6b}\,\mathrm{Vol}(B_{x_0}(1))^{-\frac{2}{p}}.$$

To get a lower bound of $\mathrm{Vol}(B_{x_0}(1))$, we use the volume comparison theorem (22) again, then

$$(41) \qquad \mathrm{Vol}(B_o(1))$$

$$\leq \mathrm{Vol}(B_{x_0}(d(x_0,o)+1))$$

$$\leq \mathrm{Vol}(B_{x_0}(1))(d(x_0,o)+1)^n e^{\frac{K}{6}[(d(x_0,o)+1)^2-1]+6\sup\limits_{B_{x_0}(3d(x_0,o)+3)}|f|}$$

$$\leq \mathrm{Vol}(B_{x_0}(1))e^{nd(x_0,o)+\frac{K}{6}(d^2(x_0,o)+2d(x_0,o))+6\sup\limits_{B_o(4d(x_0,o)+3)}|f|}$$

$$\leq \mathrm{Vol}(B_{x_0}(1))e^{(n+\frac{K}{2})d^2(x_0,o)+\frac{1}{4}n+\frac{K}{12}+6(32ad^2(x_0,o)+18a+b)}.$$

Combining (41), (40), (39) with (38), we arrive at

$$|\partial_t^{k-1}u(x_0,t_0)| \leq A_6^k k^{k-1} e^{A_7 d^2(x_0,o)}\,\mathrm{Vol}(B_o(1))^{-\frac{1}{p}}L$$

for all integers $k \geq 1$. Here A_6 and A_7 are two positive constants depending on n, K, a, b, p and n, a, K, p, respectively. $\qquad\square$

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